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# On Subgroups of Non-Commutative General Rhotrix Group

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***Abstract***

***This paper considers the pair $(GR_n(F),\circ)$ consisting of the set of all invertible rhotrices of size n over an arbitrary field F; and together with the binary operation of row-column based method for rhotrix multiplication; $'\circ'$, in order to introduce it as the concept of "non-commutative general rhotrix group". We identify a number of subgroups of $(GR_n(F),\circ)$ and then advance to show that its particular subgroup is embedded in a particular subgroup of the well-known general linear group $(GL_n(F),\cdot)$. Furthermore, we shall investigate isomorphic relationship between some subgroups of $(GR_n(F),\circ)$.***

**Keywords:** Rhotrix, matrix, group, rhotrix groups, matrix groups, general rhotrix group, general linear group.

## 1.0 Introduction

Rhotrix theory deals with the study of algebra and analysis of array of numbers in rhomboid shape. Since the introduction of the theory by Ajibade [1] as an extension of ideas on matrix-tertions and matrix-noitrets suggested by Atanassov and Shannon [2], there have been many demonstration of interest by researchers in the usage of rhotrix set as an underlying set in the study of various forms of algebraic structures (see [3], [4], [5], [6]). The addition and multiplication for heart-based rhotrices of size *3* were defined in [1]. Sani [4] defined a rhotrix *R* of size *n* as a rhomboidal array of numbers which can be expressed as a couple of two square matrices A and C of sizes $(t\times t)$ and $(t-1)\times(t-1)$, where $t=\frac{n+1}{2}$ and $n\in 2Z^{+}+1$. That is,

$$R_n=\left\langle A_{t\times t},C_{(t-1)\times(t-1)}\right\rangle=\left\langle\begin{array}{ccccccc} & & & a_{11} & & & \\ & & a_{21} & c_{11} & a_{12} & & \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \\ a_{t1} & \ldots & \ldots & \ldots & \ldots & \ldots & a_{1t} \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \\ & & a_{t(t-1)} & c_{(t-1)(t-1)} & a_{(t-1)t} & & \\ & & & a_{tt} & & & \end{array}\right\rangle$$

$$=\left\langle\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1(t-1)} & a_{1t} \\ a_{21} & a_{22} & \ldots & a_{2(t-1)} & a_{2t} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{(t-1)1} & a_{(t-1)2} & \ldots & a_{(t-1)(t-1)} & a_{(t-1)t} \\ a_{t1} & a_{t2} & \ldots & a_{t(t-1)} & a_{tt} \end{bmatrix},\begin{bmatrix} c_{11} & \ldots & c_{1(t-1)} \\ \ldots & \ldots & \ldots \\ c_{(t-1)1} & \ldots & c_{(t-1)(t-1)} \end{bmatrix}\right\rangle,$$

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where $[a_{ij}]$ and $[c_{lk}]$ are called the major and minor matrices of $R_n$ respectively. The set of all such collections of rhotrices with entries from an arbitrary field $F$ is given as

$$R_n(F)=\left\{\left\langle \begin{array}{ccccccc} & & & a_{11} & & & \\ & & a_{21} & c_{11} & a_{12} & & \\ & \dots & \dots. & \dots & \dots & \dots & \\ a_{t1} & \dots & \dots & \dots & \dots & \dots & a_{1t} \\ & \dots & \dots & \dots & \dots & \dots & \\ & & a_{t(t-1)} & c_{(t-1)(t-1)} & a_{(t-1)t} & & \\ & & & a_{tt} & & & \end{array}\right\rangle : a_{ij}\in F, c_{lk}\in F\right\},$$

where $1\le i,\ j\le t,\ 1\le l,k\le t-1;\ t=\frac{n+1}{2}\ and\ n\in 2Z^{+}+1.$

A row-column method for multiplication of two rhotrices $R_n,\ Q_n$ having the same size was defined by Sani[4] as follows:

$$R_n\circ Q_n=\left\langle a_{i_1j_1},c_{l_1k_1}\right\rangle\circ\left\langle b_{i_2j_2},d_{l_2k_2}\right\rangle=\left\langle \sum_{i_2j_1}^{t}(a_{i_1j_1}b_{i_2j_2}),\sum_{l_2k_1}^{t-1}(c_{l_1k_1}d_{l_2k_2})\right\rangle.$$

It was noted in [4] that this rhotrix multiplication is non-commutative but associative.The identity rhotrix for any real rhotrix of size $n$ was given as

$$I_n=\left\langle I_{t\times t},I_{(t-1)\times(t-1)}\right\rangle=\left\langle \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & 1 & 0 & & \\ & \dots & \dots. & \dots & \dots & \dots & \\ 0 & \dots & \dots & 1 & \dots & \dots & 0 \\ & \dots & \dots & \dots & \dots & \dots & \\ & & 0 & 1 & 0 & & \\ & & & 1 & & & \end{array}\right\rangle.$$

It was also stated in [4] that since $R_n$ can be represented as $R_n=\left\langle a_{ij},c_{lk}\right\rangle$; if both matrices $[a_{ij}]$ and $[c_{lk}]$ are invertible, then $R_n$ is invertible and $R_n^{-1}=\left\langle q_{ij},r_{lk}\right\rangle$, where $q_{ij}$ and $r_{lk}$ are the inverse entries of $A_{t\times t}$ and $C_{(t-1)\times(t-1)}$ respectively.

The determinant of a rhotrix $R$ of size $n$ was also defined as $\det(R_n)=\det\left\langle a_{ij},c_{lk}\right\rangle$ = $\det(A_{t\times t})\cdot\det(C_{(t-1)\times(t-1)})$; and that $R_n$ is invertible if and only if $\det(R_n)\neq 0.$ Furthermore, for any rhotrix $R_n=\left\langle a_{ij},c_{lk}\right\rangle$, the transpose of $R_n$ was defined in [4] as $R_n^T=\left\langle a_{ji},c_{kl}\right\rangle$. It was also shown in [4] that $\det(R_n\circ Q_n)=\det(R_n)\circ\det(Q_n)=\det(R_n)\cdot\det(Q_n)$ and $(R_n\circ Q_n)^T=(Q_n)^T\circ(R_n)^T$.

It was noted in [4] that the set of all invertible rhotrices of size $n$ with entries in set of real numbers together with the binary operation of row and column method of rhotrix multiplication is a group. This idea of a rhotrix group given in [4] provides us with the motivation to consider its generalization for our study under the class of non-commutative general rhotrix group of size $n$ over an arbitrary field $F$. The name results from the non-commutative but associative property of the row-column multiplication method.

In this paper, we shall adopt the row-column method for rhotrix multiplication in order to consider an algebraic study of non-commutative groups of rhotrices and their generalization. This will be achieved through our consideration of the pair $(GR_n(F),\circ)$, consisting of a set of all invertible rhotrices of size $n$ having entries from an arbitrary field $F$ and together with the binary operation of row-column method for rhotrix multiplication that forms a group of all non-singular rhotrices of size $n$, which we term as '*the non-commutative general rhotrix group*. Weidentify certain subgroups of $(GR_n(F),\circ)$ and then proceed to show that its particular subgroup is embedded in a particular subgroup of the well-known general linear group. In the process, a number of theorems will be developed.

