Journal of the Nigerian Association of Mathematical Physics
Volume 30, (May, 2015), pp65 – 78

On Non-Commutative Rhotrix Groups over Finite Fields

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Abstract

This paper considers the pair $\left(FGR_n(Z_P),\circ\right)$ consisting of the set of all invertible rhotrices of size n over a finite field of integers moduloprime p and together with the binary operation of row-column based method for rhotrix multiplication; $'\circ'$, in order to introduce concrete constructions of non-commutative rhotrix groups over finite fields. Furthermore, we pick specific groups $\left(FGR_3(Z_2),\circ\right)$, $\left(FGR_3(Z_3),\circ\right)$ and analyze them, so as to obtain their elements, multiplication tables, orders and subgroups. In the process, a number of theorems were developed.

Keywords: Groups, subgroups, finite rhotrix groups.

1.0 Introduction

A rhotrix set is a set consisting of well-defined mathematical objects called rhotrices as its members. A rhotrix is a rhomboidal method of representing array of numbers. A rhotrix group is a group having rhotrix set as an underlying set. The order of a rhotrix group is the number of distinct elements in it. A finite rhotrix group is a group having a limited order. A non-empty subset *H* of a rhotrix group *G* is called a rhotrix subgroup of *G* if *H* is a group under the rhotrix operation defined on *G*. If a rhotrix group is finite then the order of any of its rhotrix subgroup divide it own order in line with the Langrage's theorem.

The concept of rhotrix of size *3* was introduced by Ajibade [1] as an extension of ideas on matrix-tertions and matrix-noitrets suggested by Atanassov and Shannon [2]. In [1], a collection of all rhotrices of size 3 was defined as

$$R_3(\Re)=\left\{\left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array}\right\rangle : a,b,c,d,e\in\Re\right\}$$

The entry at the particular intersection of the vertical and horizontal diagonalgiven by '*h(R)=c*' is called the heart of any rhotrix $R\in R_3(\Re)$.The following are the binary operations of addition (+)and multiplication (o) defined in [1], recorded respectively, as follows:

$$R+Q=\left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array}\right\rangle+\left\langle \begin{array}{ccc} & f & \\ g & h(Q) & j \\ & k & \end{array}\right\rangle=\left\langle \begin{array}{ccc} & a+f & \\ b+g & h(R)+h(Q) & d+j \\ & e+k & \end{array}\right\rangle,$$

$$R\circ Q=\left\langle \begin{array}{ccc} & ah(Q)+fh(R) & \\ bh(Q)+gh(R) & h(R)h(Q) & dh(Q)+jh(R) \\ & eh(Q)+kh(R) & \end{array}\right\rangle \qquad (1)\ .$$

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The method of rhotrix multiplication (1) is referred to 'heart based method for rhotrix multiplication'. A generalization of (1) for rhotrices of size n was given by Mohammed [3]. Sani[4] proposed that a rhotrix of size n can be expressed as a couple of two matrices $[a_{ij}]$ and $[c_{lk}]$ such that

$$R_n = \langle a_{ij}, c_{lk} \rangle = \langle A_{t\times t}, C_{(t-1)\times(t-1)} \rangle = \left\langle \begin{array}{ccccccc} & & & a_{11} & & & \\ & & a_{21} & c_{11} & a_{12} & & \\ & \dots & \dots. & \dots & \dots & \dots & \\ a_{t1} & \dots & \dots & \dots & \dots & \dots & a_{1t} \\ & \dots & \dots & \dots & \dots & \dots & \\ & & a_{t(t-1)} & c_{(t-1)(t-1)} & a_{(t-1)t} & & \\ & & & a_{tt} & & & \end{array} \right\rangle,$$

$$R_n = \left\langle \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1(t-1)} & a_{1t} \\ a_{21} & a_{22} & \dots & a_{2(t-1)} & a_{2t} \\ \dots & \dots & \dots & \dots & \dots \\ a_{(t-1)1} & a_{(t-1)2} & \dots & a_{(t-1)(t-1)} & a_{(t-1)t} \\ a_{t1} & a_{t2} & \dots & a_{t(t-1)} & a_{tt} \end{bmatrix}, \begin{bmatrix} c_{11} & \dots & c_{1(t-1)} \\ \dots & \dots & \dots \\ c_{(t-1)1} & \dots & c_{(t-1)(t-1)} \end{bmatrix} \right\rangle.$$

where a_{ij} and c_{lk} are major and minor entries respectively. Implying that

Multiplication of two rhotrices R_n *and* Q_n was defined in [4] as follows

$$R_n \circ Q_n = \langle a_{i_1 j_1}, c_{l_1 k_1} \rangle \circ \langle b_{i_2 j_2}, d_{l_2 k_2} \rangle = \left\langle \sum_{i_2 j_1}^{t} (a_{i_1 j_1} b_{i_2 j_2}), \sum_{l_2 k_1}^{t-1} (c_{l_1 k_1} d_{l_2 k_2}) \right\rangle \qquad (2)$$

It was noted in [4] that this rhotrix multiplication is non-commutative but associative.
The identity element of a rhotrix of size n was also given as

$$I_n = \langle I_{t\times t}, I_{(t-1)\times(t-1)} \rangle = \left\langle \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & 1 & 0 & & \\ & \dots & \dots. & \dots & \dots & \dots & \\ 0 & \dots & \dots & 1 & \dots & \dots & 0 \\ & \dots & \dots & \dots & \dots & \dots & \\ & & 0 & 1 & 0 & & \\ & & & 1 & & & \end{array} \right\rangle$$

Determinant of rhotrix of size n was also defined [4] as:

$\det(R_n) = \det\langle a_{ij}, c_{lk} \rangle = \det(A_{t\times t}).\det(C_{(t-1)\times(t-1)})$.

It was also presented in [4] that $R_n = \langle a_{ij}, c_{lk} \rangle$ is invertible if and only if the two matrices a_{ij} *and* c_{lk} are both invertible matrices. This means that if R_n is invertible and $R_n^{-1} = \langle q_{ij}, r_{lk} \rangle$ then q_{ij} and r_{lk} are the inverse entries of matrices $A_{t\times t}$ and $C_{(t-1)\times(t-1)}$ respectively. Also, noteworthy to mention that R_n is invertible if and only if $\det(R_n) \neq 0$.

It was also shown in [4] that $\det(R_n \circ Q_n) = \det(R_n) \circ \det(Q_n) = \det(R_n).\det(Q_n)$

If $R_n = \langle a_{ij}, c_{lk} \rangle$ then its transpose was defined in [4] as $R_n^T = \langle a_{ji}, c_{kl} \rangle$. It was noted in [4] that $(R_n \circ Q_n)^T = (Q_n)^T \circ (R_n)^T$.

In this paper, we shall adopt the row-column method of rhotrix multiplication (2) to present concrete constructions of non-commutative rhotrix groups over finite fields.

2.0 Preliminaries

2.1 Finite set of all rhotrices of size n

$$R_n(Z_P) = \left\{ \left\langle \begin{array}{ccccccc} & & & a_{11} & & & \\ & & a_{21} & c_{11} & a_{12} & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ a_{t1} & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1t} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & a_{t(t-1)} & c_{(t-1)(t-1)} & a_{(t-1)t} & & \\ & & & a_{tt} & & & \end{array} \right\rangle : a_{ij}, c_{lk} \in Z_P \text{ and } p \text{ is a prime} \right\}, \quad (6)$$

The set of all rhotrices of size n with entries from a field Z_p, is a collection of all rhotrices of size n, defined by:

where $1 \le i, j \le t, \quad 1 \le l, k \le t-1; \; t = \frac{n+1}{2}, n \in 2Z^+ + 1.$

3.0 The non-commutative general rhotrix group over finite field

3.1 Theorem (The non-commutative general rhotrix group over a finite field)

Let *p* be a positive prime integer number. Let Z_p be a field of integers modulo *p*. Let $FGR_n(Z_P)$ be define as

$$FGR_n(Z_P) = \left\{ \left\langle \begin{array}{ccccccc} & & & a_{11} & & & \\ & & a_{21} & c_{11} & a_{12} & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ a_{t1} & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1t} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & a_{t(t-1)} & c_{(t-1)(t-1)} & a_{(t-1)t} & & \\ & & & a_{tt} & & & \end{array} \right\rangle : a_{ij}, c_{lk} \in Z_P \text{ and } \det(< a_{ij}, c_{lk} >) \neq 0 \right\},$$

where $1 \le i, j \le t, \quad 1 \le l, k \le t-1; \; t = \frac{n+1}{2}, n \in 2Z^+ + 1.$ Let $'\circ'$ be the row-column based rhotrix multiplication. Then the pair $\left(FGR_n(Z_P), \; \circ\right)$ is a group.

