Journal of the Nigerian Association of Mathematical Physics
Volume 30, **(*May, 2015*), pp421 – 426**
© J. of NAMP

Application of Analytic Function in two Dimensional Horizontal Flow of Complex Potential in a Cylinder

Orukari M. A

Department of Mathematics, Niger Delta University, Wilberforce Island, Bayelsa State, Nigeria.

Abstract

The application of flow equation to analytic function enables one to see physical applicability of complex analysis. This work provides solution to physical problem in fluid flow that formalizes the idea of motion of fluid element.

Harmonic functions are used to study fluid flow under the assumption that an incompressible and frictionless fluid, flows over the complex plane and that all cross sections in planes parallel to the complex are the same. Conformal mapping is used to transform a region in which the problem is posed to the one in which the solution is easy to obtain.

Keywords: Complex Potential, Analytic function, harmonic function, conformal mapping, irrotational

1.0 Introduction

This work is basically concerned with two dimensional fluid flows, an ideal fluid and incompressible fluid, the conformal mapping method is used to transform a region where the problem is posed to one where solutions can be obtained and the problem of finding streamlines distribution with uniform irrotational flow past a cylinder. The flow pass two cylinders in a fluid was also studied by [1],solutions were obtain by streamlines distribution'

An irrotational flow and an incompressible fluid has a close relationship to a complex potential [2, 3] in which the complex potential for uniform flow in an (x,y)- plane in a cylinder is given by $w(z) = u(z + 1/z)$, where z $z = x + iy$

The motion that occurs at fixed temperature throughout the fluid or that heard transfer between regions of temperature can be ignored, we allow the density to vary for incompressible fluid $\frac{\partial p}{\partial t} + \nabla.(\ell u) = 0, \quad \ell = \text{constant.}$

in the present of gravity fieldand on streamlines

$$\Omega = gz, \;\; g = \text{constant}:$$

$$\frac{1}{2}u.v + \frac{p}{\ell} + gz = \text{constant}$$

The study of general analytical framework of motion of point vortices around distribution of obstacles was investigated by [4]. We consider an irrotational fluid flow which is characterized by requiring the line integral of the tangential component of v(x,y) along closed contour by identically zero

2.0 Mathematical Formulation of the Physical Problem

For a two dimensional fluid flow we suppose that a fluid flows over the complex plane and that the velocity at the point $z = x + iy$ is given by the velocity vector

$$v(x, y) = p(x, y) + iq(x, y) \ldots\ldots\ldots\ldots\ldots\ldots.. \qquad (1)$$

The velocity does not depend on time and $p(x, y), \;\; q(x, y)$ are continuous partial derivatives the divergence of the vector field is given by

Corresponding author: Orukari M. A, E-mail: orukarimer@gmail.com, Tel.: +2348037079340

$DivV(x,y) = px(x,y) + qy(x,y)$ ……………… (2)

The divergence of the fluid flow is zero, and this is characterizing the net flow through any simple closed contour to be identically zero [5]. Consider a rectangle in Figure 1 where the rate of outward flow equals the line integral of the exterior normal component of $v(x,y)$ taken over the sides of the rectangle. The exterior normal component is given by

$-q$ on the bottom edge,p on the right edge,q on the top edge, and$-p$ on the left edge. Integrating and setting the resulting net flow equal to zero yields

$$\int_y^{y+\Delta y}[p(x+\Delta x,t) - p(x,t)]df + [q(t,y+\Delta y) - q(t,y)]dt = 0 \text{ ……………… } (3)$$

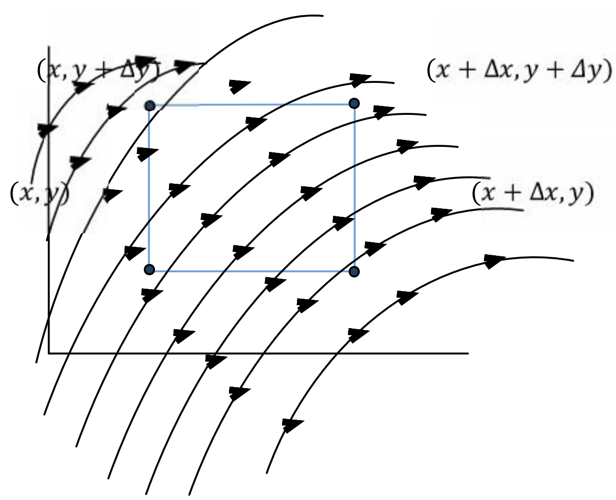

Fig. 1: Rate of outward flow

Since p and q are continuously differentiable, the mean value theorem can be used to show that

