*Journal of the Nigerian Association of Mathematical Physics*
***Volume 30, (May, 2015), pp79 – 88***
***© J. of NAMP***

# Rigid Motions of Some Regular Polygons

***Abba Sani***

***Department of Mathematics and Computer Sciences, Umaru Musa Yar'dua University, P.M.B. 2218, Katsina, Nigeria.***

## *Abstract*

***We examined permutations of vertices/sides of some regular shapes viewed as rigid motions. In particular, we use combinatorial techniques to enumerate symmetric permutations of vertices/ sides of an n-sided regular polygon $P_n$. Our results involve:***

***(1) A well knownformula, $NSYP_n = 2n$ for generating the number of symmetries in an n-sided regular polygon accomplished using permutations;***

***(2). A new formula, $NWT_n = \frac{n(n-3)}{2}$ for number of ways of triangulating $P_n$, (the number of ways of cutting $P_n$ into triangles by connecting its vertices with straight lines); thereby providing a proof for Richard and Stanley's conjecture that "All diagonals are flipped in a geodesics between two antipodes in exactly $\frac{n(n-3)}{2}$ ".We also examined the set $S = [n] = \{1, 2, ...n\}$ of vertices of $P_n$ as poset and proved some known theorems.***

***A discussion is given of lattices whose maximum length chains correspond to restricted permutations.***

**Keywords:** Triangulation, equidissection, area discrepancy polygons, pattern-avoiding permutation, restricted permutation and Symmetric permutation.

## 1.0 Introduction

The one line-notation form of a permutation ~ of a string of numbers $S = [n] = \{1, 2, 3, ..., n\}$ , is written $\sim(1)\sim(2)\sim(3)...\sim(n)$ . However, a permutation enclosed inside brackets is in cyclic form. For instance, if ~ is a permutation of $[5] = \{1, 2, 3, 4, 5\}$ whose cyclic form is $(13)$ , then ~ 's one-line notation form is $32145$ . Thus we have $(13) = 32145$ . In this paper we will use $S_n$ to refer to the set of permutations of $[n] = \{1, 2, 3, ..., n\}$ written in one-line notation. If $\sim \in S_n$ and $\in S_k$ , then ~ contains as a pattern if some subsequence of ~ of length $k$ has the same relative order as . For example, the permutation $\sim = 13524867 \in S_8$ contains the pattern $® = 2143 \in S_4$ since there is a subsequence $\sim_3\sim_4\sim_6\sim_8 = 5287$ in ~ of length **4** which has the same relative order as ® . We say ~ avoids ® whenever ~ does not contains ® . By polygon we mean a closed plane figure bounded by straight lines. The commonest among them include triangles, Quadrilateral, Pentagon, Hexagon, Heptagon Octagon, etc.

Corresponding author: Abba Sani, E-mail: saniabbawakili@gmail.com, Tel.: +2348159156335 & +2348024364794

A poset (partially ordered set) is a set P together with a binary relation ≤ which is*transitive* (x < y and y < z implies x < z) and *irreflexive* (x < x and y <y cannot both hold).We can partially order the set of all permutations of all numbers of letters by declaring that for a bigger permutation π and a smaller permutation σ (a pattern), σ≤π iff σis contained as a pattern in π. It would be interesting to study this as a poset.Ron M.A large number of articles directly or indirectly related to this Problem; for instance,Adin and Yuval Riochman in [1] reported that the diameter of the flip graph on the set of all colored triangle-free triangulations of a convex n-gon is exactly $\frac{n(n-3)}{2}$.In [2] Stein S.K reported that there is no definite formula for dividing an n-sided polygon into triangles of equal areas. See also [3 and 5] for other related areas.

## 2.0 General Notations and Preliminaries

To make this write-up clear, let us adapt unique notations throughout the write-up in order not to confuse the reader. Let us consistently denote an n-sided polygon by $P_n$; triangulation (possible number of triangles in $P_n$) by $T_n$; diagonalization (possible number of diagonals in $P_n$ ) by $NDP_n$; set of all triangulations of $P_n$ by $S_n^{\sim}$; so that $T_n = 1, 2, 3, ...$; set of bijections on $\{1, 2, ..., n\}$ by $S_n$; set of all permutations of length n that avoids $q$ by $S_n(q)$ where $q \in S_n$; $P_n$ be a regular polygon with $n$ the number of sides; $NWT_n$ be the number of ways of triangulating $P_n$; $NSYP_n$ be number of symmetries in $P_n$; $NDP_n$ be the number of diagonals in $P_n$.