This group can be termed as the non-commutative general rhotrix group over finite field of integers modulo prime *p*.

Proof

It is simple to show that the pair $\left(FGR_n(Z_P), \; \circ\right)$ satisfies all the group axioms stated in Vashishtha [5].

3.2 Finite Non-Commutativerhotrix Group of Size 3 Taking Entries From Z_2

Let $R_3(Z_2)$ denote the set of all rhotrices of size 3 with entries from Z_2

$$R_3(Z_2) = \left\{ \left\langle \begin{array}{ccc} & a & \\ b & d & e \\ & c & \end{array} \right\rangle : a, b, c, d, e \in \{0,1\} \right\} \quad (7)$$

By the rule of permutation, the rhotrix set $R_3(Z_2)$ has cardinality 2^5rhotrices. In tabular form of a set, we have

$$R_3(Z_2)=\left\{\begin{array}{l}
\left\langle\begin{array}{ccc}&1&\\1&0&1\\&1&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&1&\\1&0&0\\&1&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&1&\\1&0&1\\&0&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&1&\\0&0&1\\&1&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&1&\\0&0&0\\&0&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&1&\\0&0&1\\&0&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&1&\\0&0&0\\&1&\end{array}\right\rangle\\
\left\langle\begin{array}{ccc}&1&\\1&0&0\\&0&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&0&\\1&0&1\\&1&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&0&\\1&0&0\\&1&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&0&\\1&0&1\\&0&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&0&\\0&0&1\\&1&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&0&\\0&0&0\\&0&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&0&\\0&0&1\\&0&\end{array}\right\rangle\\
\left\langle\begin{array}{ccc}&0&\\0&0&0\\&1&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&0&\\1&0&0\\&0&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&1&\\1&1&1\\&1&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&1&\\1&1&0\\&1&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&1&\\1&1&1\\&0&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&1&\\0&1&1\\&1&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&1&\\0&1&0\\&0&\end{array}\right\rangle\\
\left\langle\begin{array}{ccc}&1&\\0&1&1\\&0&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&1&\\0&1&0\\&1&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&1&\\1&1&0\\&0&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&0&\\1&1&1\\&1&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&0&\\1&1&0\\&1&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&0&\\1&1&1\\&0&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&0&\\0&1&1\\&1&\end{array}\right\rangle\\
\left\langle\begin{array}{ccc}&0&\\0&1&0\\&0&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&0&\\0&1&1\\&0&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&0&\\0&1&0\\&1&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&0&\\1&1&0\\&0&\end{array}\right\rangle
\end{array}\right\}$$

Now, our interest is to construct a rhotrix group consisting of all invertible rhotrices in $R_3(Z_2)$ and together with non-commutative method of rhotrix multiplication, and denote it as $\left(FGR_3(Z_2),\circ\right)$.

To achieve this objective, let us start by defining the set $FGR_3(Z_2)$ as follows:

$$FGR_3(Z_2)=\left\{\left\langle\begin{array}{ccc}&a&\\b&d&e\\&c&\end{array}\right\rangle : a,b,c,d,e\in\{0,1\}\ \ and\ \det\left(\left\langle\begin{array}{ccc}&a&\\b&d&e\\&c&\end{array}\right\rangle\right)\neq 0\right\}. \qquad (8)$$

Implying that $FGR_3(Z_2)$ is a collection of all rhotrices in $R_3(Z_2)$ satisfying the condition that the sub-matrices that make up such rhotrices in $R_3(Z_2)$ must be non-singular. Thus, in tabular form, we have

$$FGR_3(Z_2)=\left\{\left\langle\begin{array}{ccc}&1&\\0&1&0\\&1&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&0&\\1&1&1\\&1&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&1&\\0&1&1\\&1&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&1&\\1&1&1\\&0&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&1&\\1&1&0\\&1&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&0&\\1&1&1\\&0&\end{array}\right\rangle\right\}.$$

Let $\circ$ be the binary operation of row-column based method for rhotrix multiplication. Then we can have the following corollary 3.3 for theorem 3.1

3.3 Corollary

The pair $\left(FGR_3(Z_2),\circ\right)$ is a finite non-commutative rhotrix group of order 6.

Proof

Let us denote the elements in $FGR_3(Z_2)$ as follows:

$$R1=\left\langle\begin{array}{ccc}&1&\\0&1&0\\&1&\end{array}\right\rangle, R2=\left\langle\begin{array}{ccc}&0&\\1&1&1\\&1&\end{array}\right\rangle, R3=\left\langle\begin{array}{ccc}&1&\\0&1&1\\&1&\end{array}\right\rangle, R4=\left\langle\begin{array}{ccc}&1&\\1&1&1\\&0&\end{array}\right\rangle, R5=\left\langle\begin{array}{ccc}&1&\\1&1&0\\&1&\end{array}\right\rangle, R6=\left\langle\begin{array}{ccc}&0&\\1&1&1\\&0&\end{array}\right\rangle.$$

The multiplication table for elements in $\left(FGR_3(Z_2),\circ\right)$ is given below by table 1.

Table 1: Multiplication table for $\left(FGR_3(Z_2),\circ\right)$

$\circ$	$R1$	$R2$	$R3$	$R4$	$R5$	$R6$
$R1$	$R1$	$R2$	$R3$	$R4$	$R5$	$R6$
$R2$	$R2$	$R4$	$R6$	$R1$	$R3$	$R5$
$R3$	$R3$	$R5$	$R1$	$R6$	$R2$	$R4$
$R4$	$R4$	$R1$	$R5$	$R2$	$R6$	$R3$
$R5$	$R5$	$R6$	$R4$	$R3$	$R1$	$R2$
$R6$	$R6$	$R3$	$R2$	$R5$	$R4$	$R1$

Next, we investigate the subgroups of $\left(FGR_3(Z_2),\circ\right)$.

Observe that there exist five proper subgroups of $\left(FGR_3(Z_2),\circ\right)$ and then the group itself. The subgroups are given by the following list:

(i) $\left(S1FGR_3(Z_2),\circ\right)=\left(\{R1,R6\},\circ\right)$, (ii) $\left(S2FGR_3(Z_2),\circ\right)=\left(\{R1,R3\},\circ\right)$,

(iii) $\left(S3FGR_3(Z_2),\circ\right)=\left(\{R1,R5\},\circ\right)$, (iv) $\left(S4FGR_3(Z_2),\circ\right)=\left(\{R1,R2,R4\},\circ\right)$,

(v) $\left(S5FGR_3(Z_2),\circ\right)=\left(\{I\},\circ\right)$ *and* (vi) $\left(FGR_3(Z_2),\circ\right)$

This is in perfect harmony with Lagrange's Theorem on subgroups of finite groups in Vashishtha [5].
Notethat :

$\circ\left(S1FGR_3(Z_2),\circ\right)=2$, $\circ\left(S2FGR_3(Z_2),\circ\right)=2$, $\circ\left(S3FGR_3(Z_2),\circ\right)=2$, $\circ\left(S4FGR_3(Z_2),\circ\right)=3$, $\circ\left(S5FGR_3(Z_2),\circ\right)=1$

Also the order of each of the element of $\left(FGR_3(Z_2),\circ\right)$ is given below: $\circ(R1)=1, \circ(R2)=3, \circ(R3)=2, \circ(R4)=3, \circ(R5)=2, \circ(R6)=2.$

3.4 Finite Non-Commutative Rhotrix Group of Size 3 Taking Entries From Z_3

Let $R_3(Z_3)$ denotes the set of all rhotrices of size 3 with entries from Z_3

$$R_3(Z_3) = \left\{\left\langle \begin{matrix} & a & \\ b & d & e \\ & c & \end{matrix}\right\rangle : a,b,c,d,e \in \{0,1,2\}\right\} \quad (9)$$

