Journal of the Nigerian Association of Mathematical Physics
Volume 30, (May, 2015), pp27 – 40
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A New Iteration Multivariate Pade´ Approximation Technique for Nonlinear Partial Differential Equations of Fractional Order

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Abstract

In this paper, the Laplace transform, the New iteration method and the Multivariate Pade´ approximation technique are employed to solve nonlinear fractional partial differential equations whose fractional derivatives are described in the sense of Caputo. The Laplace transform is used to "fully" determine the initial iteration value. The New iteration method gives a sequence of series solution which approximates the exact solution of the nonlinear equations. The Multivariate Pade´ approximation is used to accelerate the rate of convergence of solutions obtained by the New iteration method. Numerical illustrations were given to show the robustness, simplicity and efficacy of the approach. Also results obtained by the Multivariate Pade´ approximation were compared with the results obtained by the Adomian decomposition method.

AMS Subject Classification : 45L05, 65B99, 65L03
Keywords: New iteration method, Laplace transform, Multivariate Pad*e*´ approximations, nonlinear fractional partial differential equations, Exact solutions

1.0 Introduction

The area of study, Fractional Calculus, has received great attention from researchers in recent years. This is due to the fact that most modelled processes in mathematical biology, viscoelasticity, electrochemistry, physics, fluid mechanics engineering, e.t.c. results in fractional differential and integral equations. Also, these fractional equations serves as a generalization of their corresponding ordinary differential equations and partial differential equations. Many numerical and analytical methods have been proposed for these important class of problems. The most transparent and efficient of these methods are the New Iteration Method (NIM) [1-7], the Variational Iteration Method (VIM) [8-13], the Adomian Decomposition Method (ADM) [14-16] and the Homotopy Perturbation Method (HPM) [17,19]. Among these solution techniques, the NIM produces exact solutions as well as numerical approximate solution to differential equations without linearization, discretization, perturbation or the identification of a Lagrange multiplier.

The New iteration method was initially proposed by Daftardar-Gejji and Jafari [1] in 2006 and improved upon by Hemeda [2-4] and Bhalekar and Daftardar-Gejji [5-7]. These authors have successfully applied the NIM to a variety of linear and nonlinear functional equations such as al- gebraic equations, integral equations, integrodifferential equations, fractional differential equation, nth-order derivative fuzzy integrodifferential equation, systems of equations, e.t.c.

The main aim of this paper is to obtain approximate numerical solution of fractional partial differential equations via the incorporation of the Laplace transform, the NIM and the Multivariate Pad*e*´ approximation technique. Many definitions and theorems have been developed for Multivari- ate Pad*e*´ approximation (MPA) [20]. The MPA has also been used to obtain approximate solutions to linear and nonlinear differential equations [20-22].

2.0 Preliminaries

In this section, we give some definitions and properties of the fractional calculus theory which are used in the sequel. For the concept of fractional derivative, we will adopt the Caputo's defini- tion which is a modification of the Riemann-Liouville definition and has the advantage of allowing initial and boundary conditions to be included in the formulation of the problem.

Definition 1: A real function $h(t)$, $t > 0$ is said to be in the space $C\mu$, $\mu \in \mathbb{R}$, if there ex- ists a real number $p > \mu$ such that $h(t) = t^p h_1(t)$, where $h_1(t) \in C[0, \infty)$ and it is said to be in the space C_μ^m if and only if $h^{(m)} \in C\mu$, $m \in \mathbb{N}$.

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Definition 2: The fractional derivative D^{α} of $h(t)$ in the Caputo's sense is defined as;

$$D^{\alpha}h(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1}h^{(m)}(\tau)d\tau$$

for $m-1 < \alpha \leq m,\ m \in N, t > 0, h \in C_{-1}^{m}$ and $\Gamma(a)$ is the well known Gamma function.