## 3.0 Triangulation of Polygon

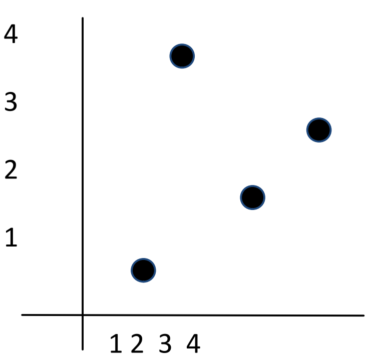
The representation of a permutation on some regular polygons such as Equilateral triangle, a square etc, further motivates three common operations on permutations. For $® \in S_n$, the reverse of ® is $®^R = ®(n)®(n-1)...®(2)®(1)$. That is, $®^R$ is the permutation whose diagram is obtained by reflecting the diagram of ® over a vertical axis.

We consider three symmetries that are natural both for the square and in the languageof permutations.

We note that:

- Reversal corresponds to flipping the graph of ® over the vertical line of symmetry:

*Graph of* $® = 1423$ *Graph of* $®^R = 3241$



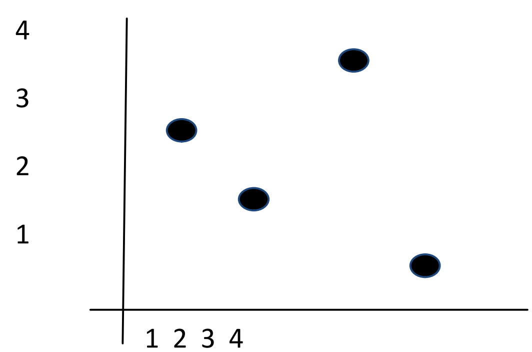




**Figure 1.1:**
The graph of the permutation $® = 1423$

**Figure 1.2:**
The graph of the permutation $®^R = 3241$

Similarly, the compliment of ® is the permutation whose entries follow the formula $®^c(j) = n+1-®(j)$. The diagram of $®^c$ is obtained from that of ® by reflection over a horizontal axis.

- Complement corresponds to flipping the graph of ® over the horizontal line of symmetry:

*Graph of* $®^c = 4132$

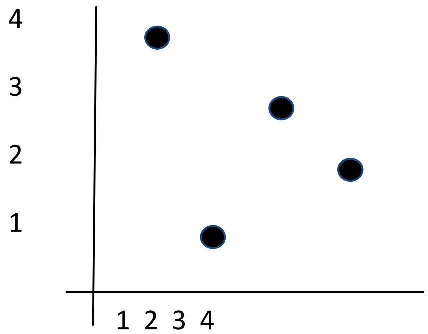



**Figure 1.3:**
The graph of the permutation ®$^c = 4132$
Finally, the inverse of ® , which we denoted by ®$^{-1}$ is the inverse of ® as a function, so if ®$(j) = k$, then ®$^{-1}(k) = j$.
The diagram of ®$^{-1}$ is that of ® , reflected over the diagonal from the lower left corner to the upper right corner.

- Inverse corresponds to flipping the graph of ® over the main diagonal line of symmetry:

*Graph of* ®$^{-1} = 1342$

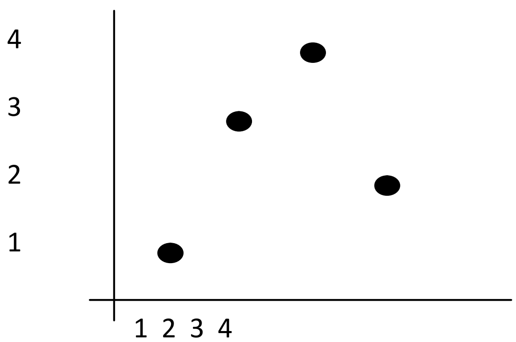



**Figure 1.4:**
The graph of the permutation ®$^{-1} = 1342$
This set of operations, $\{reverse, compliment, inverse\}$ when considered as symmetries of a regular polygon such as Equilateral triangle, Square, etc., motivates a brief foray into the algebra of *dihedral groups.* The dihedral group of eight symmetries, $D_4$ of a square, for instance is well-known to be generated by the above mappings (Reversing, Complimenting and Inverting). We say that a permutation is *preserved* under some symmetry $\pm \in D_4$ if its diagram is unchanged by $\pm$. Equivalently, if we consider $D_4$ to be a group of actions on the set of diagrams of permutations in $S_n$, then ® $\in S_n$ is preserved by $\pm \in D_4$, if $\pm$ is in the stabilizer of the diagram of ® . Since the stabilizer of a diagram is a subgroup of $D_4$, we can consider the possible symmetries of a permutation by considering the $10$ distinct subgroups of $D_4$.