By the rule of permutation, the rhotrix set $R_3(Z_3)$ has cardinality 3^5rhotrices. In tabular form of a set, the set $R_3(Z_3)$ is given by:

$$\left\{\begin{array}{l}
\left\langle\begin{smallmatrix}&1&\\1&0&1\\&1&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&1&\\1&0&0\\&1&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&1&\\1&0&2\\&1&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&1&\\1&0&0\\&0&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&1&\\1&0&2\\&0&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&1&\\1&0&0\\&2&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&1&\\1&0&1\\&2&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&1&\\1&0&2\\&2&\end{smallmatrix}\right\rangle \\
\left\langle\begin{smallmatrix}&1&\\1&0&1\\&0&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&1&\\2&0&2\\&2&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&1&\\2&0&1\\&2&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&1&\\2&0&0\\&2&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&1&\\2&0&1\\&1&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&1&\\2&0&0\\&1&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&1&\\2&0&1\\&0&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&1&\\2&0&2\\&0&\end{smallmatrix}\right\rangle \\
\left\langle\begin{smallmatrix}&1&\\2&0&0\\&0&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&1&\\2&0&2\\&1&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&1&\\0&0&0\\&0&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&1&\\0&0&2\\&0&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&1&\\0&0&1\\&0&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&1&\\0&0&2\\&2&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&1&\\0&0&1\\&2&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&1&\\0&0&2\\&1&\end{smallmatrix}\right\rangle \\
\left\langle\begin{smallmatrix}&1&\\0&0&0\\&1&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&1&\\0&0&1\\&1&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&1&\\0&0&0\\&2&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&0&\\1&0&1\\&1&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&0&\\1&0&0\\&1&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&0&\\1&0&2\\&1&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&0&\\1&0&0\\&0&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&0&\\1&0&2\\&0&\end{smallmatrix}\right\rangle \\
\left\langle\begin{smallmatrix}&0&\\1&0&0\\&2&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&0&\\1&0&1\\&2&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&0&\\1&0&2\\&2&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&0&\\1&0&1\\&0&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&0&\\2&0&2\\&2&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&0&\\2&0&1\\&2&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&0&\\2&0&0\\&2&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&0&\\2&0&1\\&1&\end{smallmatrix}\right\rangle \\
\left\langle\begin{smallmatrix}&0&\\2&0&0\\&1&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&0&\\2&0&1\\&0&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&0&\\2&0&2\\&0&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&0&\\2&0&0\\&0&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&0&\\2&0&2\\&1&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&0&\\0&0&0\\&0&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&0&\\0&0&2\\&0&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&0&\\0&0&1\\&0&\end{smallmatrix}\right\rangle \\
\left\langle\begin{smallmatrix}&0&\\0&0&2\\&2&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&0&\\0&0&1\\&2&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&0&\\0&0&2\\&1&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&0&\\0&0&0\\&1&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&0&\\0&0&1\\&1&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&0&\\0&0&0\\&2&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&2&\\1&0&1\\&1&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&2&\\1&0&0\\&1&\end{smallmatrix}\right\rangle \\
\left\langle\begin{smallmatrix}&2&\\1&0&2\\&1&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&2&\\1&0&0\\&0&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&2&\\1&0&2\\&0&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&2&\\1&0&0\\&2&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&2&\\1&0&1\\&2&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&2&\\1&0&2\\&2&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&2&\\1&0&1\\&0&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&2&\\2&0&2\\&2&\end{smallmatrix}\right\rangle
\end{array}\right\}$$

$R_3(Z_3)$ Continued…

$$\left\{\begin{array}{l}
\left\langle\begin{smallmatrix}&2&\\2&0&1\\&2&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&2&\\2&0&0\\&2&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&2&\\2&0&1\\&1&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&2&\\2&0&0\\&1&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&2&\\2&0&1\\&0&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&2&\\2&0&2\\&0&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&2&\\2&0&0\\&0&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&2&\\2&0&2\\&1&\end{smallmatrix}\right\rangle \\
\left\langle\begin{smallmatrix}&2&\\0&0&0\\&0&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&2&\\0&0&2\\&0&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&2&\\0&0&1\\&0&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&2&\\0&0&2\\&2&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&2&\\0&0&1\\&2&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&2&\\0&0&2\\&1&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&2&\\0&0&0\\&1&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&2&\\0&0&1\\&1&\end{smallmatrix}\right\rangle \\
\left\langle\begin{smallmatrix}&2&\\0&0&0\\&2&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&1&\\1&1&1\\&1&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&1&\\1&1&0\\&1&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&1&\\1&1&2\\&1&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&1&\\1&1&0\\&0&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&1&\\1&1&2\\&0&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&1&\\1&1&0\\&2&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&1&\\1&1&1\\&2&\end{smallmatrix}\right\rangle \\
\left\langle\begin{smallmatrix}&1&\\1&1&2\\&2&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&1&\\1&1&1\\&0&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&1&\\2&1&2\\&2&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&1&\\2&1&1\\&2&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&1&\\2&1&0\\&2&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&1&\\2&1&1\\&1&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&1&\\2&1&0\\&1&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&1&\\2&1&1\\&0&\end{smallmatrix}\right\rangle \\
\left\langle\begin{smallmatrix}&1&\\2&1&2\\&0&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&1&\\2&1&0\\&0&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&1&\\2&1&2\\&1&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&1&\\0&1&0\\&0&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&1&\\0&1&2\\&0&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&1&\\0&1&1\\&0&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&1&\\0&1&2\\&2&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&1&\\0&1&1\\&2&\end{smallmatrix}\right\rangle \\
\left\langle\begin{smallmatrix}&1&\\0&1&2\\&1&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&1&\\0&1&0\\&1&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&1&\\0&1&1\\&1&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&1&\\0&1&0\\&2&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&0&\\1&1&1\\&1&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&0&\\1&1&0\\&1&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&0&\\1&1&2\\&1&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&0&\\1&1&0\\&0&\end{smallmatrix}\right\rangle \\
\left\langle\begin{smallmatrix}&0&\\1&1&2\\&0&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&0&\\1&1&0\\&2&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&0&\\1&1&1\\&2&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&0&\\1&1&2\\&2&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&0&\\1&1&1\\&0&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&0&\\2&1&2\\&2&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&0&\\2&1&1\\&2&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&0&\\2&1&0\\&2&\end{smallmatrix}\right\rangle \\
\left\langle\begin{smallmatrix}&0&\\2&1&1\\&1&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&0&\\2&1&0\\&1&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&0&\\2&1&1\\&0&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&0&\\2&1&2\\&0&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&0&\\2&1&0\\&0&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&0&\\2&1&2\\&1&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&0&\\0&1&0\\&0&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&0&\\0&1&2\\&0&\end{smallmatrix}\right\rangle \\
\left\langle\begin{smallmatrix}&0&\\0&1&1\\&0&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&0&\\0&1&2\\&2&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&0&\\0&1&1\\&2&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&0&\\0&1&2\\&1&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&0&\\0&1&0\\&1&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&0&\\0&1&1\\&1&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&0&\\0&1&0\\&2&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&2&\\1&1&1\\&1&\end{smallmatrix}\right\rangle \\
\left\langle\begin{smallmatrix}&2&\\1&1&0\\&1&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&2&\\1&1&2\\&1&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&2&\\1&1&0\\&0&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&2&\\1&1&2\\&0&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&2&\\1&1&0\\&2&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&2&\\1&1&1\\&2&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&2&\\1&1&2\\&2&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&2&\\1&1&1\\&0&\end{smallmatrix}\right\rangle \\
\left\langle\begin{smallmatrix}&2&\\2&1&2\\&2&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&2&\\2&1&1\\&2&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&2&\\2&1&0\\&2&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&2&\\2&1&1\\&1&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&2&\\2&1&0\\&1&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&2&\\2&1&1\\&0&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&2&\\2&1&2\\&0&\end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix}&2&\\2&1&0\\&0&\end{smallmatrix}\right\rangle
\end{array}\right\}$$

