Journal of the Nigerian Association of Mathematical Physics
Volume 30, (*May, 2015*), pp427 – 430

On Bond Pricing with Jumps in Interest Rates

Michael C. Anyanwu

Department of Mathematics, Michael Okpara University of Agriculture, Umudike, Abia state.

Abstract

The pricing of zero coupon bonds when the interest rate in the market is given by a jump-diffusion stochastic process of CIR-type is considered. The jump is assumed to be a Levy process of exponential type with no drift. Solving the associated partial integro-differential equation for the bond price, a semi-analytical expression, involving the Levy exponent, is obtained.Numerical experiments show that, with the same set of parameters, the bond price is higher with jump interest rate than with Gaussian interest rates.

Keywords: Levy process, zero coupon bond, characteristic exponent, variance gamma

1.0 Introduction

In literature, there exists several articles on the application of Levy processes to finance. See e.g. [1, 2, 3, 4] and the bibliography therein. Levy models were introduced in finance as an alternative to Black and Scholes[5], which assumes that the asset price dynamics is given by geometric Brownian motion. Unlike Brownian motion, Levy processes exhibit jumps in their paths, and therefore capture the asset price dynamics better than Brownian motion which has continuous sample paths.

The main aim of this article is to apply Levy processes to bond pricing. To do this, jumps (Levy process) are introduced into the dynamics of the risk-free interest rates used in the pricing of the zero coupon bond. We assume that the risk-free interest rate is given by the Cox et al [6] interest rate process with jumps. We assume that the jumps are variance gamma process, which is a purely discontinuous Levy process. This helps in the better capturing of the interest rate movement for bond pricing. When the Ito lemma is applied, a partial integro-differential equation (PIDE) involving the characteristic exponent of the variance gamma process is obtained. The PIDE is solved by converting it first to a system of first order differential equation which is easier to handle.

The rest of this paper is organized as follows: In the rest of section one, the basic facts in the theory of Levy processes are presented. Also, the variance gamma(VG) process, which is the preferred Levy process used in this paper is introduced. In section two, pricing of zero coupon bond in the presence of jumps is considered generally. Section three has the numerical implementation of the model presented in this paper. Conclusion is also made in this section.

1.1 Basic Facts about Levy Processes

By simple definition, a Levy process X_t is a stochastic process with stationary and independent increments. It is therefore obvious that every Levy process has a moment generating function which is expressed in the form:

$E[e^{iuX_t}] = e^{-t\psi(u)}, \;\; u \in \mathbb{R}, t \geq 0$

The function ψ is called the characteristic exponent for X_t. The general representation of ψ for every Levy process is given by the Levy-Khintchine formula:

$$\psi(u) = \frac{\sigma^2 u^2}{2} - ibu + \int_{\mathbb{R}\setminus 0} \left(1 + iux1_{[-1,1]}(x) - e^{iux}\right) F(dx)$$

where$\sigma \geq 0$ and $b \in \mathbb{R}$ are the volatility and drift of X_t respectively, and F is the measure on $\mathbb{R}$ satisfying

$$\int_{\mathbb{R}\setminus 0} \min\{|x|^2, 1\} F(dx) < \infty$$

Corresponding author: Michael C. Anyanwu, E-mail: manyanwu71@yahoo.com, Tel.: +2348064209376

1.2 The Variance Gamma Process

The variance gamma (VG) process, proposed by Madan and Senta, belongs to the class of Levy process of infinite activity. Unlike jump-diffusion processes, VG process does not have any continuous components and is of bounded variation. The VG process is a purely jump process and infinite activity means that the paths jumps infinitely many times, for each interval. More so, jumps that are larger than a given quantity occur only a finite number of times. A typical VG process can be obtained by subordinating the standard Brownian motion with drift θ and variance ϑ, with a gamma process. The VG process has the characteristic exponent given by

$\psi(u) = -ibu + c[\ln(-\lambda_1 - iu) - \ln(-\lambda_1) + \ln(\lambda_2 + iu) - \ln(\lambda_2)]$ (1)

Where $b \geq 0, \lambda_1 < 0 < \lambda_2$, and $c > 0$. For the purpose of this paper, we take that $b = 0$. Hence, there is no drift.

