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A Line-Tau Collocation Method for Partial Differential Equations

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Abstract

This paper deals with the numerical solution of second order linear partial differential equations with the use of the method of lines coupled with the tau collocation method. The method of lines is used to convert the partial differential equation (PDE) to a sequence of ordinary differential equations (ODEs) which is then solved by the tau collocation method to obtain an approximate continuous solution in the spatial variable x at a fixed t-level. The choice of the tau collocation method over the tau method itself was due to the presence of some transcedental functions since both methods produce approximate results. Numerical evidences show that the method performs favourably well.

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1.0 Introduction

Most physical problems are generally described in scientific and engineering terms by partial differential equations. Partial differential equations provide a mathematical description of physical spacetime and thus are one of the most widely used form of mathematics. Since analytical solutions are not always available, we seek for the solution by numerical, computational or approximation methods.

Some numerical methods by appropriate transformation could be used to reduce the problem of solv- ing PDEs to ODEs. One such method is the Method of lines (MOL) [1, 2, 3].

We consider in this work a variant of the MOL which incorporates the tau method into the solution of the resulting ODEs. The tau method which was originally proposed by Lanczos [4] for the solution of ODEs seeks the solution by solving a slightly perturbed form of the ODE where the perturbation term is a linear combination of some Chebyshev polynomials. Various modifications of the tau method have since been published. Ortiz [5] developed a recursive form of generating the so-called canonical polynomials introduced into the tau method by Lanczos [4] for flexibility. As the tau method was originally developed for linear ODEs with polynomial coefficients, the collocation approach was later combined with it so as to widen its scope of application to non-polynomial coefficient problems.

In this direction, we propose here a procedure which combines the tau method in its recursive formu- lation technique and the collocation approach for handling partial differential equations.

In what follows and in the next section, we shall briefly review and present some antecedents necessary for our dicussion in the sequel. Section 3 focusses on the description and/or development of the method. We shall be concerned with the application of the technique to some examples in section 4. Finally, the paper closes with some concluding remarks in section 5.

2.0 Literature Review

2.1 The Tau Method

The tau method was initially formulated as a tool for the approximation of special functions of math- ematical physics which could be expressed in terms of simple differential equations. It developed into a powerful and accurate tool for the numerical solution of complex differential and functional equations. The main idea in it is to approximate the solution of a given problem by solving exactly an approximate problem.

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Accurate approximate polynomial solution of linear ODEs can be obtained by the tau method intro- duced by Lanczos [4]. The method is related to the principle of economization of a differential function defined by a linear differential equation with polynomial coefficients.
To illustrate the tau method, we consider the m-th order linear differential equation.

$$Ly(x) := \sum_{r=0}^{m} P_r(x) y^{(r)}(x) = f(x) \tag{1}$$

with the smooth solution $y(x)$, $a \le x \le b$, $|a| < \infty$, $|b| < \infty$ satisfying a set of multi-point boundary conditions.

$$L^* y(x) := \sum_{r=0}^{m-1} a_{rk}(x) y^{(r)}(x_{rk}) = \alpha_k, \qquad k = 1(1)m \tag{2}$$

where a_{rk} , x_{rk} , α_k, r=0(1)(m-1), k=1(1)m are given real numbers (x_{rk} are points belonging to the interval $a \le x \le b$ at which the conditions (2) are specified), $f(x)$ and $P_r(x)$, r=1(1)m in (1) are polynomials or sufficiently close polynomials (such polynomials can be derived using the tau method itself). The idea of Lanczos is to approximate the solution of the differential system (1) and (2) by an n-th degree polynomial function.

$$y_n(x) = \sum_{r=0}^{n} a_r x^r, \quad n < \infty \tag{3}$$

which is the exact solution of a perturbed equation by adding a polynomial perturbation term to the right hand side of (1).
The polynomial $yn(x)$ satisfies, then the differential system.

$$Ly_n(x) := \sum_{r=0}^{m} P_r(x) {y_n}^{(r)}(x) = f(x) + H_n(x) \tag{4}$$

$$L^* y_n(x) := \sum_{r=0}^{m-1} a_{rk}(x) {y_n}^{(r)}(x_{rk}) = \alpha_k, \qquad k = 1(1)m \tag{5}$$

where the perturbation term is constructed in such a way that (4) and (5) has a polynomial solution of degree n.
Lanczos [4] took $Hn(x)$ to be a linear combination of powers of x multiplied by the Chebyshev poly- nomials. The choice of the Chebyshev polynomials stems from the desire to distribute the errors defined by;

$$max_{a \le x \le b} |y(x) - y_n(x)| \tag{6}$$

evenly throughout the entire range $[a, b]$.
Comparing the following two types of perturbation.