$R_3(Z_3)$ Continued…

$$\left\{\begin{array}{l}
\left\langle\begin{array}{ccc}&2&\\2&1&2\\&1&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&2&\\0&1&0\\&0&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&2&\\0&1&2\\&0&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&2&\\0&1&1\\&0&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&2&\\0&1&2\\&2&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&2&\\0&1&1\\&2&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&2&\\0&1&2\\&1&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&2&\\0&1&0\\&1&\end{array}\right\rangle \\
\left\langle\begin{array}{ccc}&2&\\0&1&1\\&1&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&2&\\0&1&0\\&2&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&1&\\1&2&1\\&1&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&1&\\1&2&0\\&1&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&1&\\1&2&2\\&1&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&1&\\1&2&0\\&0&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&1&\\1&2&2\\&0&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&1&\\1&2&0\\&2&\end{array}\right\rangle \\
\left\langle\begin{array}{ccc}&1&\\1&2&1\\&2&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&1&\\1&2&2\\&2&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&1&\\1&2&1\\&0&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&1&\\2&2&2\\&2&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&1&\\2&2&1\\&2&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&1&\\2&2&0\\&2&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&1&\\2&2&1\\&1&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&1&\\2&2&0\\&1&\end{array}\right\rangle \\
\left\langle\begin{array}{ccc}&1&\\2&2&1\\&0&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&1&\\2&2&2\\&0&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&1&\\2&2&0\\&0&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&1&\\2&2&2\\&1&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&1&\\0&2&0\\&0&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&1&\\0&2&2\\&0&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&1&\\0&2&1\\&0&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&1&\\0&2&2\\&2&\end{array}\right\rangle \\
\left\langle\begin{array}{ccc}&1&\\0&2&1\\&2&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&1&\\0&2&2\\&1&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&1&\\0&2&0\\&1&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&1&\\0&2&1\\&1&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&1&\\0&2&0\\&2&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&0&\\1&2&1\\&1&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&0&\\1&2&0\\&1&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&0&\\1&2&2\\&1&\end{array}\right\rangle \\
\left\langle\begin{array}{ccc}&0&\\1&2&0\\&0&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&0&\\1&2&2\\&0&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&0&\\1&2&0\\&2&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&0&\\1&2&1\\&2&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&0&\\1&2&2\\&2&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&0&\\1&2&1\\&0&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&0&\\2&2&2\\&2&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&0&\\2&2&1\\&2&\end{array}\right\rangle \\
\left\langle\begin{array}{ccc}&0&\\2&2&0\\&2&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&0&\\2&2&1\\&1&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&0&\\2&2&0\\&1&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&0&\\2&2&1\\&0&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&0&\\2&2&2\\&0&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&0&\\2&2&0\\&0&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&0&\\2&2&2\\&1&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&0&\\0&2&0\\&0&\end{array}\right\rangle \\
\left\langle\begin{array}{ccc}&0&\\0&2&2\\&0&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&0&\\0&2&1\\&0&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&0&\\0&2&2\\&2&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&0&\\0&2&1\\&2&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&0&\\0&2&2\\&1&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&0&\\0&2&0\\&1&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&0&\\0&2&1\\&1&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&0&\\0&2&0\\&2&\end{array}\right\rangle \\
\left\langle\begin{array}{ccc}&2&\\1&2&1\\&1&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&2&\\1&2&0\\&1&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&2&\\1&2&2\\&1&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&2&\\1&2&0\\&0&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&2&\\1&2&2\\&0&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&2&\\1&2&0\\&2&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&2&\\1&2&1\\&2&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&2&\\1&2&2\\&2&\end{array}\right\rangle \\
\left\langle\begin{array}{ccc}&2&\\1&2&1\\&0&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&2&\\2&2&2\\&2&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&2&\\2&2&1\\&2&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&2&\\2&2&0\\&2&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&2&\\2&2&1\\&1&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&2&\\2&2&0\\&1&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&2&\\2&2&1\\&0&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&2&\\2&2&2\\&0&\end{array}\right\rangle \\
\left\langle\begin{array}{ccc}&2&\\2&2&0\\&0&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&2&\\2&2&2\\&1&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&2&\\0&2&0\\&0&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&2&\\0&2&2\\&0&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&2&\\0&2&1\\&0&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&2&\\0&2&2\\&2&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&2&\\0&2&1\\&2&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&2&\\0&2&2\\&1&\end{array}\right\rangle \\
\left\langle\begin{array}{ccc}&2&\\0&2&0\\&1&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&2&\\0&2&1\\&1&\end{array}\right\rangle,\left\langle\begin{array}{ccc}&2&\\0&2&0\\&2&\end{array}\right\rangle
\end{array}\right\}$$

Now, our interest is to construct a rhotrix group consisting of all invertible rhotrices in $R_3(Z_3)$ and together with non-commutative method of rhotrix multiplication, and denote it as $\left(FGR_3(Z_3),\circ\right)$.

To achieve this objective, we start by defining the set $FGR_3(Z_3)$ as follows:

$$FGR_3(Z_3)=\left\{\left\langle\begin{array}{ccc}&a&\\b&d&e\\&c&\end{array}\right\rangle : a,b,c,d,e\in\{0,1,2\}\ \ and\ \det\left(\left\langle\begin{array}{ccc}&a&\\b&d&e\\&c&\end{array}\right\rangle\right)\neq 0\right\}. \tag{10}$$

Implying that $FGR_3(Z_3)$ is a collection of all rhotrices in $R_3(Z_3)$ satisfying the condition that the sub-matrices that make up such rhotrices in $R_3(Z_3)$ are all non-singular. Thus, in tabular form, we have $FGR_3(Z_3)$ given by the set below:

$$FGR_3(Z_3)=\left\{\begin{array}{l}
R1=\left\langle\begin{array}{ccc}&1&\\0&1&0\\&1&\end{array}\right\rangle,R2=\left\langle\begin{array}{ccc}&2&\\0&1&0\\&1&\end{array}\right\rangle,R3=\left\langle\begin{array}{ccc}&1&\\0&1&0\\&2&\end{array}\right\rangle,R4=\left\langle\begin{array}{ccc}&2&\\0&1&0\\&2&\end{array}\right\rangle,R5=\left\langle\begin{array}{ccc}&1&\\0&1&1\\&1&\end{array}\right\rangle,R6=\left\langle\begin{array}{ccc}&2&\\0&1&1\\&1&\end{array}\right\rangle, \\
R7=\left\langle\begin{array}{ccc}&1&\\0&1&2\\&1&\end{array}\right\rangle,R8=\left\langle\begin{array}{ccc}&2&\\0&1&2\\&1&\end{array}\right\rangle,R9=\left\langle\begin{array}{ccc}&1&\\0&1&1\\&2&\end{array}\right\rangle,R10=\left\langle\begin{array}{ccc}&2&\\0&1&1\\&2&\end{array}\right\rangle,R11=\left\langle\begin{array}{ccc}&1&\\0&1&2\\&2&\end{array}\right\rangle,R12=\left\langle\begin{array}{ccc}&2&\\0&1&2\\&2&\end{array}\right\rangle \\
R13=\left\langle\begin{array}{ccc}&1&\\1&1&0\\&1&\end{array}\right\rangle,R14=\left\langle\begin{array}{ccc}&2&\\1&1&0\\&1&\end{array}\right\rangle,R15=\left\langle\begin{array}{ccc}&1&\\1&1&0\\&2&\end{array}\right\rangle,R16=\left\langle\begin{array}{ccc}&2&\\1&1&0\\&2&\end{array}\right\rangle,R17=\left\langle\begin{array}{ccc}&1&\\2&1&0\\&2&\end{array}\right\rangle,R18=\left\langle\begin{array}{ccc}&2&\\2&1&0\\&2&\end{array}\right\rangle \\
R19=\left\langle\begin{array}{ccc}&1&\\2&1&0\\&1&\end{array}\right\rangle,R20=\left\langle\begin{array}{ccc}&2&\\2&1&0\\&1&\end{array}\right\rangle,R21=\left\langle\begin{array}{ccc}&0&\\1&1&1\\&1&\end{array}\right\rangle,R22\left\langle\begin{array}{ccc}&0&\\1&1&2\\&1&\end{array}\right\rangle,R23=\left\langle\begin{array}{ccc}&0&\\1&1&2\\&2&\end{array}\right\rangle,R24=\left\langle\begin{array}{ccc}&0&\\2&1&2\\&2&\end{array}\right\rangle \\
R25=\left\langle\begin{array}{ccc}&0&\\2&1&1\\&2&\end{array}\right\rangle,R26=\left\langle\begin{array}{ccc}&0&\\2&1&2\\&1&\end{array}\right\rangle,R27=\left\langle\begin{array}{ccc}&0&\\2&1&1\\&1&\end{array}\right\rangle,R28=\left\langle\begin{array}{ccc}&0&\\1&1&1\\&2&\end{array}\right\rangle,R29=\left\langle\begin{array}{ccc}&1&\\1&1&1\\&0&\end{array}\right\rangle,R30=\left\langle\begin{array}{ccc}&2&\\1&1&1\\&0&\end{array}\right\rangle \\
R31=\left\langle\begin{array}{ccc}&1&\\1&1&2\\&0&\end{array}\right\rangle,R32=\left\langle\begin{array}{ccc}&2&\\1&1&2\\&0&\end{array}\right\rangle,R33=\left\langle\begin{array}{ccc}&1&\\2&1&2\\&0&\end{array}\right\rangle,R34=\left\langle\begin{array}{ccc}&2&\\2&1&2\\&0&\end{array}\right\rangle,R35=\left\langle\begin{array}{ccc}&1&\\2&1&1\\&0&\end{array}\right\rangle,R36=\left\langle\begin{array}{ccc}&2&\\2&1&1\\&0&\end{array}\right\rangle
\end{array}\right\}$$