## 4.0 Preliminaries

Recall that by triangulation of a polygon, we are presumably referring to dissection of the polygon into triangles by connecting its vertices with straight lines. The starting point of our construction of sequences of triangulations which prove equation (1) are certain triangulations of a polygon which differ slightly from that described by Stein and Szabo in [4]. Our main concern is to find a generating function for $n$ (ie. number of sides of a polygon) given an $T_n$ (ie. number of triangles that can be dissected from $P$ ). The following figures illustrate triangulation of some convex polygons (Triangle, a Quadrilateral, a Pentagon, etc.):

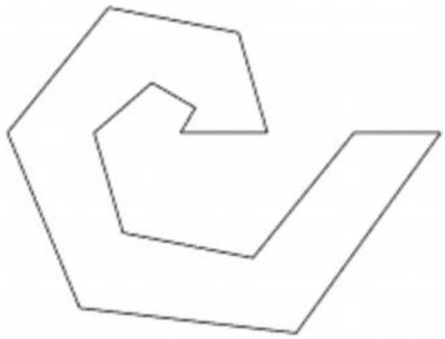


**Figure 2a**

**Figure 2b**

Partition of a 14-sided polygon into triangles.

We consider the number of ways of partitioning a convex polygon into triangles.

Let $S_n$ be a symmetric group on the letters $1,2,\ldots,n$ where the letters $1,2,\ldots,n$ are vertices of a regular polygon with n sides. Denote the permutation $\sim \in S_n$ by the sequence $[\sim(1),\sim(2),\ldots,\sim(n)]$ and transpositions by $(i, j)$ . Label the vertices of an n-sided regular convex polygon $\mathrm{P}_n$ $(n \geq 3)$ by the elements $1,2,\ldots,n$ of the multiplicative cyclic group $(S_n,*)$ . Each edge of the polygon is called an external edge of the triangulation; all other edges of the triangulation are called internal edges, or chords.

**Definitions 4.1**

*Let* $(G,*)$ *be a group, and suppose H is a nonempty subsetof* $G$ *. If* $(H,*)$ *is a group, then H is called a subgroup of G.*

**Definitions 4.2**

Consider a regular polygon $P$, such as, for example, an equilateral triangle or a square or any n-sided regular polygon. Any movement of $P$ that preserves the general shape of $P$ is called a *rigid motion*. There are two types of rigid motions: (1) rotations and (2) reflections. For a regular polygon $P$ with $n$ sides, there are $2n$ distinct rigid motions. These include the $n$ rotations of $P$ through $360\frac{i}{n}$ degrees for $i=1,2,\ldots,n$ . The remaining $n$ rigid motions are reflections. If $n$ is even, these are the reflections of $P$ across the lines that connect opposite verticesor bisect opposite sides of $P$ . If $n$ is odd, these are the reflections of $P$ across the lines that are perpendicular bisectors of the sides of $P$ . Since the rigid motions of $P$ preserve the general shape of $P$ , they can be viewed as permutations of the vertices or sides of *P*. The set of rigid motions of a regular polygon $P$ forms a group called the *symmetries* of $P$ .

**Example 1.** Consider the figure 3 below. To express the group of symmetries of this figure as permutations of its vertices (123) of the triangle, consider the figure 3:

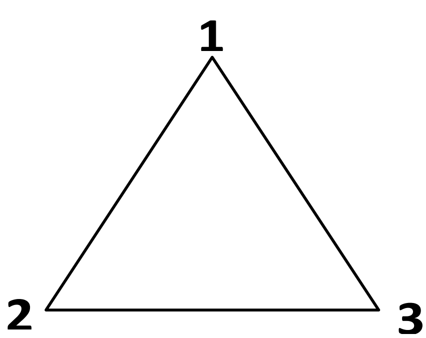



**Fig 3:** An equilateral triangle

Figure3 above (an equilateral triangle with vertices 123) is a regular polygon. Any movement of *the figure* that preserves its general shape ofis called a *rigid motion*. There are two types of rigid motions: (1) rotations and (2) reflections. For a regular polygon *P* with *n* sides, there are 2*n* distinct rigid motions. These include the *n* rotations of *P* through $360\,{}^{i}\!/_{n}$ degrees for $i=1,2,\ldots,n$ .