Definition 3: The Reimann-Liouville fractional integral operator J^{α} of order $\alpha > 0$, of a function $h(t) \in C_{\mu}, \mu \geq -1$, is defined as;

$$J^{\alpha}h(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}h(\tau)d\tau$$

$$J^{o}h(t) = h(t)$$

Some of the properties of the operator J^{α} are as follows:

1. $J^{\alpha}J^{\beta}h(t) = J^{\alpha+\beta}h(t)$
2. $J^{\alpha}J^{\beta}h(t) = J^{\beta}J^{\alpha}h(t)$
3. $J^{\alpha}t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}t^{\alpha+\gamma}$

Lemma 2.1. *If* $m-1 < \alpha \leq m, m \in N$ *and* $h \in C_{\mu}^{m}, \mu \geq -1,$ *then*

$$D^{\alpha}J^{\alpha}h(t) = h(t)$$

$$J^{\alpha}D^{\alpha}h(t) = h(t) - \sum_{k=0}^{m-1} h^{(k)}(0^{+})\frac{t^{k}}{k!}$$

3.0 Analysis of the Method

3.1 Basics of the New Iteration Method (NIM)

In order to illustrate the basic idea of the NIM, we consider the following functional equation [1-4].

$$u(x,t) = f(x,t) + N(u(x,t)) \qquad (1)$$

Where N is a nonlinear operator from a Banach space $B \rightarrow B$ and f is a known function (element) of the Banach space B. We seek a solution $u(x,t)$ $of (1)$ of the form.

$$u(x,t) = \sum_{t=0}^{\infty} u_i(x,t) \qquad (2)$$

The nonlinear operator N is decomposed as;

$$N\left(\sum_{t=0}^{\infty} u_i(x,t)\right) = N(u_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i} u_j\right)\right\} \qquad (3)$$

From (2) and (3), (1) is equivalent to

$$\sum_{t=0}^{\infty} u_i = f + N(u_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i} u_j\right)\right\} \qquad (4)$$

The required solution for (1) can be obtained recursively fiom the recurrence formula

$$u_0 = f$$
$$u_1 = N(u_0)$$
$$u_{n+1} = N(u_0 + u_1 + \cdots + u_n) - N(u_0 + u_1 + \cdots + u_{n-1}) \qquad (5)$$
$$n = 1, 2, \ldots$$

Then

$$\sum_{i=1}^{n+1} u_i = N(u_0 + u_1 + \cdots + u_n) \quad n = 1, 2, \ldots$$

$$u = f + \sum_{i=1}^{\infty} u_i \qquad (6)$$

The n-term approximate solution of (1) and (2) is given by;

$$u(x,t) = \sum_{i=0}^{n-1} u_i \qquad (7)$$

Remark 1: If N is a contraction, that is, $\|N(x,t_1)-N(x,t_2)\| \le k\|t_1-t_2\|, 0<k<1, then$
$\|u_{n+1}\| \le k^{n+1}\|u_0\| \quad n=0,1,2,\dots$
Proof: From (5), we have $u_0=f, \|u_1\|=\|N(u_0)\| \le k\|u_0\|$
$\|u_2\|=\|N(u_0+u_1)-N(u_0)\| \le k\|u_1\| \le k^2\|u_0\|$
$\|u_3\|=\|N(u_0+u_1+u_2)-N(u_0+u_1)\| \le k\|u_2\| \le k^3\|u_0\|$
$\vdots$
$\|u_{n+1}\|=\|N(u_0+u_1+\cdots+u_n)-N(u_0+u_1+\cdots+u_{n-1})\| \le k\|u_n\| \le k^{n+1}\|u_0\|,$
$n=0,1,2,\dots$
And the series $\sum_{i=0}^{\infty} u_i$ absolutely and uniformly converges to a solution of (1) which is unique in view of the Banach fixed point theorem.
Remark 2: When the general functional equation (1) is linear, the recurrence formula (5) can be simplified in the form
$u_0=f$
$u_{n+1}=N(u_n), n=0,1,2\dots$
The convergence of the NIM has also been discussed in [1-7].