$FGR_3(Z_3)$ Continued…

$$\left\{\begin{array}{l}
R37=\left\langle\begin{array}{ccc}&2&\\1&1&1\\&1&\end{array}\right\rangle, R38=\left\langle\begin{array}{ccc}&1&\\1&1&2\\&1&\end{array}\right\rangle, R39=\left\langle\begin{array}{ccc}&2&\\1&1&2\\&2&\end{array}\right\rangle, R40=\left\langle\begin{array}{ccc}&1&\\2&1&2\\&2&\end{array}\right\rangle, R41=\left\langle\begin{array}{ccc}&2&\\2&1&1\\&2&\end{array}\right\rangle, R42=\left\langle\begin{array}{ccc}&2&\\2&1&2\\&1&\end{array}\right\rangle \\
R43=\left\langle\begin{array}{ccc}&1&\\2&1&1\\&1&\end{array}\right\rangle, R44=\left\langle\begin{array}{ccc}&1&\\1&1&1\\&2&\end{array}\right\rangle, R45=\left\langle\begin{array}{ccc}&0&\\2&1&1\\&0&\end{array}\right\rangle, R46=\left\langle\begin{array}{ccc}&0&\\1&1&2\\&0&\end{array}\right\rangle, R47=\left\langle\begin{array}{ccc}&0&\\1&1&1\\&0&\end{array}\right\rangle, R48=\left\langle\begin{array}{ccc}&0&\\2&1&2\\&0&\end{array}\right\rangle, \\
R49=\left\langle\begin{array}{ccc}&1&\\0&2&0\\&1&\end{array}\right\rangle, R50=\left\langle\begin{array}{ccc}&2&\\0&2&0\\&1&\end{array}\right\rangle, R51=\left\langle\begin{array}{ccc}&1&\\0&2&0\\&2&\end{array}\right\rangle, R52=\left\langle\begin{array}{ccc}&2&\\0&2&0\\&2&\end{array}\right\rangle, R53=\left\langle\begin{array}{ccc}&1&\\0&2&1\\&1&\end{array}\right\rangle, R54=\left\langle\begin{array}{ccc}&2&\\0&2&1\\&1&\end{array}\right\rangle, \\
R55=\left\langle\begin{array}{ccc}&1&\\0&2&2\\&1&\end{array}\right\rangle, R56=\left\langle\begin{array}{ccc}&2&\\0&2&2\\&1&\end{array}\right\rangle, R57=\left\langle\begin{array}{ccc}&1&\\0&2&1\\&2&\end{array}\right\rangle, R58=\left\langle\begin{array}{ccc}&2&\\0&2&1\\&2&\end{array}\right\rangle, R59=\left\langle\begin{array}{ccc}&1&\\0&2&2\\&2&\end{array}\right\rangle, R60=\left\langle\begin{array}{ccc}&2&\\0&2&2\\&2&\end{array}\right\rangle \\
R61=\left\langle\begin{array}{ccc}&1&\\1&2&0\\&1&\end{array}\right\rangle, R62=\left\langle\begin{array}{ccc}&2&\\1&2&0\\&1&\end{array}\right\rangle, R63=\left\langle\begin{array}{ccc}&1&\\1&2&0\\&2&\end{array}\right\rangle, R64=\left\langle\begin{array}{ccc}&2&\\1&2&0\\&2&\end{array}\right\rangle, R65=\left\langle\begin{array}{ccc}&1&\\2&2&0\\&2&\end{array}\right\rangle, R66=\left\langle\begin{array}{ccc}&2&\\2&2&0\\&2&\end{array}\right\rangle \\
R67=\left\langle\begin{array}{ccc}&1&\\2&2&0\\&1&\end{array}\right\rangle, R68=\left\langle\begin{array}{ccc}&2&\\2&2&0\\&1&\end{array}\right\rangle, R69=\left\langle\begin{array}{ccc}&0&\\1&2&1\\&1&\end{array}\right\rangle, R70=\left\langle\begin{array}{ccc}&0&\\1&2&2\\&1&\end{array}\right\rangle, R71=\left\langle\begin{array}{ccc}&0&\\1&2&2\\&2&\end{array}\right\rangle, R72=\left\langle\begin{array}{ccc}&0&\\2&2&2\\&2&\end{array}\right\rangle \\
R73=\left\langle\begin{array}{ccc}&0&\\2&2&1\\&2&\end{array}\right\rangle, R74=\left\langle\begin{array}{ccc}&0&\\2&2&2\\&1&\end{array}\right\rangle, R75=\left\langle\begin{array}{ccc}&0&\\2&2&1\\&1&\end{array}\right\rangle, R76=\left\langle\begin{array}{ccc}&0&\\1&2&1\\&2&\end{array}\right\rangle, R77=\left\langle\begin{array}{ccc}&1&\\1&2&1\\&0&\end{array}\right\rangle, R78=\left\langle\begin{array}{ccc}&2&\\1&2&1\\&0&\end{array}\right\rangle \\
R79=\left\langle\begin{array}{ccc}&1&\\1&2&2\\&0&\end{array}\right\rangle, R80=\left\langle\begin{array}{ccc}&2&\\1&2&2\\&0&\end{array}\right\rangle, R81=\left\langle\begin{array}{ccc}&1&\\2&2&2\\&0&\end{array}\right\rangle, R82=\left\langle\begin{array}{ccc}&2&\\2&2&2\\&0&\end{array}\right\rangle, R83=\left\langle\begin{array}{ccc}&1&\\2&2&1\\&0&\end{array}\right\rangle, R84=\left\langle\begin{array}{ccc}&2&\\2&2&1\\&0&\end{array}\right\rangle \\
R85=\left\langle\begin{array}{ccc}&2&\\1&2&1\\&1&\end{array}\right\rangle, R86=\left\langle\begin{array}{ccc}&1&\\1&2&2\\&1&\end{array}\right\rangle, R87=\left\langle\begin{array}{ccc}&2&\\1&2&2\\&2&\end{array}\right\rangle, R88=\left\langle\begin{array}{ccc}&1&\\2&2&2\\&2&\end{array}\right\rangle, R89=\left\langle\begin{array}{ccc}&2&\\2&2&1\\&2&\end{array}\right\rangle, R90=\left\langle\begin{array}{ccc}&2&\\2&2&2\\&1&\end{array}\right\rangle \\
R91=\left\langle\begin{array}{ccc}&1&\\2&2&1\\&1&\end{array}\right\rangle, R92=\left\langle\begin{array}{ccc}&1&\\1&2&1\\&2&\end{array}\right\rangle, R93=\left\langle\begin{array}{ccc}&0&\\2&2&1\\&0&\end{array}\right\rangle, R94=\left\langle\begin{array}{ccc}&0&\\1&2&2\\&0&\end{array}\right\rangle, R95=\left\langle\begin{array}{ccc}&0&\\1&2&1\\&0&\end{array}\right\rangle, R96=\left\langle\begin{array}{ccc}&0&\\2&2&2\\&0&\end{array}\right\rangle
\end{array}\right\}$$

The following result is a corollary for theorem 3.1.

3.5 Corollary

Let $\circ$ be the binary operation of row-column based method for rhotrix multiplication and let $FGR_3(Z_3)$ be the set of all invertible rhotrices of size 3 with entries from Z_3. Then the pair $\left(FGR_3(Z_3),\circ\right)$ is a finite non-commutative rhotrix group of order 96.

4.0 Subgroups of $\left(FGR_3(Z_3),\circ\right)$

It is interesting to identify all the subgroups of $\left(FGR_3(Z_3),\circ\right)$. Observe that there exist at least 15 proper subgroups of $\left(FGR_3(Z_3),\circ\right)$ and then the group itself. The subgroups are given by the following list of lemmas

4.1 Lemma

Let $S1FGR_3(Z_3)$ be a subset of $FGR_3(Z_3)$ defined as

$$S1FGR_3(Z_3)=\{R1,R52\}$$

and let $\circ$ be a binary operation of non-commutative method of rhotrix multiplication, then $\left(S1FGR_3(Z_3),\circ\right)$ is a subgroup of $\left(FGR_3(Z_3),\circ\right)$. In particular, $\left(S1FGR_3(Z_3),\circ\right)$ is a scalar rhotrix subgroup of $\left(FGR_3(Z_3),\circ\right)$.