The remaining $n$ rigid motions are reflections. If $n$ is even, these are the reflections of $P$ across the lines that connect opposite vertices or bisect opposite sides of $P$. If $n$ is odd, these are the reflections of $P$ across the lines that are perpendicular bisectors of the sides of $P$. Since the rigid motions of $P$ preserve the general shape of $P$, they can be viewed as permutations of the vertices or sides of $P$. The set of rigid motions of a regular polygon $P$ forms a group called the *symmetries* of $P$.
The 6 symmetries of the triangle in figure 3 can be expressed as permutations of the vertices of this general figure as follows (rotations are counterclockwise). Note in this case that $n = 3;\ i = 1, 2, 3$.

**Table 1: Symmetries of an equilateral triangle**

| $i$ | **Rigid motion** $(360\, i/3)$ $n = 3$ | **Permutations** |
|---|---|---|
| 1 | $120^0$ Rotation | (123) |
| 2 | $240^0$ Rotation | (132) |
| 3 | $360^0$ Rotation | (1)(2)(3) |
| | Reflection across perpendicular bisector of side $2-3$ | (23) |
| | Reflection across perpendicular bisector of side $1-2$ | (12) |
| | Reflection across perpendicular bisector of side $1-3$ | (13) |

Note that expressing these rigid motions as permutations on the vertices of the preceding general figure yields a subgroup of $S_3$. It is easy to see that the symmetries of the regular triangle above form a group (dihedral group);
$D_3 = \{(1)(2)(3), (12), (23), (13), (123), (132)\}$, where $D_3 \subseteq S_3$.

**Example 2:**
Consider the group of symmetries of a square. To express these symmetries as permutations of vertices of a square, consider figure 4 below:

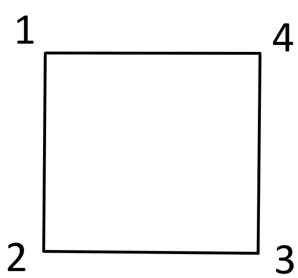



**Fig 4:** Symmetries of a Square

The 8 symmetries of a square can be expressed as permutations of the vertices of this figure as follows (rotations are counterclockwise).

**Table 2: Symmetries of a Square**

| $i$ | **Rigid motion** $(360\, i/4)$, $n = 4$ | **Permutations** |
|---|---|---|
| 1 | $90^0$ Rotation | (1234) |
| 2 | $180^0$ Rotation | (13)(24) |
| 3 | $270^0$ Rotation | (1432) |
| 4 | $360^0$ Rotation | (1)(2)(3)(4) |
| | Reflection across 1-3 diagonal | (24) |
| | Reflection across 2-4 diagonal | (13) |
| | Reflection across horizontal line of symm. | (12)(34) |
| | Reflection across vertical line of symm. | (14)(23) |

Note also that expressing these rigid motions as permutations on the vertices of the preceding general figure yields a subgroup of $S_4$. It is easy to see that the symmetries of the general shape above (square) form a group (dihedral group); $D_4 = \{(1234),(13)(24),(143),(1)(2)(3)(4),(24),(13),(12)(34),(14)(23)\}$, where $D_4 \subseteq S_4$.

**Remark**

When the symmetries of an $n-sided$ regular polygon are expressed as
Permutations on the set $\{1,2,...,n\}$, the resulting subgroup of $S_n$ is denoted by $D_n$ and called the *dihedral group on n letters*. The subgroup of
$S_3$ in Example 1 above is the dihedral group $D_3$. Similarly, in Example 2 $D_4$ is the subgroup of $S_4$. It has been observed from the preceding discussions (in Examples 1&2) that for a regular polygon *P* with *n* sides, there are $2n$ distinct rigid motions. These include the *n* rotations of *P* through $360\, {}^{i}\!/_{n}$ degrees for $\mathrm{i} = 1,\ 2,\ldots,\ n$. The remaining *n* rigid motions are reflections. If *n* is even, these are the reflections of *P* across the lines that connect opposite vertices or bisect opposite sides of *P*. If *n* is odd, these are the reflections of *P* across the lines that are perpendicular bisectors of the sides of *P*. Since the rigid motions of *P* preserve the general shape of *P*, they can be viewed as permutations of the vertices or sides of *P*. The set of rigid motions of a regular polygon *P* forms a group called the *symmetries* of *P*. Tables 1&2 demonstrate these points more. Let $P_n$ be a regular polygon with $n$ the number of sides. Let $nd$ be the number of diagonals in $P_n$ and, $NWT_n$ be the number of ways of triangulating $P_n$.

