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Determination of the Optimum Thickness of Approximately Cylindrical Top Spherical Frustrum Aluminium Cast Pot

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Abstract

In an attempt to conserve the world's scarce energy and material resources, a balance between the cost of heating a material and the optimum thickness of the material becomes vey essential. One of such materials is the local cast aluminium pot commonly used as cooking ware in Nigeria. This paper therefore sets up a model that relates the thickness of a pot with the cost of heating it. The model is then computer simulated to determine the variation of heating cost with pot's thickness. The result shows an optimum thickness of 0.064 m for a pot of volume 2.405×10^{-5} m^3. This is the thickness at which the cost of heating the material is cheapest, without an adverse effect on the thermal conductivity of the material. Above or below this thickness, the cost of heating is high making such thicknesses uneconomical. So, 0.064 m is the recommended optimum thickness for a pot of 2.405×10^{-5} m^3 capacity for the makers of cook ware. The optimum thickness for a different volume of pot can also be obtained from the model when computer simulated.

Keywords: Cost of heating, optimum thickness, cast aluminium, computer simulation

1.0 Introduction

The world population has almost exhausted much of the world's natural and un-natural resources that the world can no more tolerate wastage of materials either from natural or artificial sources. The world has therefore been driven towards the wisest use of material. The wisest use could be referred to as the planning and the executing of the modern day project in such a way as to maximize the use of material with minimum cost. This in turn leads to the economy of resources. The economy of resources invites the optimum production of material. The optimum production could be based on achieving maximum profit per unit of time or minimum cost per unit of production [1]. However, the cost of maintaining a material should not be left out in the attempt to make wise use of the scarce natural and artificial resources. This cost of maintenance is targeted mostly towards the cost of energy. In the developing countries where wastage of energy cannot be afforded but wisely utilized, the balance between the cost of energy and the cost of production becomes very essential. The cost of energy of heating a material relates to the optimum thickness of the material. It should be understood that despite the fact that wisest use of material is important, the thermal property of the material should not also be affected. Otherwise, the so-called wisest use of resources might turn into a mirage. Thus, the targeted property of material is the thickness of the material. So, an attempt could be made to balance the relationship between the cost of heating and the thickness of material. Therefore, the optimum thickness of material is that particular thickness at which the cost of heating the material is minimal, without affecting the heat retention ability or the mechanical property of the material. This leads to the optimization of both energy and material. This leads us to determining the optimum thickness of insulation of cast aluminium pot for heating purposes.

Of the vast varieties of refractory available today, clay and ceramic are the most popular. Cast aluminium is also a very good refractory with the ability to withstand temperature of about 600^0C without melting [2]. This class of material is inorganic and metallic in nature and will maintain most of its chemical and physical properties even at high temperature of about 500^oC [2]. Due to the protective coating on the Al surface, it has a good resistance to chemical attack and will be able to absorb shock and vibrations caused by the thermal stresses because it has been cast. In contact with air, aluminium rapidly becomes covered with a tough, transparent layer of aluminium oxide that resists further corrosive action.

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For this reason, materials made of aluminium do not tarnish or rust. The application of aluminium is not however stagnant at just boiling and heating purpose, it also found its application in transport, food preparation, energy generation, packaging, architecture, and electrical transmission applications. Depending upon the application, aluminium can be used to replace other materials like copper, steel, zinc, tin plate, stainless steel, titanium, wood, paper, concrete and composites. Aluminium is also used in buildings covers a wide range of applications. The applications include roofing, foil insulation, windows, cladding, doors, shop fronts, architectural hardware and guttering. Aluminium is also commonly used as the in the form of tread plate and industrial flooring [2].

This cast Aluminium cooking pot commonly made in Saki, an ancient town in south west,Nigeria, is a renowned pot in the country. This pot is known for its ability to cook food faster (high thermal conductivity) and heat retention for a relatively long time even after the heat source has been put-off. This property enables the pot to maintain the taste of the food in the pot. In order that the cost of heating be reduced, the thickness of the material should ordinarily be reduced. However, this cannot be infinitely done because the mechanical properties of material will be affected. The pot of this type is liable to have a very poor insulation which is a property of a good cookware.