## 2.0 Definitions

The following definition will serve in our discussion in subsequent sections:

## 2.1 Invertible rhotrix

A rhotrix $R_n$ is said to be invertible or non- singular if the determinant is non-zero. That is $R_n$ is invertible $iff\ \det(R_n) \neq 0$.

## 2.2 Set of all invertible rhotrices of size $n$

This is a collection of all rhotrices of size $n$ with entries from a field $F$ and satisfying the property that the determinant of all the rhotrices is non-zero. We denote such collection as $GR_n(F)$. Thus,

$$GR_n(F) = \left\{ \left\langle \begin{array}{ccccccc} & & & a_{11} & & & \\ & & a_{21} & c_{11} & a_{12} & & \\ & \cdots & \cdots\cdot & \cdots & \cdots & \cdots & \\ a_{t1} & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1t} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & a_{t(t-1)} & c_{(t-1)(t-1)} & a_{(t-1)t} & & \\ & & & a_{tt} & & & \end{array} \right\rangle : a_{ij}, c_{lk} \in F \ and\ \det([a_{ij}]) \neq 0 \neq \det([c_{lk}]) \right\},$$

where $1 \le i, j \le t,\ 1 \le l, k \le t-1;\ t = \dfrac{n+1}{2}\ and\ n \in 2Z^+ + 1.$

## 3.0 The Non-Commutative General rhotrix Group

In [4], it was noted that the set of all invertible rhotrices of size$n$ with entries from the set of real numbers is a group with respect to row-column method for rhotrix multiplication. We generalize this notion in the following theorem.

## 3.1 Theorem (A Generalization of Non-Commutativerhotrix Groups)

Let $GR_n(F)$ be the set of all invertible rhotrices with entries from an arbitrary field $F$ and let $\circ$ be the row-column method for rhotrix multiplication. Then, the pair $(GR_n(F), \circ)$ is a non-commutative general rhotrix group of size *n* over *F*.

**Proof**

We shall show that the pair $(GR_n(F), \circ)$ is a group under the binary operation of row-column multiplication of rhotrices. i.e. we shall show that the following group axioms are satisfied:

(i) Closure:

For any two rhotrices of $A_n, B_n \in GR_n(F)$, $\det(A_n) \neq 0 \Rightarrow A_n$ is invertible, and $\det(B_n) \neq 0 \Rightarrow B_n$ is invertible.

Now, $A_n \circ B_n \in GR_n(F)$ since $\det(A_n \circ B_n) = \det(A_n).\det(B_n) \neq)$

Thus, $GR_n(F)$ is closed under the group binary operation.

(ii) Associativity:

For all $A_n, B_n\ and\ C_n \in GR_n(F)$

$(A_n \circ B_n) \circ C_n = A_n \circ (B_n \circ C_n)$

(iii) Existence of identity:

For each $R_n \in GR_n(F), \exists$

$$I_n = \left\langle I_{t\times t}, I_{(t-1)\times(t-1)} \right\rangle = \left\langle \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & 1 & 0 & & \\ & \cdots & \cdots\cdot & \cdots & \cdots & \cdots & \\ 0 & \cdots & \cdots & 1 & \cdots & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & 0 & 1 & 0 & & \\ & & & 1 & & & \end{array} \right\rangle \in GR_n(F)$$

such that $I_n \circ R_n = R_n \circ I_n = R_n$

(iv) Existence of inverse:
for each $A_n \in GR_n(F), \exists\ A_n^{-1} \in GR_n(F)$ such that $A_n \circ A_n^{-1} = I_n \in GR_n(F)$.

Hence, the pair $(GR_n(F), \circ)$ is a non-commutative general rhotrix group of size *n* over *F*.

## 3.2 Corollary

Let $GR_n(\Re)$ be the set of all invertible rhotrices of size *n* with entries in $\Re$. Let $\circ$ be the row-column multiplication of rhotrices; then the pair $(GR_n(\Re), \circ)$ is the general non-commutative group of all invertible real rhotrices of size *n*.

**Proof**

By substituting $F = \Re$ in theorem 3.1 above, the result follows.

## 3.3 Corollary

The pair $(GR_3(\Re), \circ)$ is a general non-commutative group of all invertible real rhotrices of size 3.

**Proof**

Putting $F = \Re$ and $n = 3$ in theorem 3.1, it follows that the pair $(GR_3(\Re), \circ)$ is the general non-commutative group of all invertible real rhotrices of size 3.

This completes the proof.

## 3.4 Theorem

The non-commutative general rhotrixgroup $(GR_n(F), \circ)$ is embedded in the general linear group $\left(GL_n(F), \cdot\right)$ .

**Proof**

Let $(GR_n(F), \circ)$ be a group of all invertible n-dimensional rhotrices and let $\left(GL_n(F), \cdot\right)$ be the group of all invertible matrices of dimension *n x n*. We define the mapping

$\overline{\phantom{x}} : \left((GR_n(F), \circ)\right) \to \left(GL_n(F), \cdot\right)$ by

$$\overline{\left(\left\langle \begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & a_{21} & c_{11} & a_{12} & & & \\ & & a_{31} & c_{21} & a_{22} & c_{12} & a_{13} & & \\ & \dots & \dots & \dots & \dots & \dots & \dots & & \\ a_{t1} & \dots & \dots & \dots & \dots & \dots & \dots & \dots & a_{1t} \\ & \dots & \dots & \dots & \dots & \dots & \dots & & \\ & & a_{t(t-2)} & c_{(t-1)(t-2)} & a_{(t-1)(t-1)} & c_{(t-2)(t-1)} & a_{(t-2)t} & & \\ & & & a_{t(t-1)} & c_{(t-1)(t-1)} & a_{(t-1)t} & & & \\ & & & & a_{tt} & & & & \end{array} \right\rangle\right)} = \begin{bmatrix} a_{11} & 0 & a_{12} & 0 & \dots & \dots & \dots & 0 & a_{1t} \\ 0 & c_{11} & 0 & c_{12} & \dots & \dots & \dots & c_{1(t-1)} & 0 \\ a_{21} & 0 & a_{22} & 0 & \dots & \dots & \dots & 0 & a_{11} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & .... & \dots & \dots & \dots \\ a_{(t-1)1} & 0 & a_{(t-1)2} & & \dots & \dots & \dots & 0 & a_{(t-1)t} \\ 0 & c_{(t-1)1} & 0 & c_{(t-1)2} & \dots & \dots & \dots & c_{(t-1)(t-1)} & 0 \\ a_{t1} & 0 & a_{t2} & 0 & \dots & \dots & \dots & 0 & a_{tt} \end{bmatrix}$$

That is $\overline{\phantom{x}}$ maps each $R_n$ in $GR_n(F)$ to its corresponding filled coupled matrix $M_n$ in $\left(GL_n(F), \cdot\right)$

$\circ$ and $\cdot$ denote respectively, the row-column based method for multiplication of rhotrices and the usually matrix multiplication.

**Proof**

Clearly, $\overline{\phantom{x}}$ is a $1-1$ homomorphism since no two different rhotrices may have the same filled coupled matrix, hence $(GR_n(F), \circ)$ is embedded in $\left(GL_n(F), \cdot\right)$ . This completes the proof.

## 4.0 Subgroups of Non-Commutative General rhotrix Group

**Definition (Unitary rhotrix)**

A rhotrix $R_n$ is called a unitary rhotrix if the determinant of $R_n$ is equal to 1.