$$H_n^{(1)}(x) = (\tau_1 + \tau_2 x + \tau_3 x^2 + \cdots + \tau_m x^{m-1}) T_{n-m+1}(x) \tag{7}$$

$$H_n^{(2)}(x) = \tau_1 T_{n-m+1}(x) + \tau_2 T_{n-m+2}(x) + \cdots + \tau_m T_m \tag{8}$$

where τ_i, $i = 1(1)m$, are τ -parameters to be determined and $Tn(x)$ is the n-th degree Chebyshev polynomial of the first kind in $[a, b]$. $H_n^{(1)}(x)$ is more economical from the point of view of storage of Chebyshev polynomial coefficients whereas $H_n^{(2)}(x)$ is in general close to zero (being a finite Chebyshev series representation of zero) than the power series representation (8). This comparison explains the superior accuracy of the approximation $yn(x)$ obtained from the use of $H_n^{(2)}(x)$ to that of $H_n^{(1)}(x)$.
For the purpose of accuracy, the form

$$H_n(x) = \sum_{r=0}^{m+s-1} \tau_{m+s-1} T_{n-m+r+1}(x) \tag{9}$$

is considered in this paper where m is the order of (1) and s, the number of over determination of (1), is defined by

$$S = \begin{cases} max\{N_r - r : 0 \le r \le m\}, & for \ \ N_r \ge r \\ 0, & otherwise \end{cases}$$

where Nr is the degree of $P_r(x)$ and r is the order of the derivative whose coefficient is $P_r(x)$, $r = 1(1)m$
To determine the coefficients a_r, $r = 1(1)n$ in $y_n(x)$ from (3), a system of linear algebraic equation $A\underline{\tau} = B$, obtained by equating corresponding coefficients of like powers of x in (4) and then applying conditions (5), is solved for $\underline{\tau} = (a_0, a_1, a_2, \ldots, a_n, \tau_1, \tau_2, \ldots, \tau_{m+s})^T$. The tau method is of order p, in
the sense that if the the exact solution of (1) and (2) is itself a polynomial of degree less or equal to p, the method will reproduce it [5, 6].

2.2 The Method of Lines

The method of lines (MOL) is a technique that enables us to convert a partial differential equation into a set of ordinary differential equations that, in some sense, are equivalent to the former partial differential equation.
Consider the simple heat equation.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \tag{10}$$

Rather than looking at the solution $u(x, t)$ everywhere in the two dimensional space spanned by the spatial variable x and the temporal variable t, we can discretize the spatial variable , and look at the solution $u_i(t)$ where the index i denotes a particular point X_i in space. To this end, we replace the second order partial derivative of u with respect to x by a finite difference such as

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i+1} - 2u_i + u_{i+1}}{(\delta x)^2} \tag{11}$$

where δx (here equidistantly chosen) is the distance between two neighbouring discretization points in space, i.e, the so called gridth-width of the dicretization.
Inserting (11) into (10), we have

$$\frac{\partial^2 u_i}{\partial x^2} \approx k\frac{u_{i+1} - 2u_i + u_{i+1}}{(\delta x)^2} \tag{12}$$

and we have already converted the former PDE in u into a set of ODEs in ui.
In essence, the basic idea of the MOL is to dicretize all but one dimension of the PDE, i.e, discretizing the spatial derivatives and leaving the time variable continuous. However, in this paper, we shall discretize the temporal variable and leave the spatial variable undiscretized. With only one remaining independent variable, we have a system of ordinary differential equations that approximate the PDE which can be solved by any intregation algorithm for initial value solvers. Thus, one of the salient features of the MOL is the use of existing and generally well-established numerical methods for ordinary differential equations.

