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Computational Error Estimate for the Power Series Solution of Odes Using Zeros of Chebyshev Polynomial

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Abstract

This paper compares the error estimation of power series solution with recursive Tau method for solving ordinary differential equations. From the computational viewpoint, the power series using zeros of Chebyshevpolunomial is effective, accurate and easy to use.

Keywords: Lanczos Tau method, Chebyshev polynomial, initial value problems, Lanczos – Ortiz Canonical polynomial, Ordinary Differential Equations

1.0 Introduction

This paper is concerned with the error estimated of ordinary differential equation of the form:

$$Ly(x) \equiv \sum_{r=0}^{m} \left(\sum_{k=0}^{N_r} P_{rk} x^k \right) y^{(r)}(x) = \sum_{r=0}^{\pm} f_r x^r,\ a \le x \le b \tag{1}$$

together with the associated conditions:

$$L * y(x_{rk}) \equiv \sum_{r=0}^{m-1} a_{rk} y^{(r)}(x_{rk}) = \sim_k,\ k = 1(1)m \tag{2}$$

by seeking an approximate solution of the form

$$y_n(x) = \sum_{r=0}^{n} a_r x^r,\ r < +\infty$$

of $y(x)$ which is the exact solution of the corresponding perturbed system

$$Ly_n(x) = \sum_{r=0}^{\pm} f_r x^r + H_n(x) \tag{3}$$

$$L * y_n(x_r k) = \sim_k, k = 1(1)m \tag{4}$$

where L is the linear differential operator, and $\sim_k$, f_r, $P_{r,k,}$ N_r, $r = 0(1)m$, $k = 0(1)N_r$, a and b are real constants, $y^{(r)}$ denotes the derivatives of order r *of* $y(x)$, and the perturbation term $H_n(x)$ in equation (3) is defined by:

$$H_n(x) = \sum_{i=0}^{m+s-1} 2_{i+1} T_{n-m+i+1}(x) = \sum_{i=0}^{m+s-1} 2_{i+1} \sum_{i=0}^{n-m+i+1} C_r^{(n-m+i+1)} x^r \tag{5}$$

and $C_r^{(n)}$ is the coefficient of x^r in the $n-th$ degree chebyshev polynomial $T_n(x)$: that is,

$$T_n(x) = \cos\left(n \cos^1 \left\{ \frac{2x - a - b}{b - a} \right\} \right) \equiv \sum_{r=0}^{n} C_r^{(n)} x^r \tag{6}$$

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The 2's are free parameters to be determined and *s*, the number of over determination of (1), is defined in [1-10]:

$$s = \max\left\{N_r - r > 0 \middle| 0 \le r \le m\right\} \tag{7}$$

2.0 Review of Recursive formulation of the Tau approximant

In this section, we review the recursive approximant [2, 4, 11, 12, 13] and canonical polynomial [14], by adding perturbation terms to the right hand sides of (1). We have in [4]:

$$y_n(x) = \sum_{r=s}^{\pm} f_r q_r(x) + \sum_{i=0}^{m+s-1} 2_{i+1} \sum_{r=s}^{n-m+i+1} C_r^{(n-m+i+1)} q_r(x) \tag{8}$$

Assume $Q_r(x) = P_r = 1, r = 0(1)(s-1)$, then

$$\sum_{i=0}^{m+s-1} 2_{i+1} \sum_{r=0}^{n-m+i+1} C_r^{(n-m+i+1)} P_r + \sum_{r=0}^{\pm} f_r P_r = 0 \tag{9}$$

Where $q_r(x) = Q_r(x) - P_r$ and for the undetermined canonical polynomials (if any) assume $P_r = 1$, when equating the coefficient of $Q_r(x)$ to zero, otherwise $P_r = 0$ for $r = 0, 1, \cdots, (s-1)$ (that is equation (9) is the coefficient of undetermined canonical polynomials),

$$P_r = \frac{-1}{\sum_{k=0}^{m} k!\binom{r-s}{k} P_{k,k+s}} \left\{ \sum_{k=1}^{m} \left(\sum_{j=k}^{m} j!\binom{r-s}{j} P_{j,j-k} \right) P_{r-s-k} \right.$$

$$\left. + \sum_{k=0}^{s-1} \left(\sum_{j=0}^{m} j!\binom{r-s}{j} P_{j,j+k} \right) P_{r-s+k} \right\}, r \ge s \tag{10}$$

and

$$Q_n(x) = \frac{1}{\sum_{k=0}^{m} k!\binom{n-s}{k} P_{k,k+s}} \left\{ x^{n-s} - \left[\sum_{k=1}^{m} \left(\sum_{j=k}^{m} j!\binom{n-s}{j} P_{j,j-k} \right) Q_{n-s-k}(x) \right. \right.$$

$$\left. \left. + \sum_{k=0}^{s-1} \left(\sum_{j=0}^{m} j!\binom{n-s}{j} P_{j,j+k} \right) Q_{n-s+k}(x) \right] \right\} \tag{11}$$

3.0 Error estimate for the recursive form (RF)

The canonical polynomials were generated and generalized [14, 19], and this was used in error estimation of the Tau method. Adeniyi [16, 18] reported a polynomial estimate $(e_n(x))_{n+1} \cong E_{n+1}(x)$ of degree $(n+1)$ as:

$$(e_n(x))_{n+1} \cong E_{n+1}(x) = \frac{¤_n v_m(x) T_{n-m+1}(x)}{C_{n-m+1}^{(n-m+1)}} = \frac{¤_n v_m(x) \sum_{r=0}^{n-m+1} C_r^{(n-m+1)} x^r}{C_{n-m+1}^{(n-m+1)}} \tag{12}$$

where $v_m(x)$ is to ensure that $E_{n+1}(x)$ satisfies some or all the homogenous conditions of $e_n(x)$.