$p(x+\Delta x,t) - p(x,t)] - p(x,t) = px(x_1,t)\Delta x$, $q(y+\Delta y,t) - q(y,t)] - q(y,t) = py(y_1,t)\Delta y$ ……(4)where

$x < x_1 < x + \Delta x \quad and \quad y < y_1 < y + \Delta y$ …………………… (5)

Substituting equation (1) into equation (3) we will have

$$\frac{1}{\Delta y}\int_y^{y+\Delta y} px(x_1,t)dt + \frac{1}{\Delta x}\int_x^{x+\Delta x} qy(t,y_1)dt = 0 \text{ ……………………… } (6)$$

The mean value theorem for integrals is used for equation (6) and letting $\Delta x \to 0 \;\; and \;\; \Delta y \to 0$ we will have

$p_x(x,y) + q_y(x,y) = 0$ ……………………… (7)

Equation (7) gives us the continuity equation. The curl of the vector field in equation (1) has magnitude

$1/_2\, q_y(x,y) - 1/_2\, p_x(x,y) = 1/_2\, |curlv(x,y)|$ .... (8)

We will consider an irrotational fluid flow which curl is zero that is characterized by requiring the line integral of the tangential component of $v(x,y)$along closed contour by identically zero.

From Figure 1 the tangential component is given by p on the bottom edge, and –q on the left edge. Integrating and setting the resulting circulation integral equal to zero yields the equation

$$\int_y^{y+\Delta y}[q(x+\Delta x,t) - q(x,t)]dt - \int_x^{x+\Delta x}[p(t,y+\Delta y) - p(t,y)]dt = 0 \text{………… } (9)$$

Applying the mean value theorem and divided through by $\Delta x \Delta y$ we will obtain the equation

$$\frac{1}{\Delta y}\int_y^{y+\Delta y} q_x(x,t)dt - \frac{1}{\Delta x}\int_x^{x+\Delta x} p_y(t,y_1)dt = 0 \text{……………….. } (10)$$

Using the mean value theorem equation (10) reduces to

$q_x(x_1,y_1) - p_y(x,y) = 0 \; letting \; \Delta x \to 0 \; and \Delta y \to 0.$

This yield

$q_x(x,y) - p(x,y) = 0$ ……………………… (11)

Equation (7) and (11) shows that the function $f(z) = p(x,y) - iq(x,y)$ satisfies the Cauchy-Riemann's equations and is an analytic function as [6]

Let $f(z)$ denote the anti-derivative of $f(z)$.

Then $f(z) = \phi_x(x,y) + i\psi(x,y)$ … … … … … … … … … … … … …. (12)

This is called the complex potential of the flow and has the property

$f^{'}(z) = \phi_x(x,y) - i\psi_x(x,y) = p(x,y) + iq(x,y) = v(x,y)$ ……… (13)

Since $\phi_x = p\ and\ \phi_y = q$ we have

$grad\ \phi(x,y) = p(x,y) + iq(x,y) = v(x,y)\ so\ \phi(x,y)$ … … … … …. (14)

Equation (14) is the velocity potential for the flow and the curves while

$\phi(x,y) = k_1$ and $\psi(x,y) = k_2$……….………………….. (15)

The function $\psi(x,y)$ is called the stream function and the curves

We implicitly differentiate $\psi(x,y) = k_2$ and find that the slope of a vector tangent which is given by

$\frac{dy}{dx} = \frac{-\psi_x(x,y)}{\psi_y(x,y)}$……………………………………………… (16)

Using the fact that $\psi_x = \phi_x$and equation (16), we find that the tangent vector to the curve is

T= $\phi_x(x,y) - i\psi_x(x,y) = p(x,y) + iq(x,y) = v(x,y)$.………… (17)

$f(z) = \phi(x,y) + i\psi(x,y)$ … … … … … … … … … … … … (18)

Where $\{\psi(x,y) = k_2\}$ represents the streamlines of a fluid flow.