We denote the set of all such unitary rhotrices of size *n* with entries from field $F$ as $SR_n(F)$. Thus,

$$SR_n(F)=\left\{\left\langle\begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & a_{21} & c_{11} & a_{12} & & & \\ & & a_{31} & c_{21} & a_{22} & c_{12} & a_{13} & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ a_{t1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1t} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & a_{t(t-2)} & c_{(t-1)(t-2)} & a_{(t-1)(t-1)} & c_{(t-2)(t-1)} & a_{(t-2)t} & & \\ & & & a_{t(t-1)} & c_{(t-1)(t-1)} & a_{(t-1)t} & & & \\ & & & & a_{tt} & & & & \end{array}\right\rangle : a_{ij}, c_{lk}\in F, \det([a_{ij}])=1=\det([c_{lk}]).\right\}$$

**4.1 Theorem**

The pair $(SR_n(F),\circ)$ is a special rhotrix subgroup of $(GR_n(F),\circ)$ .

**Proof**

Since $I_n \in SR_n(F),$ then $SR_n(F)\neq\varnothing$ .

Now, Let $A_n$ and $B_n \in SR_n(F)$,

Then it follows that, det( $A_n$ ) $=1\neq 0$ and det( $B_n$ ) $=1\neq 0$ respectively. This implies that for each $A_n$ and $B_n \in SR_n(F),$ $\exists\ A_n^{-1}$ and $B_n^{-1}\in SR_n(F) \ni A_n\circ B_n^{-1}\in SR_n(F)$ and $\det(A_n\circ B_n^{-1})=\det(A_n)\circ\det(B_n^{-1})=1\circ 1^{-1}=1$

Hence $(SR_n(F),\circ)$ is a subgroup of $(GR_n(F),\circ)$ .

**4.2 Theorem**

The special rhotrix subgroup $(SR_n(F),\circ)$ of $(GR_n(F),\circ)$ is a embedded in the special linear subgroup $(SL_n(F),\cdot)$ of $(GL_n(F),\cdot)$.

**Proof**

Let $(SR_n(F),\circ)$ be a special rhotrix group of $(GR_n(F),\circ)$ and let $(SL_n(F),\cdot)$ be a special linear subgroup of $(GL_n(F),\cdot)$ We define a mapping $^{-}:(SR_n(F),\circ)\to(SL_n(F),\cdot)$ by

$$\overline{\left(\left\langle\begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & a_{21} & c_{11} & a_{12} & & & \\ & & a_{31} & c_{21} & a_{22} & c_{12} & a_{13} & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ a_{t1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1t} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & a_{t(t-2)} & c_{(t-1)(t-2)} & a_{(t-1)(t-1)} & c_{(t-2)(t-1)} & a_{(t-2)t} & & \\ & & & a_{t(t-1)} & c_{(t-1)(t-1)} & a_{(t-1)t} & & & \\ & & & & a_{tt} & & & & \end{array}\right\rangle\right)}=\begin{bmatrix} a_{11} & 0 & a_{12} & 0 & \cdots & \cdots & \cdots & 0 & a_{1t} \\ 0 & c_{11} & 0 & c_{12} & \cdots & \cdots & \cdots & c_{1(t-1)} & 0 \\ a_{21} & 0 & a_{22} & 0 & \cdots & \cdots & \cdots & 0 & a_{11} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots. & \cdots & \cdots & \cdots \\ a_{(t-1)1} & 0 & a_{(t-1)2} & & \cdots & \cdots & \cdots & 0 & a_{(t-1)t} \\ 0 & c_{(t-1)1} & 0 & c_{(t-1)2} & \cdots & \cdots & \cdots & c_{(t-1)(t-1)} & 0 \\ a_{t1} & 0 & a_{t2} & 0 & \cdots & \cdots & \cdots & 0 & a_{tt} \end{bmatrix}$$

Where $^{-}$ maps each $R_n$ in $SR_n(F)$ to its corresponding filled coupled matrix $M_n$ in $\left(SL_n(F),\cdot\right)$ It is clear to see that $^{-}$ is an injective homomorphism, furthermore,

$$\det\left(\left\langle\begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & a_{21} & c_{11} & a_{12} & & & \\ & & a_{31} & c_{21} & a_{22} & c_{12} & a_{13} & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ a_{t1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1t} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & a_{t(t-2)} & c_{(t-1)(t-2)} & a_{(t-1)(t-1)} & c_{(t-2)(t-1)} & a_{(t-2)t} & & \\ & & & a_{t(t-1)} & c_{(t-1)(t-1)} & a_{(t-1)t} & & & \\ & & & & a_{tt} & & & & \end{array}\right\rangle\right)=\det\left(\begin{bmatrix} a_{11} & 0 & a_{12} & 0 & \cdots & \cdots & \cdots & 0 & a_{1t} \\ 0 & c_{11} & 0 & c_{12} & \cdots & \cdots & \cdots & c_{1(t-1)} & 0 \\ a_{21} & 0 & a_{22} & 0 & \cdots & \cdots & \cdots & 0 & a_{11} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots. & \cdots & \cdots & \cdots \\ a_{(t-1)1} & 0 & a_{(t-1)2} & & \cdots & \cdots & \cdots & 0 & a_{(t-1)t} \\ 0 & c_{(t-1)1} & 0 & c_{(t-1)2} & \cdots & \cdots & \cdots & c_{(t-1)(t-1)} & 0 \\ a_{t1} & 0 & a_{t2} & 0 & \cdots & \cdots & \cdots & 0 & a_{tt} \end{bmatrix}\right)=1$$

Hence the result.

**Definition (Diagonal rhotrix)**

A rhotrix $R_n$ is called a diagonal rhotrix if all the elements in the vertical diagonal are non-zero, while others are zeros. We denote the set of all invertible diagonal rhotrices of size $n$ as $DR_n(F)$. Thus,

$$DR_n(F)=\left\{\left\langle\begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & 0 & c_{11} & 0 & & & \\ & & 0 & 0 & a_{22} & 0 & 0 & & \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & & \\ 0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & & \\ & & 0 & 0 & a_{(t-1)(t-1)} & 0 & 0 & & \\ & & & 0 & c_{(t-1)(t-1)} & 0 & & & \\ & & & & a_{tt} & & & & \end{array}\right\rangle : a_{ij}, c_{lk}\in F, \det(a_{ij})\neq 0\neq \det(c_{lk})\right\}$$

### 4.3 Theorem

The pair $(DR_n(F),\circ)$ is a rhotrix subgroup of $(GR_n(F),\circ)$ .