He also reported that:

$$L(E_{n+1}(x)) = \hat{H}_{n+1}(x) - H_n(x) \tag{13}$$

where

$$\hat{H}_{n+1}(x) = \sum_{i=0}^{m+s} 2_{i+1} \sum_{r=0}^{n-m+i-2} C_r^{(n-m+i+2)} x^r \tag{14}$$

After assuming that a transformation has been made such that $a = 0, b = 1$, equation (12) becomes:

$$E_{n+1}(x) = \frac{\text{¤}_n(x)\sum_{r=0}^{n-m+1} C_r^{(n-m+1)} x^{r+m}}{C_{n-m+1}^{(n-m+1)}} \qquad (15)$$

as an approximate to the error

$$e_n(x) = y(x) - y_n(x) \qquad (16)$$

in $y_n(x)$ obtained from the Tau approximation process [4, 5]. The parameter $\ddot{}_n$ is to be determined along with $\tilde{\text{²}}_r$'s parameters [19] and the exact error (maximum error) is defined as:

$$\text{¶} = a \leq^{\max} x \leq b \left\{\left|y(x) - y_n(x)\right|\right\} \qquad (17)$$

4.0 Error estimate for the power series form (PSF)

The error function (16), which satisfies the perturbed error problem

$$L(e_n(x)) = -H_n(x) = -\sum_{r=0}^{m+s-1} {}^{2}{}_{m+s-r} T_{n-m+r+1}(x) \qquad (18)$$

$$L^* e_n(\sim) = 0$$

Satisfies the perturbed error problem [17, 18, 19]

$$L(e_n(x))_{n+1} = -H_n(x) + H_{n+1}(x) = -\sum_{r=0}^{m+s-1} {}^{2}{}_{m+s-r} T_{n-m+r+1}(x) + \sum_{r=0}^{m+s-1} {}_{m+s-r} T_{n-m+r+2}(x) \qquad (19)$$

$$L^* e_n(\sim)_{n+1} = 0$$

where the extra $_r, r = 1, 2, \ldots, m+s$ parameters and $\ddot{}_n$ are to be determined, using zeroes of chebyshev polynomial together with the given conditions. A forward elimination process is recommended for the solution of the resulting linear system. The value of $\ddot{}_n$ is then inserted into (15) and subsequently, we get the estimate:

$$E_{n+1}(x) = a \leq^{\max} x \leq b \left|(e_n(x))_{n+1}\right| = \left|\frac{\ddot{}_n}{C_{n-m+1}^{n-m+1}}\right| \qquad (20)$$

5.0 Numerical Examples

In this section, we applied the presented method to some selected examples. The main objective here is to solve five examples by power series using zeros of chebyshev polynomial and compare the results with method discussed in section 2. All the approximant solution are subject to degree 5.

Example 4.1

$$y''(x) + y(x) = 0,\; y(0) = 1,\; y'(0) = 0,\; 0 \leq x \leq 1$$

With analytical solution $y(x) = \cos(x)$

Example 4.2

$$Ly(x) = 2(1+x)\, y'(x) + y(x) = 0,\; y(0) = 1,\; 0 \leq x \leq 1$$

with analytical solution $y(x) = (1+x)^{\frac{-1}{2}}$

Example 4.3

$$Ly(x) = y'(x) - x^2 y(x) = 0,\; y(0) = 1$$

with analytical solution $y(x) = \exp\left(\frac{1}{3}x^3\right)$

Example 4.4

$$y'''(x) - y''(x) - y'(x) + y(x) = ,\; y(0) = 2,\; y'(0) = 1,\; y''(0) = 0,0 \leq x \leq 1\,[19]$$

with analytical solution $y(x) = (2-x)\exp(x)$

Example 4.5

$$y^{iv}(x) - 3601 y^{ii}(x) + 3600 y(x) = -1 + 1800x^2,\; y(0) = y'(0) = y''(0) = y'''(0) = 1,\; 0 \leq x \leq 1\,[16]$$

with analytical solution $y(x) = \frac{1}{2}\exp(-x)\left(1 + 2\exp(x) + \exp(2x) + x^2 \exp(x)\right)$

Table 4.1: Exact Error

Method	Example 4.1 (¶)	Example 4.2 (¶)	Example 4.3 (¶)	Example 4.4 (¶)	Example 4.5 (¶)
RF	1.16×10^{-5}	3.08×10^{-5}	2.07×10^{-4}	1.70×10^{-5}	2.77×10^{-3}
PSF	1.16×10^{-5}	1.16×10^{-5}	2.07×10^{-4}	1.70×10^{-5}	2.77×10^{-3}

Table 4.2: Error Estimate (Maximum Error)

Method	Example 4.1	Example 4.2	Example 4.3	Example 4.4	Example 4.5
RF	1.34×10^{-5}	2.90×10^{-5}	4.94×10^{-4}	4.72×10^{-4}	7.68×10^{-6}
PSF	1.20×10^{-5}	2.90×10^{-5}	4.44×10^{-4}	3.93×10^{-4}	$7.68\ 10^{-6}$

6.0 Conclusion

In this work, we have studied the power series approach using zeros of chebyshev polynomial to compute the error estimate of ODEs. The error estimate obtained with the presented method are in good agreement with the recursive form, as it is favourably compared and give accurate results.

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