The boundary condition for an ideal fluid flow is that V should be parallel to the boundary curve containing the fluid (the fluid flows parallel to the walls of a containing vessel). This means if equation (18) is the complex potential for the flow and it is in agreement with Darren[7]. Then the boundary curve must be given by $\psi(x,y) = k$ for some constants k, that is, the boundary curve must be a streamline. Let

$F_1(w) = \Phi(u,v) + i\Psi(u,v)$………………………………….. (19)

denote the complex potential for a fluid flow in a domain G in the w-plane where the velocity is

$V_1(u,v) = \overline{F_1^{'}(w)}$……………………………… (20)

3.0 Cylindrical Flow Around Corners

This case study helps us understand what transpires when water flows and meets an obstruction in the path of flow which is of great importance to the environment because of its effect. Water is observed to flow downstream and on getting to a point, it is forced to turn a corner, this is the case examined in this example.

Consider a flow in the first quadrant $x > 0, y > 0$ that comes in downward parallel to the y-axis but is forced to turn in a corner near the origin, as shown in Figure 2

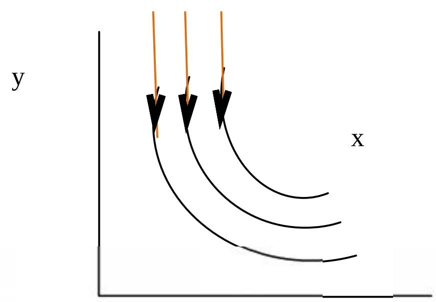

Fig 2: A downward flows

We know that $z = x + iy$ and $w = u + iv$, the mapping $w = z^2$ can be thought of as the transformation $u = x^2 - y^2$, $v = 2xy$ because water flows downward parallel to the y-axis.

(But originally, $f(z) = |z|^2; u(x,y) = x^2 + y^2; v(x,y) = 2xy$)

Hence the transformation is the conformal mapping;

$w = z^2 = u(x,y) + v(x,y) = x^2 - y^2 + i2xy$.

To determine the flow, we recall that the transformation

$w = z^2 = x^2 - y^2 + i2xy$ ………………….. (21)

maps the first quadrant onto the upper half of the uv-plane and the boundary quadrant onto the entire $u - axis$.

We know that the complex potential for a uniform flow to the upper half of the w-plane is

$F = AW$ ………………………………… (22)

where A is a positive real constant [8] the potential in the quadrant is, therefore

$F = AZ^2 = A(x^2 - y^2) + i2Axy$;………………… (23)

and it follows that the stream function for the flow there is

$\psi = 2Axy$…………………………………………. (24)

This stream function is harmonic in the first quadrant, and is vanishes at the boundary. The streamlines are branches of the rectangular hyperboles

$2Axy = c_2$ ……………………………………….. (25)

The velocity of the fluid is

$V = \overline{2AZ} = 2A(x - iy)$……………… (26)

observe that the speed

$|V| = 2A\sqrt{X^2 + y^2}$……………………… (27)

of a particle is directly proportional to its distance from the origin. The value of the stream function at any point (x, y) can be interpreted as the rate of flow across a line segment extending from the origin to that point.

The water flowing down along the canal meets anobstacles right in the middle of it, this obstacle is a circle around which the water must flow past, this is the case as illustrated in the following example.

Example

Find the complex potential for an ideal fluid flowing from left to right across the complex plane and around the unit circle $|z| = 1$

Solution:

We will use the fact that the conformal mapping

$w = s(z) = z + \frac{1}{z}$…………………………….. (28a)

maps the domain

$D = \{z; |z| < 1\}$

which is one-to-one and onto to the w-plane slit along the segment

$-2 \le u \le 2, V = 0$

with the complex potential for a uniform horizontal flow parallel to this slit in the w-plane is

$F_1(w) = AW,$……………………………… (28b)

where A is a positive real number. The stream function for the flow in the w-plane is

$$\Psi(u, v) = AV$$

so that the slit lies along the streamline $\Psi(u, v) = 0$.