**Proof**

$DR_n(F)\neq\varnothing$ since
$$I_n=\left\langle\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & 1 & 0 & & \\ & \ldots & \ldots. & \ldots & \ldots & \ldots & \\ 0 & \ldots & \ldots & 1 & \ldots & \ldots & 0 \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \\ & & 0 & 1 & 0 & & \\ & & & 1 & & & \end{array}\right\rangle \in DR_n(F).$$

Next, let
$$A_n=(p_{ij},q_{lk})=\left\langle\begin{array}{ccccccccc} & & & & p_{11} & & & & \\ & & & 0 & q_{11} & 0 & & & \\ & & 0 & 0 & p_{22} & 0 & 0 & & \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & & \\ 0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & & \\ & & 0 & 0 & p_{(t-1)(t-1)} & 0 & 0 & & \\ & & & 0 & q_{(t-1)(t-1)} & 0 & & & \\ & & & & p_{tt} & & & & \end{array}\right\rangle \in DR_n(F)$$
and
$$B_n=(r_{ij},s_{lk})=\left\langle\begin{array}{ccccccccc} & & & & r_{11} & & & & \\ & & & 0 & s_{11} & 0 & & & \\ & & 0 & 0 & r_{22} & 0 & 0 & & \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & & \\ 0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & & \\ & & 0 & 0 & r_{(t-1)(t-1)} & 0 & 0 & & \\ & & & 0 & s_{(t-1)(t-1)} & 0 & & & \\ & & & & r_{tt} & & & & \end{array}\right\rangle \in DR_n(F)$$

it follows that $\det(A_n)\neq 0$ and $\det(B_n)\neq 0$ respectively. Implying that $A_n^{-1}$ and $B_n^{-1}$ exist in $DR_n(F)$ .
So,

$$A_n \circ B_n^{-1} = \left\langle \begin{array}{ccccccccc} & & & & p_{11} & & & & \\ & & & 0 & q_{11} & 0 & & & \\ & & 0 & 0 & p_{22} & 0 & 0 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & 0 & 0 & p_{(t-1)(t-1)} & 0 & 0 & & \\ & & & 0 & q_{(t-1)(t-1)} & 0 & & & \\ & & & & p_{tt} & & & & \end{array} \right\rangle \circ \left\langle \begin{array}{ccccccccc} & & & & \frac{1}{r_{11}} & & & & \\ & & & 0 & \frac{1}{s_{11}} & 0 & & & \\ & & 0 & 0 & \frac{1}{r_{22}} & 0 & 0 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & 0 & 0 & \frac{1}{r_{(t-1)(t-1)}} & 0 & 0 & & \\ & & & 0 & \frac{1}{s_{(t-1)(t-1)}} & 0 & & & \\ & & & & \frac{1}{r_{tt}} & & & & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccccccccc} & & & & \frac{p_{11}}{r_{11}} & & & & \\ & & & 0 & \frac{q_{11}}{s_{11}} & 0 & & & \\ & & 0 & 0 & \frac{p_{22}}{r_{221}} & 0 & 0 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & 0 & 0 & \frac{p_{(t-1)(t-1)}}{r_{(t-1)(t-1)}} & 0 & 0 & & \\ & & & 0 & \frac{q_{(t-1)(t-1)}}{s_{(t-1)(t-1)}} & 0 & & & \\ & & & & \frac{p_{tt}}{r_{tt}} & & & & \end{array} \right\rangle \in DR_n(F)$$

Hence $(DR_n(F), \circ)$ is a rhotrix subgroup of $(GR_n(F), \circ)$

## 4.4 Theorem

The Diagonal rhotrix subgroup $(DR_n(F), \circ)$ of $(GR_n(F), \circ)$ is embedded in the diagonal linear subgroup $(DL_n(F), \cdot)$ of $(GL_n(F), \cdot)$

**Proof**

Let $(DR_n(F), \circ)$ be a diagonal rhotrix subgroup of $(GR_n(F), \circ)$ and let $(DL_n(F), \cdot)$ be a diagonal linear subgroup of $(GL_n(F), \cdot)$,

We define a mapping ¤ $: (DR_n(F), \circ) \rightarrow (DL_n(F), \cdot)$ by

$$¤\left(\left\langle \begin{array}{c} a_{11} \\ 0 \quad c_{11} \quad 0 \\ 0 \quad 0 \quad a_{22} \quad 0 \quad 0 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad 0 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ 0 \quad 0 \quad 0 \quad 0 \quad 0 \\ 0 \quad c_{(t-1)(t-1)} \quad 0 \\ a_{tt} \end{array} \right\rangle\right) = \begin{bmatrix} a_{11} & 0 & 0 & 0 & \ldots & \ldots & \ldots & 0 & 0 \\ 0 & c_{11} & 0 & 0 & \ldots & \ldots & \ldots & 0 & 0 \\ 0 & 0 & a_{22} & 0 & \ldots & \ldots & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & .... & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & \ldots & \ldots & 0 & 0 \\ 0 & 0 & 0 & 0 & \ldots & \ldots & \ldots & c_{(t-1)(t-1)} & 0 \\ 0 & 0 & 0 & 0 & \ldots & \ldots & \ldots & 0 & a_{tt} \end{bmatrix}$$

Where ¤ mapped each rhotrix $R_n$ in $DR_n(F)$, to its filled coupled matrix $M_n$ in $DL_n(F)$, Clearly, it ¤ is an injective homomorphism. Since no two rhotrices have the same filled coupled matrix, hence the diagonal rhotrix subgroup is embedded in the diagonal linear subgroup.

**Definition (Scalar rhotrix)**

A rhotrix $R_n$ is called a scalar rhotrix if all the elements in the vertical diagonal are non-zero scalar, while others are zero(s). Scalar rhotrices are rhotrices of the form $KI$, where $I$ is the identity rhotrix and $K$ is a non-zero constant.

We denote the set of all invertible scalar rhotrices of size $n$ as $KR_n(F)$.

Thus,

$$KR_n(F) = \left\{ \left\langle \begin{array}{c} k_{11} \\ 0 \quad ©_{11} \quad 0 \\ 0 \quad 0 \quad k_{22} \quad 0 \quad 0 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad 0 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ 0 \quad 0 \quad k_{(t-1)(t-1)} \quad 0 \quad 0 \\ 0 \quad ©_{(t-1)(t-1)} \quad 0 \\ k_{tt} \end{array} \right\rangle : a_{ij}, c_{lk} \in F, \det([a_{ij}]) \neq 0 \neq \det([c_{lk}]) \right\}$$

## 4.5 Theorem

The pair $(KR_n(F), \circ)$ is a rhotrix subgroup of $(GR_n(F), \circ)$

**Proof**

$$KR_n(F) \neq \varnothing \text{ since } I_n = \left\langle \begin{array}{c} 1 \\ 0 \quad 1 \quad 0 \\ \ldots \quad .... \quad \ldots \quad \ldots \quad \ldots \\ 0 \quad \ldots \quad \ldots \quad 1 \quad \ldots \quad \ldots \quad 0 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ 0 \quad 1 \quad 0 \\ 1 \end{array} \right\rangle \in KR_n(F).$$

Next, let $A_n = \left\langle p_{ij}, p_{lk} \right\rangle = \left\langle \begin{array}{ccccccccc} & & & & p_{11} & & & & \\ & & & 0 & p_{11} & 0 & & & \\ & & 0 & 0 & p_{11} & 0 & 0 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & 0 & 0 & p_{11} & 0 & 0 & & \\ & & & 0 & p_{11} & 0 & & & \\ & & & & p_{11} & & & & \end{array} \right\rangle \in KR_n(F)$ and

$$B_n = \left\langle r_{ij}, r_{lk} \right\rangle = \left\langle \begin{array}{ccccccccc} & & & & r_{11} & & & & \\ & & & 0 & r_{11} & 0 & & & \\ & & 0 & 0 & r_{11} & 0 & 0 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & 0 & 0 & r_{11} & 0 & 0 & & \\ & & & 0 & r_{11} & 0 & & & \\ & & & & r_{11} & & & & \end{array} \right\rangle \in KR_n(F)$$