Also, increase in the thickness of the pot layer leads to an increase in the cost of heating which might lead to unnecessary increment in the use of energy. A minimum thickness must therefore exist which ensures the required reduction of heat loss to the surroundings and ensures heating and insulating cost are brought to the minimum.

This work is therefore aimed at determining the optimum thickness of the Aluminium-cast pot. Since the local fine clay pot are molded in a hemispherical configuration [3], and the shape of the cast 'Saki' pot is like that of the clay pot, the heat transfer phenomena associated with casted-Aluminium pot is therefore analyzed using the thick spherical shell design approach [4].

2.0 Materials and Method

Assuming conduction is the predominant mode of heat transfer in the system, Fourier's law is therefore applied as a means to finding solution to the problem.

2.1 Fourier's Law

It states that the rate of heat flow, Q, through a homogeneous solid is directly proportional to the area, A, of the section at right angle to the direction of heat flow, and to the temperature difference along the path of heat flow, dt/dx [5]. It is mathematically represented as

$$Q = -KA\frac{dt}{dx} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (1)$$

Where

Q = quantity of heat transferred per second, (W)

K = thermal conductivity of the body, (W/m/K)

A = surface area of the material, (m^2)

dt = temperature on any two faces, (K)

dx = thickness of the body through which heat flows, (m)

The thermal conductivity and the area of the pot are of significant meaning in this work and must be dealt with thoroughly.

2.2 Surface Area of the Pot

In this paper, an approximate area of the Al-casted pot is found in terms of a single variable, the radius, for easier simplification and manipulation of data.

The picture of the Al-casted pot with its diagram is shown below

Fig. 1: The picture of the local Al-casted pot.

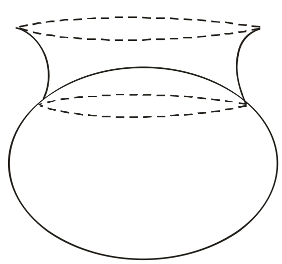

Fig. 2: The skeleton of the local Al-casted pot

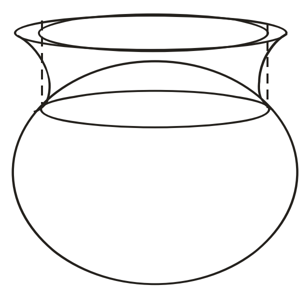

Fig.3: The skeleton of the Al-casted pot with top being

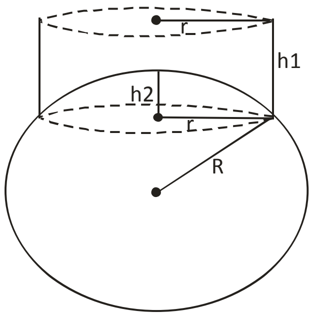

Fig. 4: The skeleton of the Al-casted pot
with top approximated to a cylinder

2.3 Calculation of Total Surface Area of the Pot

Area of the hollow mini cylinder at the top of approximated pot, A_c

$$A_c = 2\text{®}rh_1 = \frac{2}{5}\text{®}r^2 \quad \ldots\ldots\ldots\ldots \quad (2)$$

Where $h_1 = \frac{1}{5}r$

The resulting shape after the cylinder has been removed is the hemispherical frustrum which by extrapolation forms a full sphere as shown Fig. 5

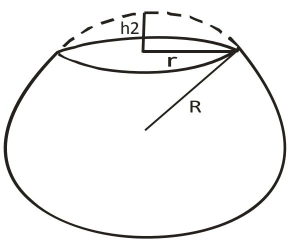

Fig. 5: The sphere formed by extrapolation of the spherical
frustrum after the removal of the cylindrical top.