**Table 3: Symmetries of a regular polygon**

| **Regular Polygon** $P_n$ | $n \geq 3$ **No. of sides of** $P_n$ | $\lvert D_n \rvert$ **No. of Symmetries in** $P_n$ |
|---|---|---|
| Equilateral Triangle | 3 | $6 = 2(3)$ |
| Quadrilateral | 4 | $8 = 2(4)$ |
| Pentagon | 5 | $10 = 2(5)$ |
| Hexagon | 6 | $12 = 2(6)$ |
| Heptagon | 7 | $14 = 2(7)$ |
| Octagon | 8 | $16 = 2(8)$ |
| … | … | … |
| … | … | … |
| $n-gon$. | $n$ | $2n$ |

$\therefore\ NSYP_n = 2n, \forall n \geq 3$

When talking about the possible number of ways of partitioning a polygon into triangles by joining vertices with straight lines (drawing the diagonals in the polygon), we observed that for the case of a polygon with three sides (triangle) there is no diagonal (thus no construction is required); for a polygon with four sides there are two diagonals (by joining the opposite vertices); for a pentagon there are five diagonals, etc.

**Table 4: Number of ways of triangulating a convex polygon**

| **Polygon ( $P_n$ )** | $n \geq 3$ **No. of sides in** $P_n$ | $NDP_n$ **No. of diagonals in** $P_n$ | **Generating** $NWT_n$ **From** $NDP_n$ **&** $n$ |
|---|---|---|---|
| Triangle | 3 | 0 = | $\frac{3(3-3)}{2}$ |
| Quadrilateral | 4 | 2 = | $\frac{4(4-3)}{2}$ |
| Pentagon | 5 | 5 = | $\frac{5(5-3)}{2}$ |
| Hexagon | 6 | 9 = | $\frac{6(6-3)}{2}$ |
| Heptagon | 7 | 14 = | $\frac{7(7-3)}{2}$ |
| Octagon | 8 | 20 = | $\frac{8(8-3)}{2}$ |
| … | … | … | … |
| … | … | … | … |
| $n-gon$. | $n$ | | $\frac{n(n-3)}{2}$ |

$$\therefore NWT_n = \frac{n(n-3)}{2}, \forall n \geq 3$$

It has been observed that by the preceding discussions

## 5.0 Symmetry Class:

Let $S_n = \{1,2,...,n\}$ and $q \in S_n$. If $q$ is a permutation and $q^{-1}$ is its group theoretic inverse, then by elementary arguments $|S_n(q)| = |S_n(q^{-1})|$ for all n (see [6, 7 and 8]). The same hold between $q$ and its reverse $q^R$, where $q_i^R = q_{n+1-i}$, and $1 \leq \mathrm{i} \leq \mathrm{n}$.

Example; Let $S = \{1,2,3\}$ be a set and $S_n$ the set of permutations of length $n$. If $q = 132 = (23)$; $\forall q \in S_3$, then $q^R = 231 \text{ or } (123)$ and $q^{-1} = 132 \text{ or } (23)$

These two operations generate the dihedral group of order 6.

$D_3 = \{(1)(2)(3),(12),(23),(13),(123),(132)\}$.

Where $D_3 \subseteq S_3$ and $S_n$ the set of permutations of length $n$. Another useful operation is known as complementation of $q$ denoted by $q_i^C = n+1-q_i$, where $1 \leq \mathrm{i} \leq \mathrm{n}$.

Example if $q = 132 = (23)$ then $q^C = 312 \text{ or } (132)$.