2.3 The Collocation Method

Basically, a collocation method is a method which involves the determination of an approximate solution in a suitable set of functions sometimes called the trial or basis functions in which the approximate solution is required to satisfy the equation and the conditions associated with it at certain points of the domain of definition called the collocation points [7].
The collocation method is not new but dates back to the 1930's. According to Kantorovich and Akilov [8], the method was first proposed by Kantorovich. The method of Kantorovich was actually a method of lines collocation procedure for the solution of partial differential equations in two variables with the collocation being applied in one variable for each fixed value of the second. Also, the work of Frazer et al reported the applicability of collocation to the solution of PDEs. Collatz [9] also dicussed collocation for both ordinary and partial differential equations and provided numerical examples.
The standard collocation method requires equal spacing of collocation points within specified range of the problem at hand, i.e.

$$x_k \in [a,b], \quad x_k = kh, \quad k = 1(1)(k+1)$$

$$h = \frac{b-a}{n+1}$$

3.0 Description of the Method

In the description of the method, we shall consider the three major types of PDEs

3.1 Parabolic Equations

The simplest parabolic equation is given by;

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{13}$$

and has the associated initial condition

$$u(x, t_0) = g(x), \quad a \le x \le b \tag{14}$$

and the boundary conditions

$$u(a,t) = f_1(t), \quad u(b,t) = f_2(t), \quad t \ge t_0 \tag{15}$$

This problem is commonly solved by the finite difference scheme. In particular, we might use the Crank- Nicolson's formula which approximate (13) by the approximation.

$$\frac{u(x,t_r+\delta t)-u(x,t_r)}{\delta t}=\frac{1}{2}\left\{\frac{d^2u(x,t_r+\delta t)}{dx^2}+\frac{d^2u(x,t_r)}{dx^2}\right\}$$

$$\Rightarrow k_1[u(x,t_r+\delta t)-u(x,t_r)]=\frac{d^2u(x,t_r+\delta t)}{dx^2}+\frac{d^2u(x,t_r)}{dx^2}$$

where $k_1 = 2/\delta t$

For $r = 0$

$k_1[u(x, t_0 + \delta t) - u(x, t_0)] = u''(x, t_0 + \delta t) + u''(x, t_0)$

$\Rightarrow u''(x, t_0 + \delta t) - k_1 u(x, t_0 + \delta t) = -[u''(x, t_0) + k_1 u(x, t_0)]$

$u(a, t_0 + \delta t) = f_1(t_0 + \delta t),\ u(b, t_0 + \delta t) = f_2(t_0 + \delta t)$

Let $v(x) = u(x, t_0 + \delta t)$, we have

$v''(x) - k_1 v(x) = -[g''(x) + k_1 g(x)]$

$\Rightarrow v''(x) - k_1 v(x) = h_1(x)$

where $h_1(x) = -[g''(x) + k_1 g(x)]$

Thus, we have the two point boundary value problem

$$V''(x) - k_1 v(x) = h_1(x) \tag{16}$$

$$V(a) = \alpha_1,\quad v(b) = \alpha_2 \tag{17}$$

where $\alpha_1 = f_1(t_0 + \delta t)$ and $\alpha_2 = f_2(t_0 + \delta t)$ are constants.

Thus the PDE (13) has been reduced to an equivalent two point boundary value problem. (This is the method of lines). We therefore have a step by step process in it, the initial conditions being used at the very first step.

Then, we use the tau collocation method to obtain an approximate continuous solution of (16) and (17) in *x* at a fixed *t*-step.

3.2 The Hyperbolic Equation

We consider the hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2} \tag{18}$$

with the conditions

$u(x, t_0) = g_1(x)$

$ut(x, t_0) = g_2(x) \quad x \in [a, b]$

$u(a, , t) = f_1(t)$

$u(b, t) = f_2(t) \qquad t \in [t_0, c] \qquad or \qquad t \in [t_0, \infty)$

By using the technique of MOL, (18) reduces to

$$\frac{u(x,t_r+\delta t)-2U(x,t_r)+u(x,t_r+\delta t)}{(\delta t)^2}=\frac{c^2}{2}\left\{\frac{d^2u(x,t_r+\delta t)}{dx^2}+\frac{d^2u(x,t_r)}{dx^2}\right\}$$

$$k_2\,[u(x, t_r + \delta t) - 2U(x,t_r) + u(x, t_r - \delta t)] = u''(x, t_r + \delta t) + u''(x, t_r)$$

where $k_2 = 2/c^2(\delta t)^2$

For $r = 0$, we have

$$k_2\,[u(x, t_0 + \delta t) - 2u(x, t_0) + u(x, t_0 - \delta t)] = u''(x, t_0 + \delta t) + u''(x, t_0)$$