3.2 Lapace New Iteration Method

To illustrate the pertinent features of this technique, we consider the general nonlinear time fractional partial differential equation

$$D_{*t}^{\alpha}u(x,t)+L_1\big(u(x,t)\big)+N\big(u(x,t)\big)=g(x,t) \quad t>0 \qquad (8)$$

Where L_1 is a linear operator which might include other fractional derivatives of order less than α, N is a nonlinear operator which also might include other fractional derivatives of order less than $\alpha, g(x,t)$ is the source term and $D_{*t}^{\alpha}=\frac{\partial^{\alpha}}{\partial t^{\alpha}}$ is the Caputo fractional derivative of order α.
The initial condition associated with (8) are of the form

$$u(x,0)=f_0(x), \quad \frac{\partial^k u(x,0)}{\partial t^k}=f_k(x), k=1,2,\dots,m-1 \qquad (9)$$

The Laplace transform [23] of the term $D_{*t}^{\alpha}u(x,0)$ with respect to t holds:

$$L[D_{*t}^{\alpha}u(x,t)]=s^{\alpha}U(x,s)-\sum_{k=0}^{m-1}u^{(k)}(x,0)\,s^{\alpha-k-1}, \quad m-1<\alpha\le m \qquad (10)$$

Where $U(x,s)$ is the Laplace transform of $u(x,t)$ with respect to t.
Applying the Laplace transform on both sides of (8) yields:

$$s^{\alpha}U(x,s)-\sum_{k=0}^{m-1}u^{(k)}(x,0)s^{\alpha-k-1}+L\big[L_1\big(u(x,t)\big)\big]+L\big[N\big(u(x,t)\big)\big]=L[g(x,t)]$$

$$\Rightarrow U(x,s)=\frac{1}{s^{\alpha}}\left[\sum_{k=0}^{m-1}u^{(k)}(x,0)s^{\alpha-k-1}\right]-\frac{1}{s^{\alpha}}\left\{L\big[L_1\big(u(x,t)\big)\big]+L\big[N\big(u(x,t)\big)\big]-L[g(x,t)]\right\}$$

Assuming the solution

$$U(x,s)=\sum_{i=0}^{\infty}U_i(x,s) \qquad (12)$$

And decomposing $N\big(u(x,t)\big)$ as

$$N\left(\sum_{i=0}^{\infty}u_i(x,t)\right)=N(u_0)+\sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i}u_j\right)-N\left(\sum_{j=0}^{i-1}u_j\right)\right\}$$

So that (11) becomes

$$\sum_{i=0}^{\infty}U_i(x,s)=g_0(x)+\frac{1}{s^{\alpha}}\left\{L\left[L_1\left(\sum_{i=0}^{\infty}U_i(x,s)\right)\right]+L\left[N(u_0)+\sum_{i=0}^{\infty}\left(N\left(\sum_{j=0}^{i}u_j\right)-N\left(\sum_{j=0}^{i-1}u_j\right)\right)\right]+L[g(x,t)]\right\} \quad (13)$$

Where $g_0(x)=\frac{u(x,0)}{s}+\cdots+\frac{u^{(m-1)}(x,0)}{s^m}$
From (13), the successive iterates are determined by the recursive relation

$$U_0(x,s)=g_0(x)+\frac{1}{s^{\alpha}}\big[L[g(x,t)]\big]$$

$$U_1(x,s) = \frac{1}{s^\alpha}\{L[L_1(u_0)] + L[N(u_0)]\}$$

$$U_{n+1}(x,s) = \frac{1}{s^\alpha}\{L[L_1(u_n)] + L[N(u_0+\cdots+u_n) - N(u_0+\cdots+u_{n-1})]\} \quad n = 1,2,\ldots$$