It follows that $\det(A_n) \neq 0$ and $\det(B_n) \neq 0$ respectively. Implying that $A_n^{-1}$ and $B_n^{-1}$ exist in $KR_n(F)$.
So,

$$A_n \circ B_n^{-1} = \left\langle \begin{array}{ccccccccc} & & & & p_{11} & & & & \\ & & & 0 & p_{11} & 0 & & & \\ & & 0 & 0 & p_{11} & 0 & 0 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & 0 & 0 & p_{11} & 0 & 0 & & \\ & & & 0 & p_{11} & 0 & & & \\ & & & & p_{11} & & & & \end{array} \right\rangle \circ \left\langle \begin{array}{ccccccccc} & & & & \frac{1}{r_{11}} & & & & \\ & & & 0 & \frac{1}{r_{11}} & 0 & & & \\ & & 0 & 0 & \frac{1}{r_{11}} & 0 & 0 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & 0 & 0 & \frac{1}{r_{11}} & 0 & 0 & & \\ & & & 0 & \frac{1}{r_{11}} & 0 & & & \\ & & & & \frac{1}{r_{11}} & & & & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccccccccc} & & & & \frac{p_{11}}{r_{11}} & & & & \\ & & & 0 & \frac{p_{11}}{r_{11}} & 0 & & & \\ & & 0 & 0 & \frac{p_{11}}{r_{11}} & 0 & 0 & & \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & & \\ 0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & & \\ & & 0 & 0 & \frac{p_{11}}{r_{11}} & 0 & 0 & & \\ & & & 0 & \frac{p_{11}}{r_{11}} & 0 & & & \\ & & & & \frac{p_{11}}{r_{11}} & & & & \end{array} \right\rangle \in KR_n(F)$$

Hence $(KR_n(F),\circ)$ is a rhotrix subgroup of $(GR_n(F),\circ)$

## 4.6 Theorem

The scalar rhotrix subgroup $(KR_n(F),\circ)$ of $(GR_n(F),\circ)$ is embedded in the Scalar linear subgroup $(KL_n(F),\cdot)$ of $(GL_n(F),\cdot)$

**Proof**

Let $(KR_n(F),\circ)$ be a scalar rhotrix subgroup of $(GR_n(F),\circ)$ and let $(KL_n(F),\cdot)$ be a scalar linear subgroup of $(GL_n(F),\cdot)$,

We define a mapping $\text{«} : (KR_n(F),\circ) \to (KL_n(F),\cdot)$ by

$$\text{«}\left( \left\langle \begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & 0 & a_{11} & 0 & & & \\ & & 0 & 0 & a_{11} & 0 & 0 & & \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & & \\ 0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & & \\ & & 0 & 0 & 0 & 0 & 0 & & \\ & & & 0 & a_{11} & 0 & & & \\ & & & & a_{11} & & & & \end{array} \right\rangle \right) = \begin{bmatrix} a_{11} & 0 & 0 & 0 & \ldots & \ldots & \ldots & 0 & 0 \\ 0 & a_{11} & 0 & 0 & \ldots & \ldots & \ldots & 0 & 0 \\ 0 & 0 & a_{11} & 0 & \ldots & \ldots & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots. & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & \ldots & \ldots & 0 & 0 \\ 0 & 0 & 0 & 0 & \ldots & \ldots & \ldots & a_{11} & 0 \\ 0 & 0 & 0 & 0 & \ldots & \ldots & \ldots & 0 & a_{11} \end{bmatrix}$$

Where « maps each rhotrix $R_n$ in $KR_n(F)$, to its filled coupled matrix $M_n$ in $KL_n(F)$, clearly, it follows that:

$$\text{«}(A_n \circ B_n) = \text{«}(A_n)\cdot \text{«}(B_n)\ \forall\ A_n, B_n \in GR_n(F)$$

« is a homomorphism. Also, « is 1 – 1 since no two rhotrices have the same filled coupled matrix.

**Definition ( Left triangular rhotrix)**

A rhotrix $R_n$ is called a left triangular rhotrix if all the elements in the right of the vertical diagonal are all zero.

We denote the set of all invertible left triangular rhotrices of size $n$ as $LTR_n(F)$ .

Thus,

$$LTR_n(F) = \left\{ \left\langle \begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & a_{21} & c_{11} & 0 & & & \\ & & a_{31} & c_{21} & a_{22} & 0 & 0 & & \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & & \\ a_{t1} & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & & \\ & & a_{t(t-2)} & c_{(t-1)(t-2)} & a_{(t-1)(t-1)} & 0 & 0 & & \\ & & & a_{t(t-1)} & c_{(t-1)(t-1)} & 0 & & & \\ & & & & a_{tt} & & & & \end{array} \right\rangle : a_{ij}, c_{lk} \in F, \det(a_{ij}) \neq 0 \neq \det(c_{lk}); \right\}$$

$$where\ \ a_{ij} = 0\ \ if\ \ i < j\ \ \ and\ \ \ c_{lk} = 0\ if\ \ l < k$$

## 4.7 Proposition

If $A_n$ and $B_n$ are left triangular rhotrices, then their product $A_n \circ B_n$ , is a left triangular rhotrix.

**Proof**

Suppose $A_n = \left\langle a_{i_1 j_1}, c_{l_1 k_1} \right\rangle$ and $B_n = \left\langle b_{i_2 j_2}, d_{l_2 k_2} \right\rangle$ are left triangular rhotrices such that $i < j \ and \ l < k$ , we will show that $A_n \circ B_n = O_n$ such that $O_n$ is the zero rhotrix.

i.e. $a_{ij} \circ b_{ij} = 0 \ if \ \ i < j \ \ and \ c_{lk} \circ d_{lk} = 0 \ if \ \ l < k$ ,

From the multiplication of rhotrices,

$$A_n \circ B_n = \left\langle a_{i_1 j_1}, c_{l_1 k_1} \right\rangle \circ \left\langle b_{i_2 j_2}, d_{l_2 k_2} \right\rangle$$

$$= \left\langle \sum_{i_2 j_1}^{t} (a_{i_1 j_1} b_{i_2 j_2}), \sum_{l_2 k_1}^{t-1} (c_{l_1 k_1} d_{l_2 k_2}) \right\rangle$$

$$= \left\langle \sum_{i_2 j_1 = 1}^{i-1} (a_{i_1 j_1} b_{i_2 j_2}), \sum_{l_2 k_1 = 1}^{l-1} (c_{l_1 k_1} d_{l_2 k_2}) \right\rangle + \left\langle \sum_{i_2 j_1 = i}^{t} (a_{i_1 j_1} b_{i_2 j_2}), \sum_{l_2 k_1 = l}^{t-1} (c_{l_1 k_1} d_{l_2 k_2}) \right\rangle$$

Observe that for each term of the first sum $i < i_2 j_1, \quad l < l_2 k_1$ , so $\left\langle b_{i_2 j_2}, d_{l_2 k_2} \right\rangle$ =0 since $B_n$ is a left triangular rhotrix.

For each term of the second sum, $i_2 j_1 < i, \quad l_2 k_1 < l$ , so $\left\langle a_{i_1 j_1}, c_{l_1 k_1} \right\rangle$ =0 since $A_n$ is a left triangular rhotrix.

Therefore each term in the sum is zero so we get $(A_n \circ B_n) = O_n$ hence the proof.

## 4.8 Theorem

The pair $(LTR_n(F), \circ)$ is a rhotrix subgroup of $(GR_n(F), \circ)$ .