The multiplication table for $\left(S1FGR_3(Z_3),\circ\right)$ is given by Table 2

Table 2: Multiplication table for $\left(S1FGR_3(Z_3),\circ\right)$

$\circ$	$R1$	$R54$
$R1$	$R1$	$R54$
$R54$	$R54$	$R1$

Note that:

$\circ\left(S1FGR_3(Z_3),\circ\right)=2$

4.2 Lemma

Let $S2FGR_3(Z_3)$ be a subset of $FGR_3(Z_3)$ defined as

$S2FGR_3(Z_3)=\{R1,R2,R3,R4\}$

and let $\circ$ be a binary operation of non-commutative method of rhotrix multiplication, then $\left(S2FGR_3(Z_3),\circ\right)$ is a subgroup of $\left(FGR_3(Z_3),\circ\right)$. In particular, $\left(S2FGR_3(Z_3),\circ\right)$ is the diagonal rhotrix subgroup of $\left(FGR_3(Z_3),\circ\right)$ with unit heart.

The multiplication table for $\left(S2FGR_3(Z_3),\circ\right)$ is given by Table 3

Table 3: Multiplication table for $\left(S2FGR_3(Z_3),\circ\right)$

$\circ$	$R1$	$R2$	$R3$	$R4$
$R1$	$R1$	$R2$	$R3$	$R4$
$R2$	$R2$	$R1$	$R4$	$R3$
$R3$	$R3$	$R4$	$R1$	$R2$
$R4$	$R4$	$R3$	$R2$	$R1$

Note that

$\circ\left(S2FGR_3(Z_3),\circ\right)=4$

4.3 Lemma

Let $S3FGR_3(Z_3)$ be a subset of $FGR_3(Z_3)$ defined as

$S3FGR_3(Z_3)=\{R1,R4,R50,R51\}$

and let $\circ$ be a binary operation of non-commutative method of rhotrix multiplication, then $\left(S3FGR_3(Z_3),\circ\right)$ is a subgroup of $\left(FGR_3(Z_3),\circ\right)$.in particular, $\left(S3FGR_3(Z_3),\circ\right)$ is a special diagonal rhotrix subgroup of $\left(FGR_3(Z_3),\circ\right)$

The multiplication table for $\left(S3FGR_3(Z_3),\circ\right)$ is given by Table 4

Table 4: Multiplication table for $\left(S3FGR_3(Z_3),\circ\right)$

$\circ$	$R1$	$R4$	$R50$	$R51$
$R1$	$R1$	$R4$	$R50$	$R51$
$R4$	$R4$	$R1$	$R51$	$R50$
$R50$	$R50$	$R51$	$R1$	$R4$
$R51$	$R51$	$R50$	$R4$	$R1$

Note that $\circ\left(S3FGR_3(Z_3),\circ\right)=4$

4.4 Lemma

Let $S4FGR_3(Z_3)$ be a subset of $FGR_3(Z_3)$ defined as

$S4FGR_3(Z_3)=\{R1,R2,R3,R4,R49,R50,R51,R52\}$

and let $\circ$ be a binary operation of non-commutative method of rhotrix multiplication, then $\left(S4FGR_3(Z_3),\circ\right)$ is a subgroup of $\left(FGR_3(Z_3),\circ\right)$. In particular, $\left(S4FGR_3(Z_3),\circ\right)$ is the diagonal rhotrix subgroup of $\left(FGR_3(Z_3),\circ\right)$.

The multiplication table for $\left(S4FGR_3(Z_3),\circ\right)$ is given by Table 5.

Table 5: Multiplication table for $\left(S4FGR_3(Z_3),\circ\right)$

$\circ$	$R1$	$R2$	$R3$	$R4$	$R49$	$R50$	$R51$	$R52$
$R1$	$R1$	$R2$	$R3$	$R4$	$R49$	$R50$	$R51$	$R52$
$R2$	$R2$	$R1$	$R4$	$R3$	$R50$	$R49$	$R52$	$R51$
$R3$	$R3$	$R4$	$R1$	$R2$	$R51$	$R52$	$R49$	$R50$
$R4$	$R4$	$R3$	$R2$	$R1$	$R52$	$R51$	$R50$	$R49$
$R49$	$R49$	$R50$	$R51$	$R52$	$R1$	$R2$	$R3$	$R4$
$R50$	$R50$	$R49$	$R52$	$R51$	$R2$	$R1$	$R4$	$R3$
$R51$	$R51$	$R52$	$R49$	$R50$	$R3$	$R4$	$R1$	$R2$
$R52$	$R52$	$R51$	$R50$	$R49$	$R4$	$R3$	$R2$	$R1$

Note that $\circ\left(S4FGR_3(Z_3),\circ\right)=8$

4.5 Lemma

Let $S5FGR_3(Z_3)$ be a subset of $FGR_3(Z_3)$ defined as

$$S5FGR_3(Z_3)=\{R1,R2,R3,R4,R13,R14,R15,R16,R17,R18,R19,R20\}$$

and let $\circ$ be a binary operation of non-commutative method of rhotrix multiplication, then $\left(S5FGR_3(Z_3),\circ\right)$ is a subgroup of $\left(FGR_3(Z_3),\circ\right)$. In particular, $\left(S5FGR_3(Z_3),\circ\right)$ is the Left Triangular rhotrix subgroup of $\left(FGR_3(Z_3),\circ\right)$ with unit heart.

The multiplication table for $\left(S5FGR_3(Z_3),\circ\right)$ is given by Table 6

Table 6: The multiplication table for $\left(S5FGR_3(Z_3),\circ\right)$

$\circ$	$R1$	$R2$	$R3$	$R4$	$R13$	$R14$	$R15$	$R16$	$R17$	$R18$	$R19$	$R20$
$R1$	$R1$	$R2$	$R3$	$R4$	$R13$	$R14$	$R15$	$R16$	$R17$	$R18$	$R19$	$R20$
$R2$	$R2$	$R1$	$R4$	$R3$	$R14$	$R13$	$R16$	$R15$	$R18$	$R17$	$R20$	$R19$
$R3$	$R3$	$R4$	$R1$	$R2$	$R17$	$R18$	$R19$	$R20$	$R13$	$R14$	$R15$	$R16$
$R4$	$R4$	$R3$	$R2$	$R1$	$R18$	$R17$	$R20$	$R19$	$R14$	$R13$	$R16$	$R15$
$R13$	$R13$	$R20$	$R15$	$R18$	$R19$	$R2$	$R17$	$R4$	$R3$	$R16$	$R1$	$R14$
$R14$	$R14$	$R19$	$R16$	$R17$	$R20$	$R1$	$R18$	$R3$	$R4$	$R15$	$R2$	$R13$
$R15$	$R15$	$R18$	$R13$	$R20$	$R3$	$R16$	$R1$	$R14$	$R19$	$R2$	$R17$	$R4$
$R16$	$R16$	$R17$	$R14$	$R19$	$R4$	$R15$	$R2$	$R13$	$R20$	$R1$	$R18$	$R3$
$R17$	$R17$	$R16$	$R19$	$R14$	$R15$	$R4$	$R13$	$R2$	$R1$	$R20$	$R3$	$R18$
$R18$	$R18$	$R15$	$R20$	$R13$	$R16$	$R3$	$R14$	$R1$	$R2$	$R19$	$R4$	$R17$
$R19$	$R19$	$R14$	$R17$	$R16$	$R1$	$R20$	$R3$	$R18$	$R15$	$R4$	$R13$	$R2$
$R20$	$R20$	$R13$	$R18$	$R15$	$R2$	$R19$	$R4$	$R17$	$R16$	$R3$	$R14$	$R1$

Note that $\circ\left(S5FGR_3(Z_3),\circ\right)=12$

4.6 Lemma

Let $S6FGR_3(Z_3)$ be a subset of $FGR_3(Z_3)$ defined as

$$S6FGR_3(Z_3)=\left\{\begin{matrix}R1,R2,R3,R4,R13,R14,R15,R16,R17,R18,R19,R20\\R49,R50,R51,R52,R61,R62,R63,R64,R65,R66,R67,R68\end{matrix}\right\}$$

and let $\circ$ be a binary operation of non-commutative method of rhotrix multiplication, then $\left(S6FGR_3(Z_3),\circ\right)$ is a subgroup of $\left(FGR_3(Z_3),\circ\right)$. In particular, $\left(S6FGR_3(Z_3),\circ\right)$ is the left triangular rhotrix subgroup of $\left(FGR_3(Z_3),\circ\right)$.