The radius of the sphere, R is related to the radius r of the open top by equation 3:

$$R = \frac{3}{2}r \quad \ldots\ldots\ldots\ldots\ldots \quad (3)$$

Thus the surface area of the full sphere, A_s, from Fig. 5 is

$$A_s = 9\text{®}r^2 \quad \ldots\ldots\ldots\ldots\ldots \quad (4)$$

Area of the spherical cap, A_{sc} is given in [6] as

$$A_{sc} = \text{®}(r^2 + h_2^2) = \frac{5}{4}\text{®}r^2 \quad \ldots\ldots\ldots\ldots\ldots \quad (5)$$

Where $h_2 = \frac{1}{2}r$ ………….. (6)

Area of the spherical frustrum, A_{sf} is given by

$$A_{sf} = A_s - A_{sc} = \frac{31}{4}\text{®}r^2 \quad \ldots\ldots\ldots\ldots \quad (7)$$

The total surface area of the pot, A_p, is therefore ,

$$A_p = A_c + A_{sf} = 8.15\text{®}r^2 \quad \ldots\ldots\ldots.. \quad (8)$$

2.4 Calculation of Volume of the Pot

Volume of the mini-cylinder at the top,

$$V_c = \text{®}r^2 = 0.2\text{®}r^3 \quad \ldots\ldots\ldots\ldots \quad (9)$$

Where $h_1 = \frac{1}{5}r$

Volume of the sphere,

$$V_s = \frac{4}{3}\pi R^3 = 4.5\pi r^3 \qquad \ldots\ldots\ldots\ldots \qquad (10)$$

Volume of the spherical cap,

$$V_{sc} = \frac{1}{6}\pi h_2\left(3r^2 + h_2^2\right) = 0.27\pi r^3 \qquad \ldots\ldots\ldots\ldots.. \qquad (11)$$

Where $h_2 = \frac{1}{2}r$

The volume of the spherical frustrum,

$V_{sf} = V_s - V_{sc} = 4.23\pi r^3$ ………… (12)

Therefore the volume of the pot,

$V_p = V_c + V_{sf} = 4.43\pi r^3$ ………… (13)

3.0 Formulation of the Mathematical Model

3.1 Calculation of the thickness of the pot

The cross section of a spherical solid pot is shown Fig. 6. This is the related to the cross section of the pot.

t_1 = temperature of the inside wall of the pot, (K)

t_2 = outside temperature of the pot, (K)

r_1 = inside radius of the hemispherical pot, (m)

r_2 = outside radius of the pot, (m)

dt = temperature across the pot thickness, (K)

dr = small element of thickness, (m)

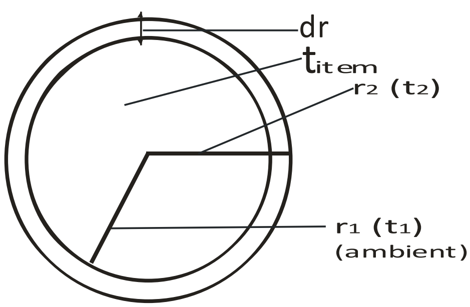

Fig. 6: The cross section of a solid spherical pot.

Substituting the area of the pot, A_p from equation 8, for A in equation 1 gives the quantity of heat transferred through the pot,

$$Q = -K\ A_p \frac{dt}{dr} = -8.15K\pi r^2 \frac{\mathrm{dt}}{\mathrm{dr}} \qquad \ldots\ldots\ldots.. \qquad (14)$$

By separation of variables, equation 14 becomes,

$$\frac{dr}{r^2} = \left(\frac{-8.15\pi K}{Q}\right)dt \qquad \ldots\ldots\ldots \qquad (15)$$

Integrating r from r_1 to r_2 and t from t_1 to t_2 and making Q the subject gives

$$Q = 8.15\pi K\ r_1 r_2\left(\frac{t_1 - t_2}{r_2 - r_1}\right) \qquad \ldots\ldots\ldots \qquad (16)$$

Taking the thickness of the pot, $r_2 - r_1$ = X

Then

$$Q = 8.15\pi K\ r_1 r_2\left(\frac{t_1 - t_2}{X}\right) \qquad \ldots\ldots\ldots.. \qquad (17)$$

Equation 17 denotes the heat transferred across the pot's thickness by conduction.