**Proof**

Since
$$I_n = \left\langle \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & 1 & 0 & & \\ & \dots & \dots. & \dots & \dots & \dots & \\ 0 & \dots & \dots & 1 & \dots & \dots & 0 \\ & \dots & \dots & \dots & \dots & \dots & \\ & & 0 & 1 & 0 & & \\ & & & 1 & & & \end{array} \right\rangle \in LTR_n(F), \ then \ LTR_n(F) \neq \varnothing.$$

Let

$$A_n = \left\langle a_{ij}, c_{lk} \right\rangle = \left\langle \begin{array}{cccccccccc} & & & & a_{11} & & & & & \\ & & & a_{21} & c_{11} & 0 & & & & \\ & & a_{31} & c_{21} & a_{22} & 0 & 0 & & & \\ & \dots & \dots & \dots & \dots & \dots & \dots & & & \\ a_{t1} & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ & \dots & \dots & \dots & \dots & \dots & \dots & & & \\ & & a_{t(t-2)} & c_{(t-1)(t-2)} & a_{(t-1)(t-1)} & 0 & 0 & & & \\ & & & a_{t(t-1)} & c_{(t-1)(t-1)} & 0 & & & & \\ & & & & a_{tt} & & & & & \end{array} \right\rangle$$

and

$$B_n = \langle b_{ij}, d_{lk} \rangle = \left\langle \begin{array}{ccccccccc} & & & & b_{11} & & & & \\ & & & b_{21} & d_{11} & 0 & & & \\ & & b_{31} & d_{21} & b_{22} & 0 & 0 & & \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & & \\ b_{t1} & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & & \\ & & b_{t(t-2)} & d_{(t-1)(t-2)} & b_{(t-1)(t-1)} & 0 & 0 & & \\ & & & b_{t(t-1)} & d_{(t-1)(t-1)} & 0 & & & \\ & & & & b_{tt} & & & & \end{array} \right\rangle$$

be two rhotrices of size *n* in $LTR_n(F)$, it follows that $(A_n \circ B_n) \in LTR_n(F)$ from proposition 1.

So the set $LTR_n(F)$ is closed under the operation of rhotrix multiplication.

Next, for any $A_n \in LTR_n(F)$ , $A_n^{-1} \in LTR_n(F)$ since $\det(A_n) \neq 0$

Now we have $(A_n \circ B_n^{-1}) \in LTR_n(F) \, \forall \, A_n, B_n \in LTR_n(F)$

Hence $(LTR_n(F), \circ)$ is a subgroup of $(GR_n(F), \circ)$

## 4.9 Theorem

Let $(LTR_n(F), \circ)$ be the left triangular rhotrix subgroup of $(GR_n(F), \circ)$ and let $(LTL_n(F), \cdot)$ be the lower triangular linear subgroup of $(GL_n(F), \cdot)$ then $(LTR_n(F), \circ)$ is embedded in $(LTM_n(F), \cdot)$ .

**Proof**

Let $(LTR_n(F), \circ)$ be a Left triangular rhotrix subgroup and let $(LTL_n(F), \cdot)$ lower triangular linear subgroup,

We define a mapping ¨ $: (LTR_n(F), \circ) \to (LTM_n(F), \cdot)$ by

$$\text{¨}\left( \left\langle \begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & a_{21} & c_{11} & 0 & & & \\ & & a_{31} & c_{21} & a_{22} & 0 & 0 & & \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & & \\ a_{t1} & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & & \\ & & a_{t(t-2)} & c_{(t-1)(t-2)} & a_{(t-1)(t-1)} & 0 & 0 & & \\ & & & a_{t(t-1)} & c_{(t-1)(t-1)} & 0 & & & \\ & & & & a_{tt} & & & & \end{array} \right\rangle \right) = \begin{bmatrix} a_{11} & 0 & 0 & 0 & \ldots & \ldots & 0 & 0 & 0 \\ 0 & c_{11} & 0 & 0 & \ldots & \ldots & 0 & 0 & 0 \\ a_{21} & 0 & a_{22} & 0 & \ldots & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots. & \ldots & \ldots & \ldots \\ a_{(t-1)1} & 0 & a_{(t-1)2} & 0 & \ldots & \ldots & a_{(t-1)(t-1)} & 0 & 0 \\ 0 & c_{(t-1)1} & 0 & c_{(t-1)2} & \ldots & \ldots & 0 & c_{(t-1)(t-1)} & 0 \\ a_{t1} & 0 & a_{t2} & 0 & \ldots & \ldots & a_{t(t-1)} & 0 & a_{tt} \end{bmatrix}$$

Where ¨ maps every left triangular rhotrix to its correspondence filled coupled lower triangular matrix. We observe that ¨ is an injective homomorphism Hence, the left triangular rhotrix subgroup is embedded in the left triangular matrix group.

**Definition Special left triangular rhotrix**

A rhotrix $R_n$ is called a special left triangular rhotrix if all the elements in the right of the vertical diagonal are all zero and $\det(R_n) = 1$ .

We denote the set of all special left triangular rhotrices of size $n$ as $LTR_n^*(F)$ .

$$Thurs,\ LTR_n^*(F) = \left\{ \begin{array}{l} \left\langle \begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & a_{21} & c_{11} & 0 & & & \\ & & a_{31} & c_{21} & a_{22} & 0 & 0 & & \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & & \\ a_{t1} & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & & \\ & & a_{t(t-2)} & c_{(t-1)(t-2)} & a_{(t-1)(t-1)} & 0 & 0 & & \\ & & & a_{t(t-1)} & c_{(t-1)(t-1)} & 0 & & & \\ & & & & a_{tt} & & & & \end{array} \right\rangle : a_{ij}, c_{lk} \in F, \det(a_{ij}) = 1 = \det(c_{lk}); \\ where\ \ a_{ij} = 0\ \ if\ \ i < j\ \ \ and\ \ \ c_{lk} = 0\ if\ \ l < k \end{array} \right\}$$

## 4.10 Theorem

Let $(LTR_n^*(F),\circ)$ be the special right triangular rhotrix subgroup of $(GR_n(F),\circ)$ and let $(SR_n(F),\circ)$ be the special rhotrix subgroup of $(GR_n(F),\circ)$, then the pair $(LTR_n^*(F),\circ)$ is a rhotrix subgroup of $(SR_n(F),\circ)$ .

**Proof**

Since $I_n \in LTR_n^*(F)$ then $RTR_n^*(F) \neq \varnothing$ .

Now, Let $A_n$ and $B_n \in LTR_n^*(F)$,

Then det( $A_n$ ) =1 $\neq 0$ det( $B_n$ ) =1 $\neq 0$ respectively. This implies that for each $A_n, B_n \in LTR_n^*(F)$ $\exists A_n^{-1}$ and $B_n^{-1} \in LTR_n^*(F) \ni A_n \circ B_n^{-1} \in LTR_n^*(F)$ and $\det(A_n \circ B_n^{-1}) = \det(A_n).\det(B_n^{-1}) = 1.1^{-1} = 1$

Hence $LTR_n^*(F)$ is a subgroup of $(SR_n(F),\circ)$

**Definition (Right triangular rhotrix)**

A rhotrix $R_n$ is called a right triangular rhotrix if all the elements in the left of the vertical diagonal are all zero.