Note that $\circ\left(S6FGR_3(Z_3),\circ\right)=24$

4.7 Lemma

Let $S7FGR_3(Z_3)$ be a subset of $FGR_3(Z_3)$ defined as

$$S7FGR_3(Z_3)=\{R1,R2,R3,R4,R5,R6,R7,R8,R9,R10,R11,R12\}$$

and let $\circ$ be a binary operation of non-commutative method of rhotrix multiplication, then $\left(S7FGR_3(Z_3),\circ\right)$ is a subgroup of $\left(FGR_3(Z_3),\circ\right)$. In particular, $\left(S7FGR_3(Z_3),\circ\right)$ is the right triangular rhotrix subgroup of $\left(FGR_3(Z_3),\circ\right)$ with unit heart.

The multiplication table for $\left(S7FGR_3(Z_3),\circ\right)$ is given by Table 7

Table 7: Multiplication table for $\left(S7FGR_3(Z_3),\circ\right)$

$\circ$	$R1$	$R2$	$R3$	$R4$	$R5$	$R6$	$R7$	$R8$	$R9$	$R10$	$R11$	$R12$
$R1$	$R1$	$R2$	$R3$	$R4$	$R5$	$R6$	$R7$	$R8$	$R9$	$R10$	$R11$	$R12$
$R2$	$R2$	$R1$	$R4$	$R3$	$R8$	$R7$	$R6$	$R5$	$R12$	$R11$	$R10$	$R9$
$R3$	$R3$	$R4$	$R1$	$R2$	$R9$	$R10$	$R11$	$R12$	$R5$	$R6$	$R7$	$R8$
$R4$	$R4$	$R3$	$R2$	$R1$	$R7$	$R11$	$R10$	$R9$	$R8$	$R7$	$R6$	$R5$
$R5$	$R5$	$R6$	$R11$	$R12$	$R12$	$R8$	$R1$	$R2$	$R3$	$R4$	$R9$	$R10$
$R6$	$R6$	$R5$	$R12$	$R11$	$R2$	$R1$	$R8$	$R7$	$R10$	$R9$	$R4$	$R3$
$R7$	$R7$	$R8$	$R9$	$R10$	$R1$	$R2$	$R5$	$R6$	$R11$	$R12$	$R3$	$R4$
$R8$	$R8$	$R7$	$R10$	$R9$	$R6$	$R5$	$R2$	$R1$	$R4$	$R3$	$R12$	$R11$
$R9$	$R9$	$R10$	$R7$	$R8$	$R11$	$R12$	$R3$	$R4$	$R1$	$R2$	$R5$	$R6$
$R10$	$R10$	$R9$	$R8$	$R7$	$R4$	$R3$	$R12$	$R11$	$R6$	$R5$	$R2$	$R1$
$R11$	$R11$	$R12$	$R5$	$R6$	$R3$	$R4$	$R9$	$R10$	$R7$	$R8$	$R1$	$R2$
$R12$	$R12$	$R11$	$R6$	$R5$	$R10$	$R9$	$R4$	$R3$	$R2$	$R1$	$R8$	$R7$

Note that $\circ\left(S7FGR_3(Z_3),\circ\right)=12$

4.8 Lemma

Let $S8FGR_3(Z_3)$ be a subset of $FGR_3(Z_3)$ defined as

$$S8FGR_3(Z_3)=\begin{Bmatrix} R1,R2,R3,R4,R5,R6,R7,R8,R9,R10,R11,R12 \\ R49,R50,R51,R52,R53,R54,R55,R56,R57,R58,R59,R60 \end{Bmatrix}$$

and let $\circ$ be a binary operation of non-commutative method of rhotrix multiplication, then $\left(S8FGR_3(Z_3),\circ\right)$ is a subgroup of $\left(FGR_3(Z_3),\circ\right)$. In particular, $\left(S8FGR_3(Z_3),\circ\right)$ is the right triangular rhotrix subgroup of $\left(FGR_3(Z_3),\circ\right)$ .Note that $\circ\left(S8FGR_3(Z_3),\circ\right)=24$.

4.9 Lemma

Let $S9FGR_3(Z_3)$ be a subset of $FGR_3(Z_3)$ defined as

$$S9FGR_3(Z_3)=\{R1,R4,R13,R16,R18,R19\}$$

and let $\circ$ be a binary operation of non-commutative method of rhotrix multiplication, then $\left(S9FGR_3(Z_3),\circ\right)$ is a subgroup of $\left(FGR_3(Z_3),\circ\right)$. In particular, $\left(S9FGR_3(Z_3),\circ\right)$ is the special left triangular rhotrix subgroup of $\left(FGR_3(Z_3),\circ\right)$ with unit heart.

The multiplication table for $\left(S9FGR_3(Z_3),\circ\right)$ is given by Table 8

Table 8: Multiplication table for $\left(S9FGR_3(Z_3),\circ\right)$

$\circ$	$R1$	$R4$	$R13$	$R16$	$R18$	$R19$
$R1$	$R1$	$R4$	$R13$	$R16$	$R18$	$R19$
$R4$	$R4$	$R1$	$R18$	$R19$	$R13$	$R16$
$R13$	$R13$	$R18$	$R19$	$R4$	$R16$	$R1$
$R16$	$R16$	$R19$	$R4$	$R13$	$R1$	$R18$
$R18$	$R18$	$R13$	$R16$	$R1$	$R19$	$R4$
$R19$	$R19$	$R16$	$R1$	$R18$	$R4$	$R13$

Note that $\circ\left(S9FGR_3(Z_3),\circ\right)=6$

4.10 Lemma

Let $S10FGR_3(Z_3)$ be a subset of $FGR_3(Z_3)$ defined as

$$S10FGR_3(Z_3)=\left\{\begin{matrix}R1,R4,R13,R16,R18,R19,\\ R50,R51,R62,R63,R65,R68\end{matrix}\right\}$$

and let $\circ$ be a binary operation of non-commutative method of rhotrix multiplication, then $\left(S10FGR_3(Z_3),\circ\right)$ is a subgroup of $\left(FGR_3(Z_3),\circ\right)$. In particular, $\left(S10FGR_3(Z_3),\circ\right)$ is the special left triangular rhotrix subgroup of $\left(FGR_3(Z_3),\circ\right)$.

The multiplication table for $\left(S10FGR_3(Z_3),\circ\right)$ is given by Table 9

Table 9: The multiplication table for $\left(S10FGR_3(Z_3),\circ\right)$

$\circ$	$R1$	$R4$	$R13$	$R16$	$R18$	$R19$	$R50$	$R51$	$R62$	$R63$	$R65$	$R68$
$R1$	$R1$	$R4$	$R13$	$R16$	$R18$	$R19$	$R50$	$R51$	$R62$	$R63$	$R65$	$R68$
$R4$	$R4$	$R1$	$R18$	$R19$	$R13$	$R16$	$R51$	$R50$	$R65$	$R68$	$R62$	$R63$
$R13$	$R13$	$R18$	$R19$	$R4$	$R16$	$R1$	$R68$	$R63$	$R50$	$R65$	$R51$	$R62$
$R16$	$R16$	$R19$	$R4$	$R13$	$R1$	$R18$	$R65$	$R62$	$R63$	$R50$	$R68$	$R51$
$R18$	$R18$	$R13$	$R16$	$R1$	$R19$	$R4$	$R63$	$R68$	$R51$	$R62$	$R50$	$R65$
$R19$	$R19$	$R16$	$R1$	$R18$	$R4$	$R13$	$R62$	$R65$	$R68$	$R51$	$R63$	$R50$
$R50$	$R50$	$R51$	$R62$	$R63$	$R65$	$R68$	$R1$	$R4$	$R13$	$R16$	$R18$	$R19$
$R51$	$R51$	$R50$	$R65$	$R68$	$R62$	$R63$	$R4$	$R1$	$R18$	$R19$	$R13$	$R16$
$R62$	$R62$	$R65$	$R68$	$R51$	$R63$	$R50$	$R19$	$R16$	$R1$	$R18$	$R4$	$R13$
$R63$	$R63$	$R68$	$R51$	$R62$	$R50$	$R65$	$R18$	$R13$	$R16$	$R1$	$R19$	$R4$
$R65$	$R65$	$R62$	$R63$	$R50$	$R68$	$R51$	$R16$	$R19$	$R4$	$R13$	$R1$	$R18$
$R68$	$R68$	$R63$	$R50$	$R65$	$R51$	$R62$	$R13$	$R18$	$R19$	$R4$	$R16$	$R1$

Note that $\circ\left(S10FGR_3(Z_3),\circ\right)=12$.