The fictitious value $u(x, t_0 - \delta t)$ is eliminated using the central difference formula

$$\frac{\partial u}{\partial t}|(x,t_0)=\frac{u(x,t_0+\delta t)-u(x,t_0-\delta t)}{2(\delta t)}=g_2(x)$$

$$=u(x,t_0-\delta t)=u(x,t_0+\delta t)-2(\delta t)g_2(x)$$

Thus, we have

$$k_2\,[u(x, t_0 + \delta t) - 2u(x, t_0) + u(x, t_0 + \delta t) - 2(\delta t)g_2(x)] = u''(x, t_0 + \delta t) + u(x, t_0)$$

$$\Rightarrow u''(x, t_0 + \delta t) - 2k_2 u(x, t_0 + \delta t) = -[u''(x, t_0) + 2u(x, t_0) + 2(\delta t)g_2(x)]$$

$$\Rightarrow u''(x, t_0 + \delta t) - 2k_2 u(x, t_0 + \delta t) = h_2(x)$$

where $h_2(x) = -[u''(x, t_0) + 2u(x, t_0) + 2(\delta t)g_2(x)]$

By letting $v(x) = u(x, t_0 + \delta t)$, we obtain

$$v''(x) - 2k_2 v(x) = h_2(x) \tag{19}$$

$$v(a) = \alpha_1,\qquad v(b) = \alpha_2 \tag{20}$$

where $\alpha_1 = f_1(t_0 + \delta t)$ and $\alpha_2 = f_2(t_0 + \delta t)$ are constants.

We, then, solve the two point boundary value problem by the tau collocation method to obtain an approximate continuous solution of (18) at a fixed t-step.

9.0 The Elliptic Equation

We consider the Laplace's equation in two variables x and t

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = 0 \tag{21}$$

with the conditions

$$u(x, t_0) = g_1(x)$$
$$u_t(x, t_0) = g_2(x)$$
$$u(a, t) = f_1(t)$$
$$u(b, t) = f_2(t)$$

Following the same procedure as in section (2.2), we obtain

$$v''(x) + 2k_{3v}(x) = h_2(x) \tag{22}$$
$$v(a) = \alpha_1 \qquad v(b) = \alpha_2 \tag{23}$$

where $k_3 = 2/(\delta t)^2$ and α_1, α_2 and $h_2(x)$ are as defined in section (2.2)

We see that the second of the conditions associated with (18) and (21) (i.e $u_t(x, t_0) = g_2(\mathrm{x})$) is always required to be able to eliminate the fictitious value $u(x, t_0 - \delta t)$ for both hyperbolic and elliptic equations.

Also, solutions of further t-step can be obtained in a similar manner.

4.0 Numerical Examples

In this section, we give numerical examples to illustrate the accuracy of the Line-Tau Collocation method . We give the exact errors (for different values of n) at some selected points calculated as Error $= |u(x, t_0 + \delta t) - u_n(x, t_0 + \delta t)|$.

Example 4.1. *We consider the parabolic PDE*

$$\frac{\partial u}{\partial t} = \frac{1}{10\pi^2}\frac{\partial^2 u}{\partial x^2} \tag{24}$$

subject to the conditions

$$u(x, 0) = \cos(\pi x) \qquad x \in [0, 1]$$
$$u(0, t) = exp(-t/10)$$
$$u_x(1, t) = 0 \qquad t \in [0, \infty)$$

whose theoretical solution is $u(x, t) = exp(-t/10)\cos(\pi x)$

The computational results for this example is given in Table 1.

Table 1: Exact Errors for Example 4.1

x	$n = 5$	$n = 6$	$n = 7$	$n = 8$
0.1	6.28×10^{-3}	2.41×10^{-3}	1.86×10^{-3}	1.84×10^{-4}
0.2	4.20×10^{-3}	7.90×10^{-4}	3.29×10^{-3}	1.96×10^{-3}
0.3	2.01×10^{-3}	2.97×10^{-3}	1.55×10^{-3}	1.42×10^{-3}
0.4	5.60×10^{-3}	1.33×10^{-3}	2.46×10^{-3}	1.68×10^{-3}
0.5	4.81×10^{-3}	1.41×10^{-3}	2.51×10^{-3}	1.04×10^{-3}
0.6	1.05×10^{-3}	2.31×10^{-3}	6.86×10^{-4}	1.59×10^{-3}
0.7	2.77×10^{-3}	7.92×10^{-3}	2.35×10^{-3}	5.88×10^{-4}
0.8	3.81×10^{-3}	1.17×10^{-3}	3.21×10^{-4}	1.13×10^{-3}
0.9	1.18×10^{-3}	1.05×10^{-3}	1.40×10^{-3}	6.63×10^{-4}
1.0	1.61×10^{-3}	4.06×10^{-3}	2.73×10^{-4}	3.38×10^{-5}