1. **Bond Price with Jumps in CIR Interest Rates.**

Let (Ω, f_t, Q)be the underlying probability space equipped with the filtration $\{f_t\}$, Ω is the set of all possible events and Q is the risk-neutral measure. Under the measure Q, the interest rate dynamics is given by the mean-reverting process with jumps:

$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}d\omega_t + dX_t$ (2)

$\boldsymbol{\kappa} > 0$is the coefficient of mean reversion, $\theta \geq 0$ is the central tendency of the interest rate process, $\sigma > 0$ is the volatility, $d\omega_t$ is the increment of the standard Brownian motion, and dX_t is the increment of the Levy process. It is assumed here that ω_t and X_t are independent and that has no drift and is of finite variation such that its Levy measure $F(dx)$satisfies

$$\int_{\mathbb{R}\backslash 0} \min\{|x|, 1\} F(dx) < \infty$$

The characteristic exponent ψ of $X_t$is given by

$\psi(u) = \int_{\mathbb{R}\backslash 0}(1 - e^{iux})\, F(dx)$ (3)

And the infinitesimal generator L of X_t acts as follows

$Lg(x) = \int_{\mathbb{R}\backslash 0}\big(g(x + x') - g(x)\big)F(dx')$ (4)

Under the measure Q, the time t price $P(t, r)$ of a zero coupon bond that pays one unit of currency at maturity date T is given by

$P(t, r) = E^{Q,r}[e^{-\int_t^T r(s)ds}]$ (5)

Applying Ito lemma, we see that $P(t, r)$ solves the partial integro-differential equation

$\left(\partial_t + \kappa(\theta - r)\partial_r + \frac{\sigma^2 r}{2}\partial_r^2 + L - r\right)P(t, r) = 0$ (6)

subject to $P(T, r) = 1$.

where

$LP(t, r) = \int_{\mathbb{R}\backslash 0}\big(P(t, r + r') - P(t, r)\big)F(dr')$ (7)

Let the solution to (6) be given by

$P(r, \tau) = \exp\big(A(\tau)r + B(\tau)\big)$, $\tau = T - t$ (8)

Using (7) on (8) we have that

$$LP(t, r) = \int_{\mathbb{R}\backslash 0} (\exp\Big(A(\tau)(r + r') + B(\tau)\Big) - \exp(A(\tau)r + B(\tau)))F(dr')$$

$$= \exp(A(\tau)r + B(\tau)) \int_{\mathbb{R}\backslash 0} (\exp(A(\tau) - 1)F(dr')$$

$= \psi(-iA)P(\tau, r)$ (9)

Substituting (8) and (9) into the PIDE (6), and comparing the coefficients, it is easily seen that $A(\tau)$ and $B(\tau)$satisfy the system of first order ordinary differential equations:

$-A_\tau(\tau) - \kappa A(\tau) + \frac{\sigma^2}{2}A^2(\tau) - 1 = 0,\ A(0) = 0$ (10)

$-B_\tau + \kappa\theta A(\tau) - \psi\big(-iA(\tau)\big) = 0, \quad B(0) = 0$ (11)

The solution to (10) is known analytically:

$$A(\tau) = \frac{\kappa + \alpha}{\sigma^2}\left(1 - exp\left(\frac{2\alpha\tau}{\sigma^2}\right)\right)\left(1 - \beta exp\left(\frac{2\alpha\tau}{\sigma^2}\right)\right)^{-1}$$

$\alpha = \sqrt{\kappa^2 + 2\sigma^2}$, $\beta = \frac{\kappa+\alpha}{\kappa-\alpha}$.

Whereas because of the presence of the Levy exponent ψ, we use Runge-Kutta method order 4 to approximately solve (11) at different points of time to maturity τ.