We denote the set of all invertible right triangular rhotrices of size $n$ as $RTR_n(F)$

$$RTR_n(F) = \left\{ \left\langle \begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & 0 & c_{11} & a_{12} & & & \\ & & 0 & 0 & a_{22} & c_{12} & a_{13} & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1t} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & 0 & 0 & a_{(t-1)(t-1)} & c_{(t-2)(t-1)} & a_{(t-2)t} & & \\ & & & 0 & c_{(t-1)(t-1)} & a_{(t-1)t} & & & \\ & & & & a_{tt} & & & & \end{array} \right\rangle : a_{ij}, c_{lk} \in F, \det(a_{ij}) \neq 0, \det(c_{lk}) \neq 0 \\ where \quad a_{ij} = 0 \ \ if \ \ i > j \ \ and \ \ c_{lk} = 0 \ if \ \ l > k \right\},$$

## 4.11 Proposition

If $A_n$ and $B_n$ are right triangular rhotrices, then their product $A_n \circ B_n$ , is a right triangular rhotrix.

**Proof**

Suppose $A_n = \langle a_{i_1 j_1}, c_{l_1 k_1} \rangle$ and $B_n = \langle b_{i_2 j_2}, d_{l_2 k_2} \rangle$ are right triangular rhotrices such that $i > j \ and \ l > k$ , we will show that $A_n \circ B_n = O_n$ such that $O_n$ is the zero rhotrix.

i.e. $a_{ij} \circ b_{ij} = 0 \ \ if \ \ i > j \ \ and \ c_{lk} \circ d_{lk} = 0 \ \ if \ \ l > k$ ,

From the multiplication of rhotrices,

$$A_n \circ B_n = \langle a_{i_1 j_1}, c_{l_1 k_1} \rangle \circ \langle b_{i_2 j_2}, d_{l_2 k_2} \rangle$$

$$= \left\langle \sum_{i_2 j_1}^{t} (a_{i_1 j_1} b_{i_2 j_2}), \sum_{l_2 k_1}^{t-1} (c_{l_1 k_1} d_{l_2 k_2}) \right\rangle$$

$$= \left\langle \sum_{i_2 j_1 = 1}^{i-1} (a_{i_1 j_1} b_{i_2 j_2}), \sum_{l_2 k_1 = 1}^{l-1} (c_{l_1 k_1} d_{l_2 k_2}) \right\rangle + \left\langle \sum_{i_2 j_1 = i}^{t} (a_{i_1 j_1} b_{i_2 j_2}), \sum_{l_2 k_1 = l}^{t-1} (c_{l_1 k_1} d_{l_2 k_2}) \right\rangle$$

Observe that for each term of the first sum $i > i_2 j_1, \quad l > l_2 k_1$ , so $\langle b_{i_2 j_2}, d_{l_2 k_2} \rangle$ =0 since $B_n$ is a right triangular rhotrix.

For each term of the second sum, $i_2 j_1 > i, \quad l_2 k_1 > l$ , so $\langle a_{i_1 j_1}, c_{l_1 k_1} \rangle$ =0 since $A_n$ is a left triangular rhotrix.

Therefore each term in the sum is zero so we get $(A_n \circ B_n) = O_n$ hence the proof.

## 4.12 Theorem

The pair $(RTR_n(F),\circ)$ is a rhotrix subgroup of $(GR_n(F),\circ)$ .

**Proof**

Since
$$I_n = \left\langle \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & 1 & 0 & & \\ & \dots & \dots & \dots & \dots & \dots & \\ 0 & \dots & \dots & 1 & \dots & \dots & 0 \\ & \dots & \dots & \dots & \dots & \dots & \\ & & 0 & 1 & 0 & & \\ & & & 1 & & & \end{array} \right\rangle \in RTR_n(F),\ then\ RTR_n(F) \neq \varnothing.$$

Let
$$A_n = \langle a_{ij}, c_{lk} \rangle = \left\langle \begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & 0 & c_{11} & a_{12} & & & \\ & & 0 & 0 & a_{22} & c_{12} & a_{13} & & \\ & \dots & \dots & \dots & \dots & \dots & \dots & & \\ 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & a_{1t} \\ & \dots & \dots & \dots & \dots & \dots & \dots & & \\ & & 0 & 0 & a_{(t-1)(t-1)} & c_{(t-2)(t-1)} & a_{(t-2)t} & & \\ & & & 0 & c_{(t-1)(t-1)} & a_{(t-1)t} & & & \\ & & & & a_{tt} & & & & \end{array} \right\rangle$$

and

$$B_n = \langle b_{ij}, d_{lk} \rangle = \left\langle \begin{array}{ccccccccc} & & & & b_{11} & & & & \\ & & & 0 & d_{11} & b_{12} & & & \\ & & 0 & 0 & b_{22} & d_{12} & b_{13} & & \\ & \dots & \dots & \dots & \dots & \dots & \dots & & \\ 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & b_{1t} \\ & \dots & \dots & \dots & \dots & \dots & \dots & & \\ & & 0 & 0 & b_{(t-1)(t-1)} & d_{(t-2)(t-1)} & b_{(t-2)t} & & \\ & & & 0 & d_{(t-1)(t-1)} & b_{(t-1)t} & & & \\ & & & & b_{tt} & & & & \end{array} \right\rangle$$

be two rhotrices of size n in $RTR_n(F)$, it follows that $(A_n \circ B_n) \in RTR_n(F)$ from proposition 2.

So the set $RTR_n(F)$, is closed under the operation of rhotrix multiplication.

Next, for any $A_n \in RTR_n(F),\ A_n^{-1} \in RTR_n(F)$ since $\det(A_n) \neq 0$

Now we have $(A_n \circ B_n^{-1}) \in RTR_n(F)\ \forall\ A_n, B_n \in RTR_n(F)$ .

Hence $(RTR_n(F),\circ)$ is a subgroup of $(GR_n(F),\circ)$

## 4.13 Theorem

Let $(RTR_n(F),\circ)$ be the right triangular rhotrix subgroup of $(GR_n(F),\circ)$ and let $(UTM_n(F),\cdot)$ be the upper triangular linear subgroup of $(GL_n(F),\cdot)$ , then $(RTR_n(F),\circ)$ is embedded in $(UTM_n(F),\cdot)$

**Proof**

Let $(RTR_n(F),\circ)$ be a Left triangular rhotrix subgroup and let $(UTM_n(F),\cdot)$ upper triangular matrix group,

We define a mapping ¨ $:(RTR_n(F),\circ) \to (UTM_n(F),\cdot)$ by

$$\ddot{}\left(\left\langle\begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & 0 & c_{11} & a_{12} & & & \\ & & 0 & 0 & a_{22} & c_{12} & a_{13} & & \\ & \dots & \dots & \dots & \dots & \dots & \dots & & \\ 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & a_{1t} \\ & \dots & \dots & \dots & \dots & \dots & \dots & & \\ & & 0 & 0 & a_{(t-1)(t-1)} & c_{(t-2)(t-1)} & a_{(t-2)t} & & \\ & & & 0 & c_{(t-1)(t-1)} & a_{(t-1)t} & & & \\ & & & & a_{tt} & & & & \end{array}\right\rangle\right)=\left[\begin{array}{ccccccccc} a_{11} & 0 & a_{12} & 0 & \dots & \dots & a_{1(t-1)} & 0 & a_{1t} \\ 0 & c_{11} & 0 & c_{12} & \dots & \dots & 0 & c_{1(t-1)} & 0 \\ 0 & 0 & a_{22} & 0 & \dots & \dots & a_{2(t-1)} & 0 & a_{2t} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \dots & a_{(t-1)(t-1)} & 0 & a_{(t-1)t} \\ 0 & 0 & 0 & 0 & \dots & \dots & 0 & c_{(t-1)(t-1)} & 0 \\ 0 & 0 & 0 & 0 & \dots & \dots & 0 & 0 & a_{tt} \end{array}\right]$$

Where $\ddot{}$ maps every right triangular rhotrix to its correspondence filled coupled upper triangular matrix. We observe that $\ddot{}$ is an injective homomorphism hence the right triangular rhotrix group is embedded in the upper triangular matrix group.