These lead to the natural symmetries:

Let $\mathrm{p} = \mathrm{p}_1 p_2 ... \mathrm{p}_n \in \mathrm{S}_n$. Then: $\forall p, q \in \mathrm{S}_n$

$p$ avoids q $\Leftrightarrow$ $p^R$ avoids $q^R$,

$p$ avoids q $\Leftrightarrow$ $p^C$ avoids $q^C$,

$p$ avoids q $\Leftrightarrow$ $p^{-1}$ avoids $q^{-1}$;

And moreover $|\mathrm{S}_n(\mathrm{Q})| = |\mathrm{S}_n(\mathrm{Q}^R)| = |\mathrm{S}_n(\mathrm{Q}^C)| = |\mathrm{S}_n(\mathrm{Q}^{-1})|$, where $\mathrm{Q}^*$ is the set obtained by applying the operation * to all patterns in the set $\mathrm{Q}$.

By repeatedly applying the operations of reverse, complement, and inverse, which generate the symmetries of the square, we see that we can partition sets of patterns into equivalence classes up to size 8 that will necessarily have the same enumeration.

## 6.0 Properties of Posets:

Elements x and y of a poset $X$ are *comparable* if x < y and/or y < x hold. A *chain* in a poset $X$ is a subset $C \subseteq X$ such that any two elements in C are comparable. An *antichain* in a poset $X$ is a subset $A \subseteq X$ such that no two elements in A are comparable.

An element *x* of a poset (*X*,≤) is called *maximal* if there is no element *y*ϵ*X*satisfying *x* < *y*. Dually, *x* is *minimal* if no element satisfies *y* < *x*.

In a general poset there may be no maximal element, or there may be more than one. But in a finite poset there is always at least one maximal element, which can be found as follows: choose any element *x*, if it is not maximal, replace it by an element *y* satisfying *x* < *y*, repeat until a maximal element is found. The process must terminate, since by the irreflexive and transitive laws the chain can never revisit any element. Dually, a finite poset must contain minimal elements.

An element *x* is an *upper bound* for a subset *Y* of *X* if *y* ≤ *x* for all *y*ϵ*Y*. *Lower bounds* are defined similarly. We say that *x* is a *least upper bound* or *l.u.b. (supremum)*of *Y* if it is an upper bound and satisfies $x \le x^!$ for any upper bound $x^!$. The concept of a *greatest lower bound* or *g.l.b.(infinimum)* is defined similarly.

The *height* of a poset is the largest cardinality of a chain, and its *width* is the largest cardinality of an antichain. We denote the height and width of (*X*,≤) by *h*(*X*) and *w*(*X*) respectively (suppressing as usual the relation *R* in the notation). In a finite poset (*X*, ≤), a chain *C* and an antichain *A* have at most one element in common. Hence the least number of antichains whose union is *X* is not less than the size *h*(*X*) of the largest chain in *X*. In fact there is a partition of *X* into *h*(*X*) antichains.

We examined a set of vertices/sides of a regular convex polygon as a poset. The number of subsets of an n-element set is $2^X$. An important poset is the set $2^X$ (all subsets of the set X with |X| = n) with set inclusion: x < y if $X \subseteq Y$. Note the bijections between subsets of the set $2^X$ $(i.e. \varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, X)$ and the subperms of the permutation 6=123 $(i.e. \varnothing, 1, 2, 3, 12, 13, 23, 123)$.

This poset can be visualized by a Hasse diagram (see figure 5a) for the set $X = \{1,2,3\}$ in example 1(set of vertices of the triangle 123).

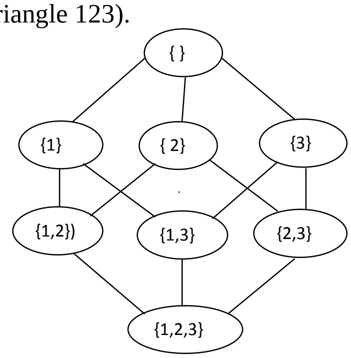



Figure 5a: A hasse diagram for $X = \{1,2,3\}$.

## 6.1 Decomposition of Posets Using Antichain

We now partition the poset into antichains as shown in figure 5b.

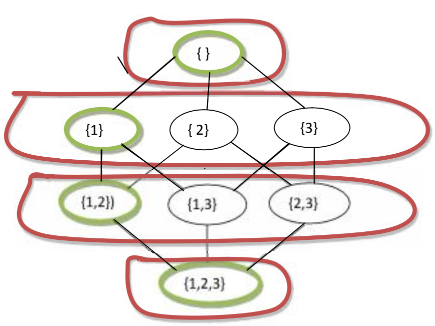



**Figure 5b:** A partition of the poset $(X, \le)$ into antichains.

A poset with a chain of size *r* cannot be partitioned into fewer than *r* antichains (Anytwo elements of the chain $2^X = (\{a \in X\}, \leq)$ must be in a different antichain).