4.11 Lemma

Let $S11FGR_3(Z_3)$ be a subset of $FGR_3(Z_3)$ defined as

$$S11FGR_3(Z_3)=\left\{R1,R4,R5,R7,R10,R12\right\}$$

and let $\circ$ be a binary operation of non-commutative method of rhotrix multiplication, then $\left(S11FGR_3(Z_3),\circ\right)$ is a subgroup of $\left(FGR_3(Z_3),\circ\right)$. In particular, $\left(S11FGR_3(Z_3),\circ\right)$ is the special right triangular rhotrix subgroup of

$(FGR_3(Z_3),\circ)$ with unit heart.

The multiplication table for $(S11FGR_3(Z_3),\circ)$ is given by Table 10

Table 10: Multiplication table for $(S11FGR_3(Z_3),\circ)$

$\circ$	$R1$	$R4$	$R5$	$R7$	$R10$	$R12$
$R1$	$R1$	$R4$	$R5$	$R7$	$R10$	$R12$
$R4$	$R4$	$R1$	$R12$	$R10$	$R7$	$R5$
$R5$	$R5$	$R12$	$R7$	$R1$	$R4$	$R10$
$R7$	$R7$	$R10$	$R1$	$R5$	$R12$	$R4$
$R10$	$R10$	$R7$	$R4$	$R12$	$R5$	$R1$
$R12$	$R12$	$R5$	$R10$	$R4$	$R1$	$R7$

Note that $\circ(S11FGR_3(Z_3),\circ)=6$.

4.12 Lemma

Let $S12FGR_3(Z_3)$ be a subset of $FGR_3(Z_3)$ defined as

$$S12FGR_3(Z_3)=\left\{\begin{matrix} R1, R4, R5, R7, R10, R12, \\ R50, R51, R54, R56, R57, R59 \end{matrix}\right\}$$

and let $\circ$ be a binary operation of non-commutative method of rhotrix multiplication, then $(S12FGR_3(Z_3),\circ)$ is a subgroup of $(FGR_3(Z_3),\circ)$. In particular, $(S12FGR_3(Z_3),\circ)$ is the special right triangular rhotrix subgroup of $(FGR_3(Z_3),\circ)$.

The multiplication table for $(S12FGR_3(Z_3),\circ)$ is given by Table 11

Table 11: Multiplication table for $(S12FGR_3(Z_3),\circ)$

$\circ$	$R1$	$R4$	$R5$	$R7$	$R10$	$R12$	$R50$	$R51$	$R54$	$R56$	$R57$	$R59$
$R1$	$R1$	$R4$	$R5$	$R7$	$R10$	$R12$	$R50$	$R51$	$R54$	$R56$	$R57$	$R59$
$R4$	$R4$	$R1$	$R12$	$R10$	$R7$	$R5$	$R51$	$R50$	$R59$	$R57$	$R56$	$R54$
$R5$	$R5$	$R12$	$R7$	$R1$	$R4$	$R10$	$R54$	$R59$	$R56$	$R50$	$R51$	$R57$
$R7$	$R7$	$R10$	$R1$	$R5$	$R12$	$R4$	$R56$	$R57$	$R50$	$R54$	$R59$	$R51$
$R10$	$R10$	$R7$	$R4$	$R12$	$R5$	$R1$	$R57$	$R56$	$R51$	$R59$	$R54$	$R50$
$R12$	$R12$	$R5$	$R10$	$R4$	$R1$	$R7$	$R59$	$R54$	$R57$	$R51$	$R50$	$R56$
$R50$	$R50$	$R51$	$R56$	$R54$	$R59$	$R57$	$R1$	$R4$	$R7$	$R5$	$R12$	$R10$
$R51$	$R51$	$R50$	$R57$	$R59$	$R54$	$R56$	$R4$	$R1$	$R10$	$R12$	$R5$	$R7$
$R54$	$R54$	$R59$	$R50$	$R56$	$R57$	$R51$	$R5$	$R12$	$R1$	$R7$	$R10$	$R4$
$R56$	$R56$	$R57$	$R54$	$R50$	$R51$	$R59$	$R7$	$R10$	$R5$	$R1$	$R4$	$R12$
$R57$	$R57$	$R56$	$R59$	$R51$	$R50$	$R54$	$R10$	$R7$	$R12$	$R4$	$R1$	$R5$
$R59$	$R59$	$R54$	$R51$	$R57$	$R56$	$R50$	$R12$	$R5$	$R4$	$R10$	$R7$	$R1$

Note that $\circ(S12FGR_3(Z_3),\circ)=12.$

4.13 Lemma

Let $S13FGR_3(Z_3)$ be a subset of $FGR_3(Z_3)$ defined as

$$S13FGR_3(Z_3)=\{R1, R4, R47, R48\}$$

and let $\circ$ be a binary operation of non-commutative method of rhotrix multiplication, then $(S13FGR_3(Z_3),\circ)$ is a subgroup of $(FGR_3(Z_3),\circ)$.

The multiplication table for $(S13FGR_3(Z_3),\circ)$ is given by Table 12

Table 12: Multiplication table for $(S13FGR_3(Z_3),\circ)$

$\circ$	$R1$	$R4$	$R47$	$R48$
$R1$	$R1$	$R4$	$R47$	$R48$
$R4$	$R4$	$R1$	$R48$	$R47$
$R47$	$R47$	$R48$	$R4$	$R1$
$R48$	$R48$	$R47$	$R1$	$R4$

Note that $\circ(S13FGR_3(Z_3),\circ) = 4$.

4.14 Lemma

Let $S14FGR_3(Z_3)$ be a subset of $FGR_3(Z_3)$ defined as $S14FGR_3(Z_3) = \{R1\}$

and let $\circ$ be a binary operation of non-commutative method of rhotrix multiplication, then $(S14FGR_3(Z_3),\circ)$ is a subgroup of $(FGR_3(Z_3),\circ)$.

4.15 Lemma

Let $S15FGR_3(Z_3)$ be a subset of $FGR_3(Z_3)$ defined as

$$S15FGR_3(Z_3) = \left\{\begin{matrix} R1, R2, R3, R4.R5, R6, R7, R8, R9, R10, R11, R12, R13, R14, R15, \\ R16, R17, R18, R19, R20,, R21, R22, R23, R24, R25, R26, R27, R28, \\ R29, R30, R31, R32, R33, R34, R35, R36, R37, R38, R39, R40, R41, \\ R42, R43, R44, R45, R46, R47, R48. \end{matrix}\right\}$$

and let $\circ$ be a binary operation of non-commutative method of rhotrix multiplication, then $(S15FGR_3(Z_3),\circ)$ is a subgroup of $(FGR_3(Z_3),\circ)$.In particular $(S15FGR_3(Z_3),\circ)$ is a rhotrix subgroup of $(FGR_3(Z_3),\circ)$ with unit heart.

Note that: $\circ(S15FGR_3(Z_3),\circ) = 48$

4.16 Lemma

The pair $(FGR_3(Z_3),\circ)$ is an improper subgroup of $(FGR_3(Z_3),\circ)$.

5.0 Conclusion

This paper considers the pair $(FGR_3(Z_P),\circ)$ consisting of the set of all invertible rhotrices of size *3* over a finite field of integer modulo and prime *P* and together with the binary operation of row-column based method for rhotrix multiplication; $'\circ'$, in order to develop concrete constructions of finite non-commutative rhotrix groups. More importantly, specific cases of $(FGR_3(Z_P),\circ)$ for *p=2* and *p=3* were algebraically analyzed in details and their subgroups were identified to be in harmony with the well known Lagrange's Theorem on finite groups.In the future, it may be worthy to consider a number of topics on non-commutative rhotrix groups such as development of finite cyclic groups and composition series for non-commutative groups of rhotrices over finite fields.

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