Another form of heat transferred is applied in cooking the food in the pot. Since liquid is used for most cooking, the heat transfer from the inner layer of the pot to the food is by convection. The quantity of the heat is generally given by

$$Q = UA\Delta T \qquad \ldots\ldots\ldots\ldots \qquad (18)$$

Apply this equation to this problem gives,

$$Q = U \times \left(t_{item} - t_1\right) = 8.15®r_1^2 U \left(t_{item} - t_1\right) \qquad …………. \qquad (19)$$

Where U is the overall coefficient of heat transfer.
Equating (17) with (19) and solving for X, yields

$$X = \frac{K r_2}{U r_1}\left(\frac{t_1 - t_2}{t_{item} - t_1}\right) \qquad ……….. \qquad (20)$$

The above result is the thickness of the pot. However, the cost of heating through this thickness has to be obtained to get the optimum thickness. Since the thickness does not depend directly on area, it implies that the model will be useful in determination of thickness of various size of approximated cylindrical top spherical frustrum Aluminium casted pot.

3.2 Calculation of Total Cost of Heating

The first is the cost of transferring heat from firewood burning in a local stove to the item in the pot. Let this cost of heating be C_{fp}

$$C_{fp} = QC_w \times 3600T \qquad ……….. \qquad (21)$$

T = time of operation of the pot per year, (s), in seconds.
C_w = the cost of dry wood; equivalent to 1 joule of heat i.e. cost of 0.0556kg of dry wood [7]

$$C_{fp} = 3600\ QC_wT \qquad ………… \qquad (22)$$

Another cost is associated with the cost of heating through the pot's thickness. This is previewed as the cost of heating the pot itself. This decreases as the years of operation of the pot increases. Let this be denoted by C_{pot}

$$C_{pot} = \frac{C_{Al} X}{S_1} \qquad …………. \qquad (23)$$

[1]

C_{Al} = cost per unit volume of Aluminium, (Nm^{-3})
S_1 = life span of the pot (years)

The total cost of heating, $C_{HT} = \dfrac{C_{Al} X}{S_1} + 3600\ QC_wT$ ……….. (24)

By substituting (19) and (20) into (24)

$$C_{HT} = \frac{C_{Al} K\ r_2 \left(t_1 - t_2\right)}{S_1 r_1 H \left(t_1 - t_{item}\right)} + 29340\ ®r_1^2 U\left(t_1 - t_{item}\right) C_w T \qquad …….. \qquad (25)$$

The above equation gives the total cost of heating in a particular period of year, T.
From equation 20

$$10.15®K\ r_1 r_2\left(\frac{t_1 - t_2}{X}\right) = 10\ .15\ ®r_1^2 U\ \left(t_{item} - t_1\right) \qquad …… \qquad (26)$$

The above equation is then reduced to

$$K\ r_2 \frac{\left(t_1 - t_2\right)}{X} = r_1 U\left(t_{item} - t_1\right)$$

From where

$$t_1 = \frac{K\ r_2 t_2 + r_1 U X t_{item}}{K\ r_2 + r_1 U X} \qquad ……… \qquad (27)$$

The above result is the inner surface temperature of the pot.

3.3 Computer Simulation

A non-differential numerical method is employed in determining the thickness of insulation of the casted Aluminium pot. With the predetermined volume of the pot by the manufacturer, the value of 'r_1' can be obtained from

$$V_p = 4.43\ ®r_1^3 \qquad ………. \qquad (28)$$

The value of the outside radius, 'r_2' is then guessed which results in guessed value for the thickness 'X' of the pot from the equation

$$r_2 - r_1 = X$$

Since U, K_p, t_{item} are the properties of heating system and the casted Aluminium pot, the value of the inner temperature of the pot can be found from the equation;

$$t_1 = \frac{K \quad r_2 t_2 + r_1 UXt_{item}}{K \quad r_2 + r_1 UX} \qquad \ldots\ldots\ldots\ldots \quad (29)$$

Hence, the total cost of heating can be deduced since C_{Al}, S_1, and C_w are also properties of cast Aluminium and heating system by the equation

$$C_{HT} = \frac{C_{Al} K \quad r_2 (t_1 - t_2)}{S_1 r_1 U \ (t_{item} - t_1)} + 29340 \quad ® r_1^2 U \ (t_{item} - t_1) C_w T \qquad \ldots\ldots\ldots\ldots \quad (30)$$