**Definition Special right triangular Rhotrix**

A rhotrix $R_n$ is called a special right triangular rhotrix if all the elements in the left of the vertical diagonal are all zero and $\det(R_n)=1$.

We denote the set of all special right triangular rhotrices of size $n$ as $RTR_n^*(F)$ . Thus,

$$RTR_n^*(F)=\left\{\begin{array}{l}\left\langle\begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & 0 & c_{11} & a_{12} & & & \\ & & 0 & 0 & a_{22} & c_{12} & a_{13} & & \\ & \dots & \dots & \dots & \dots & \dots & \dots & & \\ 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & a_{1t} \\ & \dots & \dots & \dots & \dots & \dots & \dots & & \\ & & 0 & 0 & a_{(t-1)(t-1)} & c_{(t-2)(t-1)} & a_{(t-2)t} & & \\ & & & 0 & c_{(t-1)(t-1)} & a_{(t-1)t} & & & \\ & & & & a_{tt} & & & & \end{array}\right\rangle : a_{ij}, c_{lk}\in F, \det(a_{ij})=1, \det(c_{lk})=1 \\ where \quad a_{ij}=0 \;\; if \;\; i>j \;\; and \;\; c_{lk}=0 \; if \;\; l>k \end{array}\right\},$$

## 4.14 Theorem

Let $(RTR_n^*(F),\circ)$ be the special right triangular rhotrix subgroup of $(GR_n(F),\circ)$ and let $(SR_n(F),\circ)$ be the special rhotrix subgroup of $(GR_n(F),\circ)$, then the pair $(RTR_n^*(F),\circ)$ is a rhotrix subgroup of $(SR_n(F),\circ)$

**Proof**

Since $I_n\in RTR_n^*(F)$ then $RTR_n^*(F)\neq\varnothing$ .

Now, Let $A_n$ and $B_n\in$ $RTR_n^*(F)$ ,

Then det( $A_n$ ) =1 $\neq 0$ det( $B_n$ ) =1 $\neq 0$ respectively. This implies that for each $A_n$ an $B_n\in RTR_n^*(F)$ $\exists\, A_n^{-1}$ and $B_n^{-1}\in RTR_n^*(F)\ni A_n\circ B_n^{-1}\in RTR_n^*(F)$ and $\det(A_n\circ B_n^{-1})=\det(A_n).\det(B_n^{-1})=1.1^{-1}=1$

Hence $RTR_n^*(F)$ is a subgroup of $(SR_n(F),\circ)$ .

## 5.0 Isomorphisms Between some Subgroups of Non-Commutative General rhotrix Group

## 5.1 Theorem

Let $\ddot{}$ be a mapping from $(LTR_n(F),\circ)$ to $(RTR_n(F),\circ)$ defined by

$$\ddot{}\left(\left\langle\begin{array}{ccccccc} & & & a_{11} & & & \\ & & a_{21} & c_{11} & 0 & & \\ & \dots & \dots & \dots & \dots & \dots & \\ a_{t1} & \dots & \dots & \dots & \dots & \dots & 0 \\ & \dots & \dots & \dots & \dots & \dots & \\ & & a_{t(t-1)} & c_{(t-1)(t-1)} & 0 & & \\ & & & a_{tt} & & & \end{array}\right\rangle\right)=\left\langle\begin{array}{ccccccc} & & & a_{11} & & & \\ & & 0 & c_{11} & a_{12} & & \\ & & \dots & \dots & \dots & & \\ 0 & \dots & \dots & \dots & \dots & \dots & a_{1t} \\ & & \dots & \dots & \dots & & \\ & & 0 & c_{(t-1)(t-1)} & a_{(t-1)t} & & \\ & & & a_{tt} & & & \end{array}\right\rangle$$

Then the mapping ¨ is an isomorphism.

**Proof**

Let $(LTR_n(F),\circ)$ and $(RTR_n(F),\circ)$ be the group of all left triangular rhotrices of size $n$ and the group of all right triangular rhotrices of size $n$ respectively, we define a mapping

$$¨ : (LTR_n(F),\circ) \to (RTR_n(F),\circ)$$

*by*

$$¨\left(R_n\right) = ¨\left(\left\langle a_{ij}, c_{lk}\right\rangle\right) = \left\langle a_{ji}, c_{kl}\right\rangle$$

This is a homomorphism since if $R_n = \left\langle a_{i_1 j_1}, c_{l_1 k_1}\right\rangle$ and $Q_n = \left\langle b_{i_2 j_2}, d_{l_2 k_2}\right\rangle$ then

$$¨(R_n \circ Q_n) = ¨\left(\left\langle a_{i_1 j_1}, c_{l_1 k_1}\right\rangle \circ \left\langle b_{i_2 j_2}, d_{l_2 k_2}\right\rangle\right)$$

$$= ¨\left(\sum_{i_2 j_1 = 1}^{t} a_{i_1 j_1} b_{i_2 j_2}, \sum_{l_2 k_1 = 1}^{t-1} c_{l_1 k_1} d_{l_2 k_2}\right)$$

$$= \left(\sum_{i_2 j_1 = 1}^{t} a_{j_1 i_1} b_{j_2 i_2}, \sum_{l_2 k_1 = 1}^{t-1} c_{k_1 l_1} d_{k_2 l_2}\right)$$

$$= \left\langle a_{j_1 i_1}, c_{k_1 l_1}\right\rangle \circ \left\langle b_{j_2 i_2}, d_{k_2 l_2}\right\rangle$$

$$= ¨\left(\left\langle a_{i_1 j_1}, c_{l_1 k_1}\right\rangle\right) \circ ¨\left(\left\langle b_{i_2 j_2}, d_{l_2 k_2}\right\rangle\right)$$

$$= ¨\left(R_n\right) \circ ¨\left(Q_n\right)$$

Next, ¨ is a bijectionsince $\ker(¨) = \left\{ I_n \in (LTR_n(F),\circ) : ¨\left(I_n\right) = I_n^T \in (RTR_n(F),\circ)\right\}$.

## 6.0 Conclusion

We have presented an algebraic study of non-commutative rhotrix groups and their generalizations as $(GR_n(F),\circ)$. We have identified the subgroups of $(GR_n(F),\circ)$ and showed the embedment of its particular subgroupto a particular subgroup of the well known general linear group. Furthermore, we investigated some isomorphic relationship between subgroups of $(GR_n(F),\circ)$. In the future, it may interesting to consider a number of topics on non-commutative rhotrix groups such as computingfinite groups of rhotrices, development of finite cyclic groups, as well as construction of composition series for non-commutative finite group of rhotrices.

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