Applying the inverse Laplace transform L^{-1} on both sides (14) yields the required successive components of solutions of (8).

$$u_0(x,t) = u(x,0) + \cdots + \frac{u^{(m-1)}(x,0)t^{m-1}}{(m-1)!} + L^{-1}\left\{\frac{1}{s^\alpha}\left[L[g(x,t)]\right]\right\}$$

$$u_1(x,t) = -L^{-1}\left\{\frac{1}{s^\alpha}\left[L[L_1(u_0)] + L[N(u_0)]\right]\right\}$$

$$u_{n+1}(x,s) = -L^{-1}\left\{\frac{1}{s^\alpha}\left[L[L_1(u_n)]\right] + L[N(u_0+\cdots+u_n) - N(u_0+\cdots+u_{n-1})]\right\} \quad n = 1,2,\ldots \qquad (15)$$

The nth-term approximate solution of (8) is thus given as

$$u(x,t) = \sum_{i=0}^{n-1} u_i(x,t) \qquad (16)$$

Lemma 3.1. *If $U(x,s)$ is the Laplace transform of $u(x,t)$ with respect to t, then*

$$L^{-1}\left[\frac{U(x,s)}{s^\alpha}\right] = \frac{1}{\Gamma(\alpha)}\int_0^t (t-r)^{\alpha-1}u(x,r)dr$$

Proof: The proof has been given in [13].

4.0 Applying the Multivariate Pad*e* Approximation Technique

The Pad*e* approximants of a function is derived by expanding the function as a ratio of two power series. The univariate Pad*e* approximation of order (m, n) to a function f(t) is defined to be a rational function $R_{m,n}(t)$ expressed in the form

$$R_{m,n}(t) = \frac{p(t)}{q(t)} \qquad (17)$$

Where $f(t)$ has the Maclaurin's series expansion

$$f(t) = \sum_{r=0}^{\infty} C_r t^r$$

P(t) and q(t) are power series of order m and n respectively given by;

$$\frac{p(t)}{q(t)} = \frac{\begin{vmatrix} \sum_{i=0}^{m} C_i t^i & \sum_{i=0}^{m-1} C_i t^i & \cdots & \sum_{i=0}^{m-n} C_i t^i \\ C_{m+1}t^{m+1} & C_m t^m & \cdots & C_{m+1-n}t^{m+1-n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{m+n}t^{m+n} & C_{m+n-1}t^{m+n-1} & \cdots & C_m t^m \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ C_{m+1}t^{m+1} & C_m t^m & \cdots & C_{m+1-n}t^{m+1-n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{m+n}t^{m+n} & C_{m+n-1}t^{m+n-1} & \cdots & C_m t^m \end{vmatrix}} \qquad (18)$$

In a similar manner, the Multivariate Pad*e* approximation of order (m,n) to a function $f(x,t)$ is defined to be a rational function

$$r_{m,n}(x,t) = \frac{p(x,t)}{q(x,t)} \qquad (19)$$

Where $f(x,t)$ has the Maclaurin's series development

$$f(x,t) = \sum_{i,j=0}^{\infty} c_{ij}x^i t^j$$

And

$$\frac{p(x,t)}{q(x,t)} = \frac{\begin{vmatrix} \sum_{i+j=0}^{m} c_{ij}x^i t^j & \sum_{i+j=0}^{m-1} c_{ij}x^i t^j & \cdots & \sum_{i+j=0}^{m-n} c_{ij}x^i t^j \\ \sum_{i+j=m+1} c_{ij}x^i t^j & \sum_{i+j=m} c_{ij}x^i t^j & \ddots & \sum_{i+j=m+1-n} c_{ij}x^i t^j \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{i+j=m+n} c_{ij}x^i t^j & \sum_{i+j=m+n-1} c_{ij}x^i t^j & & \sum_{i+j=m} c_{ij}x^i t^j \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \sum_{i+j=m+1} c_{ij}x^i t^j & \sum_{i+j=m} c_{ij}x^i t^j & \ddots & \sum_{i+j=m+1-n} c_{ij}x^i t^j \\ \vdots & \vdots & & \vdots \\ \sum_{i+j=m+n} c_{ij}x^i t^j & \sum_{i+j=m+n-1} c_{ij}x^i t^j & \cdots & \sum_{i+j=m} c_{ij}x^i t^j \end{vmatrix}} \qquad (20)$$