2. Numerical Example and Conclusion

To test this model, the interest rate range $r \in [0,1]$ is used, with the following parameters; κ=1.25, θ=0.72, $\sigma = 1.9$. The parameters of the variance gamma process are c=0.78, $\lambda_1 = -.75$, $\lambda_2 = 0.5855$ and the maturity date of the bond is one year. For comparison purposes, the same parameters are used for the cases when there are jumps in the interest rate and when there are no jumps. The results of the calculations are given graphically in Figure 1 and Figure 2. From the numerical results obtained, one sees that the bond prices obtained when there is presence of jumps are higher than the prices when there are no jumps. For example, when the interest rate $r$is 0.01, we have the following bond prices (0.7347,0.9965), (0.7375,0.9975), (0.9931,0.9986) and (1.0000,0.9986) at
$\tau = 1, 0.99, 0.01, 0.00$respectively. The first number in the brackets is the price when there are no jumps in the interest rate, while the second number is the price when there are jumps. At maturity date, $\tau = 0$, the price when there are no jumps is slightly higher than the price when there are jumps. This is because the price at maturity date which is 1.00 is already known, and is used in the derivation of the analytical formula when there are no jumps.

2.0 Conclusion

It is shown in this article that it is possible to incorporate jumps in interest rates when zero coupon bonds are priced. Moreover, interest rates dynamics are not always continuous in nature, but exhibit jumps at some points in time. Therefore, adding jumps to interest rates helps in capturing the real evolution of interest rate, and hence, helps in efficient pricing of zero coupon bond

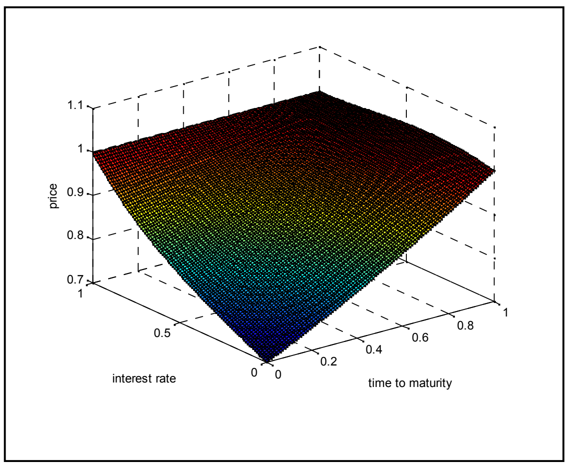

Fig 1: Bond price with jumps in interest rate

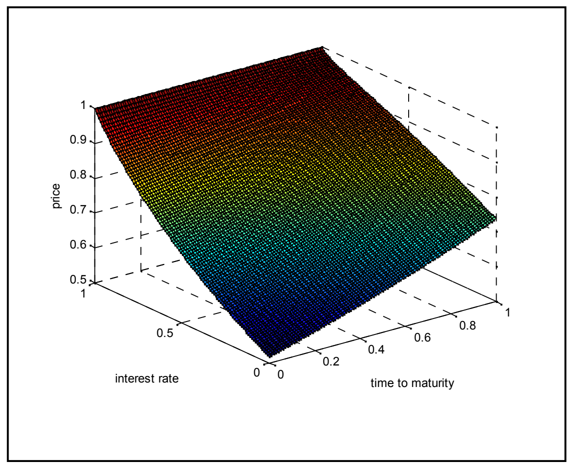

Fig 2: Bond price without jumps in interest rate

3.0 Bibliography

[1] Madan, D. and Seneta, E. (1990). The variance gamma model for share market returns, Journal of Business 63, 511-524.

[2] Carr, P. and Wu, L.(2004). Time-changed Levy processes and option pricing. Journal of Financial Economics 7, 113-141.

[3] Boyarchenko, S. I. and Levendorskii, S. Z.(2002). Perpetual American options under Levy processes. SIAM Journal on Control and Optimization 40(6), 1663-1696.

[4] DE Innocentis, M. and Levendorskii, S.Z.(2012). Pricing Discrete Barrier Options and Credit Default Swaps under Levy processes, Working paper. Available at SSRN: http://ssrn.com/abstract=2080215

[5] Black, F., and Scholes, M. (1973). The pricing of options andcorporate liabilities, Journal of Political Economy, 81, 673-659.

[6] Cox, J.C., Ingersol, J.E. and Ross, S.A., (1985). A Theory of the Term Structure of Interest Rates. Econometrica 53, 385-407