**Theorem1:** Suppose that the largest chain in theposet $X$ has size *r*. Then $X$ can be partitioned into *r* antichains.

**Proof:** Define *l(x)* as the size of the longest chain in $X$ whose greatest element is x. Define $A_i$ as $A_i = \{x : l(x) = r\}$. Then $A_1 \cup \ldots \cup A_r$ is a partition of $X$ into *r* mutually disjoint sets. Every $A_i$ is an antichain otherwise there exists two points x, y in $A_i$ such that $x < y$ which implies $l(x) < l(y)$.

## 6.2 Decomposition of Posets Using Chains

We now partition the poset into chains as shown in Figure 5c.

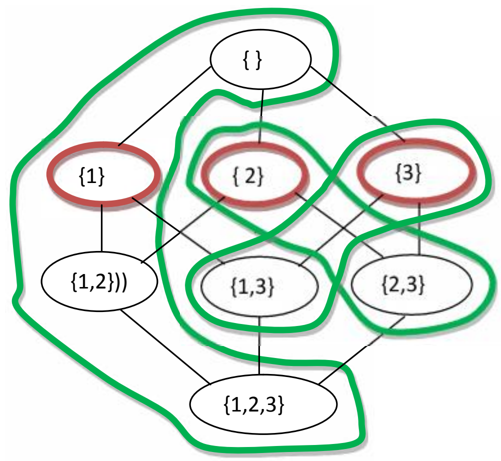



**Figure 5c:** A partition of the poset $(X, \leq)$ into chains.

A poset with an antichain of size *r* cannot be partitioned into fewer than *r* chains (Any two elements of the antichain $2^X = (\{a \in X\}, \leq)$ must be in a different chain).

## 7.0 Conclusion

We proved most importantly, Richard Stanley's conjecture which states that "All diagonals are flipped in a geodesics between two antipodes in exactly $\frac{n(n-3)}{2}$". We accomplished this assertion by rigid motions approach which was not done anywhere by anyone before this work/research was conducted, and subsequently got a generation function (formula) for the number of ways of triangulating a convex polygon as $NWT_n = \frac{n(n-3)}{2}, \forall n \geq 3$, which we discovered no one accomplished using this approach. We also established some Algebraic, Geometric and Number Theoretic properties of $P_n$ (a regular polygon with $n$ the number of sides) in relation to some other numbers $NWT_n$ (the number of ways of triangulating $P_n$). As a result of our comparison, we observed the following results:

We accomplished a well known important result (a formula/relation for generating the number of symmetries in an n-sided regular polygon) as $NSYP_n = 2n, \forall n \geq 3$.

## 8.0 Acknowledgments

I am grateful to my research supervisor Professor Aminu Alhaji Ibrahim of the Department of Mathematics Usmanu Danfodiyo University, Sokoto (UDUS) who indicated to me very useful references. I am also indebted to **Dr. A.I. Garbah** of the Department of Mathematics, UDUS and **Dr. A.U. Moreh of the Department of Physics,** UDUS for their helpful remarks and suggestions.

## 9.0 References

[1] Ron M. Adin and Yuval Riochman (2012), Triangle-free Triangulations, Hyper plane Arrangements and Shifted Tableaux, Electron. J. Combin. 19(3) (2012), P32.

[2] Stein S.K. (1999), Cutting a polygon into triangles of equal a reas, Amer Math. Monthly **106**,255-257 .

[3] Stein S.K. (2000), A generalized conjecture about cutting a polygon into triangles of equal areas, Discrete. Comp. Geom. **24**,141-145 .

[4] Stein S.K. (1994), Szabo S. Algebra and Tiling: Homomorphisms in the service of Geometry, MAA, Washington DC.

[5] Su Z. and Ding R. (2003), Dissecting of polygons into triangles of equal areas, J. Appl. Math. and Computing, **13**, No. (1-2), 29-36 .

[6] Schulze B. (2011), On the area discrepancy of triangulations of squares and trapezoids, The electronic Journal of Combinatorics **18**, P 137.

[7] Richard E. Klima, Neil P. Sigmon and Ernest Stitzinger;' Application of abstract algebra with Maple'. CRC Press Boca Raton London New York Washington, D.C

[8] David L. and Jonah O. (**2000),** Mathematics Subject Classification: Primary 05A05, 05A15; Secondary 05A19.