The composite function $F_2(z) = F_1(s(z))$ will determine a fluid flow in the domain D where the complex potential is

$F_2(z) = A\left(z + {}^1/_z\right),$……………………… (29)

where $A > 0$.

Polar coordinates can be used to express $F_2(z)$ by the equation

$F_2(z) = A\left(r + \frac{1}{r}\right)\cos\theta + iA\left(r - \frac{1}{r}\right)\sin\theta$……………….. (30)

The streamline

$\Psi(r, \theta) = A(r - Y_r)\sin\theta = 0$…………… (31)

consist of the rays

$r > 1, \theta = 0 \; and \; r > 1, \theta = \pi$…………….………….. (32)

along the x-axis and the curve $r - {}^1/_r = 0$, which is easily seen to be the unit circle $r = 1$.

This shows that the unit circle can be considered as a boundary curve of the fluid flow, since the approximation

$F_2(z) = A\left(z + {}^1/_Z\right) = AZ$…………….. (33)

is valid for large values of z, from[5] we see that the flow is approximated by a uniform horizontal flow with speed as

$|V| = A$ ………………………………….. (34)

at points that are distant from the origin. The streamlines are

$\psi(x, y) = constant$…………………… (35)

and their images are

$\Psi(u, v) = constant$

under the mapping given by eqn. (27).

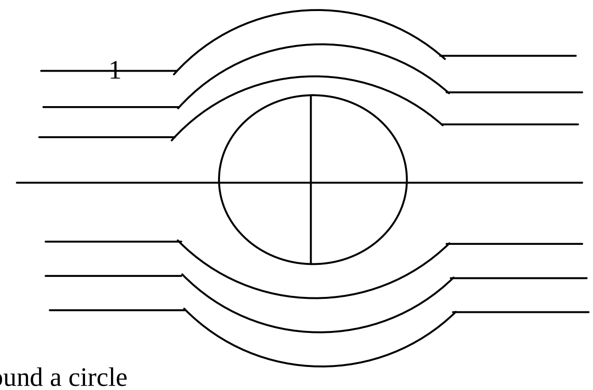

Fig 3: The flow around a circle

The streamlines are symmetric to the y-axis and have asymptotes parallel to the x-axis. The streamline consist of the circle r =1.It is notice that the unit circle is in the path of the flow but not a total hindrance to the flow, then since it is in the middle the streamlines have been expanded making them wider and the path lines are forced to turn curves.

4.0 Conclusion

Fluid (ideal) flowing over a complex plane is considered and all cross sections in planes parallel to the complex plane are the same. The study of the flow is governed by certain conditions which gives rise to equations on interpretation describes that a particular flow.

A conformal mapping is carried out on the complex plane upon which the fluid flow is considered and it has been transformed to a plane where physical properties of fluid such as velocity potential, complex potential, stream function etc are discussed. The velocity potential of a fluid flow equation generally reduces to the real part of an analytic function. Hence from any fluid flow equation an analytic function can always be pictured. The study of the application of flow equations to analytic function is essential especially when viewed from the complex analysis angle.

5.0 References

[1] Hicks W.M. (1879): On the motion of two cylinders in a fluid. Quart J. Pure Appl Math. 16, 113-140

[2] James W. B and Churchill R.V. (1996): Complex variables and Applications. Mc Graw Hill Publishing, Second Edition.

[3] Batchelor, G. (1967): An introduction to Fluid Dynamics, Cambridge University Press

[4] Crowdy D. G. and Marshall J.S. (2005): The motion of a point vertex around multiple circular island. Phys Fluids vol. 17 056602

[5] Leif, M. (2008): Complex Function Theory. Ventus Publishing APS ISBN 978-87-7681-753-

[6] Johnson R.S. (2012). Fluid Mechanics and Theory of Flight. R.S. Johnson & bookboon.com, RISBN 978-87-7681-975-0

[7] Darren C. (2006): Analytical solution for uniform potential flow past multiple cylinders. European Journal of Mechanics B/Fluids; 25:459-470.

[8] Fokas, A.S. and Ablowitz, M.J. (2003):Complex Variables, Introduction and Applications Cambridge University Press, second edition