Example 4.2. *We consider the hyperbolic equation*

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \tag{25}$$

with the conditions

$u(x, 0) = sin\left(\frac{\pi}{2}x\right)$

$u_t(x, 0) = 0, \qquad x \in [0, 1]$

$u(0, t) = 0$

$u_x(1, t) = 0 \qquad t \in [0, \infty)$

whose theoretical solution is $sin\left(\frac{\pi}{2}x\right) cos\left(\frac{\pi}{2}t\right)$

Table 2: shows the accuracy of our method for Example 4.2

x	$n = 5$	$n = 6$	$n = 7$	$n = 8$
0.1	1.27×10^{-4}	1.39×10^{-6}	7.65×10^{-6}	7.87×10^{-6}
0.2	8.61×10^{-5}	1.65×10^{-5}	1.58×10^{-5}	1.55×10^{-5}
0.3	1.12×10^{-6}	2.83×10^{-5}	2.30×10^{-5}	2.28×10^{-5}
0.4	5.65×10^{-5}	3.28×10^{-5}	2.94×10^{-5}	2.96×10^{-5}
0.5	4.85×10^{-5}	3.36×10^{-5}	3.53×10^{-5}	3.56×10^{-5}
0.6	1.15×10^{-5}	3.60×10^{-5}	4.07×10^{-5}	4.07×10^{-5}
0.7	8.14×10^{-5}	4.22×10^{-5}	4.50×10^{-5}	4.48×10^{-5}
0.8	1.12×10^{-4}	4.95×10^{-5}	4.79×10^{-5}	4.78×10^{-5}
0.9	7.93×10^{-5}	5.22×10^{-5}	4.96×10^{-5}	4.97×10^{-5}
1.0	3.65×10^{-5}	5.04×10^{-5}	5.03×10^{-5}	5.03×10^{-5}

Example 4.3. *Lastly, we consider the Laplace's equation in the rectangle with sides* $x = 0$, $t = 0$, $x = 1$ *and* $t = 1$ *with the associated initial condition*

$$u(x, 0) = \sin(\pi x) \qquad u_t(x, 0) = 0, \qquad 0 \le x \le 1$$

and the boundary conditions

$$u(0, t) = 0 \qquad u(1, t) = 0, \qquad 0 \le t \le 1$$

whose analytical solution is sin(πx) cosh(πt)

Table 3: Gives the computational results for Example 4.3.

x	$n = 5$	$n = 6$	$n = 7$	$n = 8$
0.1	2.25×10^{-2}	2.35×10^{-4}	2.68×10^{-4}	2.60×10^{-4}
0.2	2.75×10^{-2}	5.18×10^{-4}	3.61×10^{-4}	4.94×10^{-4}
0.3	1.24×10^{-2}	7.13×10^{-4}	6.56×10^{-4}	6.81×10^{-4}
0.4	1.22×10^{-2}	7.89×10^{-4}	8.91×10^{-4}	8.00×10^{-4}
0.5	3.11×10^{-2}	8.02×10^{-4}	9.04×10^{-4}	8.40×10^{-4}
0.6	3.11×10^{-2}	7.89×10^{-4}	7.62×10^{-4}	8.00×10^{-4}
0.7	8.26×10^{-3}	7.13×10^{-4}	6.08×10^{-4}	6.81×10^{-4}
0.8	2.75×10^{-2}	5.18×10^{-4}	4.95×10^{-4}	4.94×10^{-4}
0.9	4.73×10^{-2}	2.35×10^{-4}	3.27×10^{-4}	2.60×10^{-4}
1.0	8.88×10^{-16}	0	1.11×10^{-16}	7.22×10^{-16}

5.0 Conclusion

A method which combines the idea of the MOL, the tau method and the collocation method for the solution of PDEs has been presented.

The method has been illustrated with some examples and the numerical results show that it is effective and good for handling PDEs. One advantage of the technique developed here is that the resulting solution is in continuous form, thus allowing for several output of numerical values at no extra cost when compared to many other numerical schemes which are of the discrete form.

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