The Pad*e* equations $p(x,t)$ and $q(x,t)$ are of the form

$$p(x,t) = \sum_{i+j=mn}^{mn+m} a_{ij}x^i t^j$$

$$q(x,t) = \sum_{i+j=mn}^{mn+m} a_{ij}x^i t^j \qquad (21)$$

More details on Multivariate Pad*e* approximations are given in [20-22].

5.0 Illustrative Examples

In this section, we shall illustrate the solution techniques by some nonlinear fractional partial differential equations. All results were obtained using the symbolic calculus software Mathematica.

Example 5.1. Consider the nonlinear time-fractional advection equation [16].

$D_t^\alpha u(x,t) + u(x,t)u_x(x,t) = x + xt^2, u(x,0) = 0$

$t > 0, x \in R, \quad 0 < \alpha \le 1$

The analytic solution, is only known for $\alpha = 1$ *and given by*

Exact: $u(x,t) = xt$

Taking the Laplace transform of the problem yields

$$s^\alpha U(x,s) - s^{\alpha-1}u(x,0) = -L[N(u)] + L[x + xt^2] \qquad (22)$$

where $N(u) = uu_x$

Assuming the solution

$$U(x,s) = \sum_{i=0}^{\infty} U_i(x,s)$$

And decomposing $N(u)$ as

$$N\left(\sum_{i=0}^{\infty} u_i(x,t)\right) = N(u_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right\}$$

Eqn. (22) can be written as

$$\sum_{i=0}^{\infty} U_i(x,s) = \frac{u(x,0)}{s} - \frac{1}{s^\alpha}\left\{L\left[N(u_0) + \sum_{i=1}^{\infty}\left(N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i} u_j\right)\right)\right] - L[x + xt^2]\right\} \quad (23)$$

From Eqn. (23) we obtain the successive iterates

$$U_0(x,t) = \frac{u(x,0)}{s} + \frac{1}{s^\alpha}L[x + xt^2]$$

$$U_1(x,s) = \frac{1}{s^\alpha}N(u_0) \qquad (24)$$

$$U_{n+1}(x,s) = \frac{1}{s^\alpha}[N(u_0 + \cdots + u_{n-1}) - N(u_0 + \cdots + u_n)]$$

Applying the inverse Laplace transform on both sides of eqn. (24) yields

$$U_0(x,s) = x\left(\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\right)$$

$$u_1(x,t) = x\left(\frac{-96\mathrm{d}(2\alpha+1)t^{3\alpha}}{(\Gamma(\alpha+3))^2\Gamma(3\alpha+5)} - \frac{192\Gamma(2\alpha+1)t^{3\alpha+2}}{(\Gamma(\alpha+3))^2\Gamma(3\alpha+5)} - \frac{96I(2\alpha+1)t^{3\alpha+4}}{(\Gamma(\alpha+3))^2\Gamma(3\alpha+5)} - \cdots\right)$$

$$u_2(x,t) = x\left(\frac{1776\Gamma(2\alpha+1)\Gamma(4\alpha+1)t^{5\alpha}}{(\Gamma(\alpha)\Gamma(\alpha+3))^2\Gamma(3\alpha+5)\mathrm{d}(5\alpha+1)} + \frac{192\Gamma(2\alpha+1)\Gamma(4\alpha+1)t^{5\alpha}}{(\alpha\Gamma(\alpha)\mathrm{d}(\alpha+3))^2\Gamma(3\alpha+5)\mathrm{d}(5\alpha+1)} + \cdots\right) \quad \vdots \qquad (25)$$