Therefore, writing the cost of heating as a function of just the thickness by substituting the values of t_1, Q and r_2 into equation (24), gives

$$C_{HT} = \frac{C_{Al} X}{S_1} + 29340® r_1^2 U C_w T \left[\frac{(k_p t_{item} - k_p t_2) X + (k_p t_{item} r_1 - k_p t_2 r_1)}{(k_p + r_1 U) X + k_p r_1} \right] \qquad \ldots\ldots\ldots \quad (31)$$

The plot of the total cost of heating against the thickness of the pot from which the optimum thickness can be determined is shown in Fig.7. Equation (31) is easier for calculation since it contains just the two important variables of interest which are C_{HT} and X.
A computer simulation of the above procedure was carried out. The program, written in C++ language and run on the Dev C++ version 4.9.9.2 [8] compiler is presented with this paper. It carries out 32 iterations and the graph of the thickness of the pot was plotted against the cost of heating the pot from which the optimum thickness of the pot is obtained.
The data used are: t_{item}= **100^{0}C**, t_2=**27.83^{0}C**, r_1= **0.012m**, r_2= **0.02m**, C_{Al}=**0.00000002 N/m^3,** Outside radius increment=**0.004m**, S_1= **5yrs**, T=**2180hrs/yr**, C_w= **N20.00** [1], U=**1000.0,**
kp =**238.24**

4.0 Results and Discussion

The above results were fed into the program and run using Dev cpp version 4.9.9.2 and the results of the computation is shown in Table 1.
Initially, the cost of heating decreases as the thickness of the pot increases. This is because heat is transferred from the heat source to the through the pot's thickness to the item in the pot. Since the pot is also a conductor, it losses some part of heat transferred to its surrounding. This heat loss depends also on the thickness of the pot. So, at a small thickness, the rate of transfer of heat is almost equal to the rate heat loss to the surrounding making the effective heat gained by the pot very small and therefore requires higher energy consumption at that particular thickness which makes the cost of heating high.
However, as the thickness of the pot increases, the cost of heating decreases till the minimum thickness is obtained before the cost of heating starts increasing with the thickness of the pot.
The thickness of the pot later increases as the cost of heating the pot increases. Provided the same amount of energy is used for the heating. As the thickness increases, the rate of flow of heat through the pot's thickness decreases with the rate of heat loss to the surrounding. So, higher energy is required to heat the pot through that particular thickness. Hence, the cost of heating is high

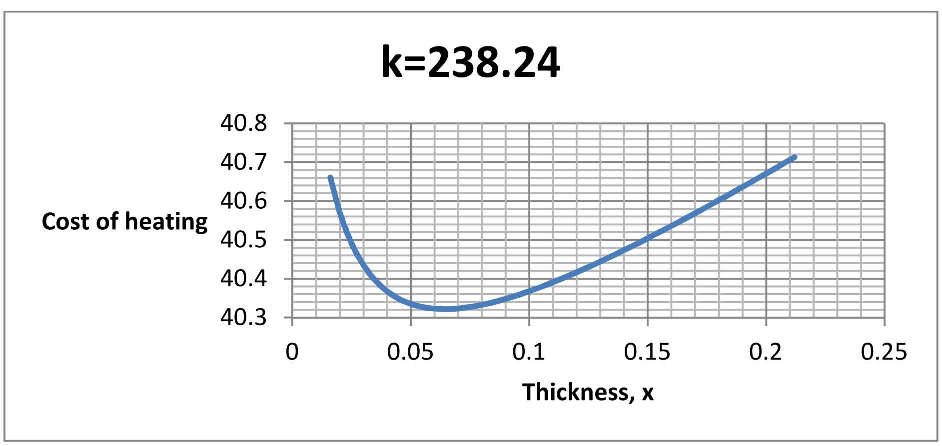

Fig. 7: Plot of Cost of Heating against Pot's thickness

Hence, the thickness at which the cost of heating is most minimum is taken as the optimum thickness and this corresponds to the turning point of the graph.

1. The Computer Program

```
#include<iostream>
#include<cmath>
#include<fstream>
using namespace std;
class proc{
  private:
     double r1,r2,dr,c_w,c_al,k_p,t_item,c1,c2,c3,c4,t2,T,s1,U,x_arr[33],t1_arr[33],c_ht_arr[33];
     const int cnst;
  public:
     proc():cnst(29340){
        cout<<"Please enter some varibles before program execution.\n"
          <<"\n\t Enter Kp:______\b\b\b\b";
          cin>> k_p;//we got value for k_p
        cout<<"\n\t Enter U:______\b\b\b\b";
          cin>> U;//we got value for U
        cout<<"\n\t Enter t_item:______\b\b\b\b";
          cin>> t_item;//we got value for t_item
        cout<<"\n\t Enter t2:______\b\b\b\b";
          cin>> t2;//we got value for t2
        cout<<"\n\t Enter Cal:______\b\b\b\b";
          cin>> c_al;//we got value for c_al
        cout<<"\n\t Enter s1:______\b\b\b\b";
          cin>> s1;//we got value for s1
        cout<<"\n\t Enter Cw:______\b\b\b\b";
          cin>> c_w;//we got value for c_w
        cout<<"\n\t Enter T:______\b\b\b\b";
          cin>> T;//we got value for T
        cout<<"\n\t Enter dr:______\b\b\b\b";
          cin>> dr;
        cout<<"\n\t Enter r2:______\b\b\b\b";
          cin>> r2;
        cout<<"\n\t Enter r1:______\b\b\b\b";
          cin>> r1;
        cin.get();
           }
     int start_proc(){
       //starting the iteration
         for(int i=0;i<=32;i++){
            r2+=dr;
            x_arr[i]=r2-r1;//we got x here
            c1=(c_al*x_arr[i])/s1;
            c2=cnst*M_PI*r1*r1*U*c_w*T;
            c3=((k_p*t_item)-(k_p*t2))+((k_p*t_item*r1)-(k_p*t2*r1));
            c4=((k_p+r1*U)*x_arr[i])+(k_p*r1);
            c_ht_arr[i]=c1+(c2*(c3/c4));//we got c_ht here
         }
          }
       int output(char filename[]){ ofstream pfile(filename);
      if(!pfile){cout<<"\nError! Failed to open file: "<<filename<<endl; return 0;}
      pfile << "\tExternal temperature(t1)\t|\tTotal cost of heating(C_ht)\t|\tThickness of the pot(X)\t\n"
         << "-----------------------------------------------------------------------------------------------------\n";
      pfile.setf(ios_base::fixed, ios_base::floatfield);//used to avoid most output of exponentials
       for(int i=0;i<=32;i++){
          pfile <<"\t|\t"<<c_ht_arr[i]<<"\t\t\t|\t"<<x_arr[i]<<"\t|\n";
```

```
            }
        pfile.close();
         for(int i=0;i<=32;i++){
              cout <<"\t|\t"<<c_ht_arr[i]<<"\t|\t"<<x_arr[i]<<"\t|\n";
              }
        cout<<"Program finished!!!";
    }
            };
int main(){
   proc pro; pro.start_proc(); pro.output("output.txt");
   cin.get();
   return 0;
}
```

Table 1: Results of the iteration

Pot's Thickness (m)	Heating Cost(₦)
0.012	40.78730
0.016	40.66083
0.020	40.57063
0.024	40.50441
0.028	40.45487
0.032	40.41741
0.036	40.38897
0.040	40.36744
0.044	40.35133
0.048	40.33954
0.052	40.33126
0.056	40.32586
0.060	40.32285
0.064	40.32186
0.068	40.32258
0.072	40.32477

Table 1 cont.

0.076	40.32822
0.080	40.33277
0.084	40.33828
0.088	40.34463
0.092	40.35173
0.096	40.35949
0.100	40.36785
0.104	40.37673
0.108	40.38609
0.112	40.39588
0.116	40.40606
0.120	40.41660
0.124	40.42745
0.128	40.43861
0.132	40.45003
0.136	40.46170

5.0 Conclusion

The model which relates directly the cost of heating with the thickness of the pot has been obtained in this paper. Therefore, the paper sets up an algorithm for computing the optimum thickness of the pot. The optimum thickness for a local aluminium casted pot (the pure Al sample) of volume 2.405×10^{-5} m^3 has been derived to be 0.064m using the data presented in this paper and the algorithm for determine the optimum thickness of different various sizes of pot has also been obtained. The optimum thickness of pots of other shapes and materials can also be determined from the model.

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