The first three terms of series solution is;

$$u(x,t) = x\left(\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{2t^{\alpha+2}}{\Gamma(\alpha+3)} - \frac{96\mathrm{d}(2\alpha+1)t^{3\alpha}}{(\Gamma(\alpha+3))^2\Gamma(3\alpha+5)} - \frac{192\mathrm{d}(2\alpha+1)t^{3\alpha+2}}{(\Gamma(\alpha+3))^2\Gamma(3\alpha+5)} - \cdots\right) \qquad (26)$$

For $\alpha = 1$, (26) becomes

$$u(x,t) = x\left(t + \frac{4t^7}{105} + \frac{2t^9}{567} - \frac{4t^{11}}{2475} - \frac{4t^{13}}{12285} - \frac{t^{15}}{59535}\right) \qquad (27)$$

Using (20), we obtain the Multiple approximation of (27) of order (14, 2) as follows;

$$p(x,t)=\begin{vmatrix} K & L & L \\ 0 & \frac{-4xt^{13}}{12285} & 0 \\ \frac{-xt^{15}}{59535} & 0 & \frac{-4xt^{13}}{12285} \end{vmatrix}$$

$$=\frac{16x^3t^{27}}{150921225}-\frac{4x^3t^{29}}{731387475}-\frac{64x^3t^{33}}{15846728625}+\frac{496x^3t^{35}}{2995031710125}-\frac{282616x^3t^{37}}{1482540696511875}-\frac{7856x^3t^{39}}{305921096105625}$$

$$q(x,t)=\begin{vmatrix} 1 & 1 & 1 \\ 0 & \frac{-4xt^{13}}{12285} & 0 \\ \frac{-xt^{15}}{59535} & 0 & \frac{-4xt^{13}}{12285} \end{vmatrix}=\frac{16x^3t^{26}}{150921225}-\frac{4x^3t^{28}}{731387475}$$

$$where\ K=xt+\frac{4xt^7}{105}+\frac{2xt^9}{567}-\frac{4xt^{11}}{2475}-\frac{4xt^{13}}{12285}\ and\ L=xt+\frac{4xt^7}{105}+\frac{2xt^9}{567}-\frac{4xt^{11}}{2475}$$

So the multivariate pade approximation of order (14, 2) is

$$r(x,t)=\frac{\frac{16x^3t^{27}}{150921225}-\frac{4x^3t^{29}}{731387475}-\frac{64x^3t^{33}}{15846728625}+\frac{496x^3t^{35}}{2995031710125}-\frac{282616x^3t^{37}}{1482540696511875}-\frac{7856x^3t^{39}}{305921096195625}}{\frac{16x^3t^{26}}{150921225}-\frac{4x^3t^{29}}{731387475}} \quad (28)$$

$For\ \alpha=0.5,(26)yields$

$$u(x,t)=1.1283xt^{0.5}-0.957798xt^{1.5}-1.90261xt^{2.5}-1.17377xt^{3.5}-1.25349xt^{4.5}-0.710349xt^{5.5}$$
$$-0.455522xt^{6.5}-0.2801xt^{7.5}+0.0614131xt^{8.5}-0.0677285xt^{9.5}-0.00664769xt^{11.5}$$

For simplicity, we let $t^{0.5}=y$ to obtain

$$u(x,t)=1.12838xy-0.957798xy^3-1.90261xy^5-1.17377xy^7-1.25349xy^9-0.710349xy^{11}-0.455522xy^{13}$$
$$-0.2801xy^{15}+0.0614131xy^{17}-0.0677285xy^{19}-0.00664769xt^{23}$$

We calculate the Pade equations of order (20,4) using (20) to obtain

$$p(x,t)=\begin{vmatrix} M & N & N & \emptyset & \emptyset \\ 0 & -0.0677285xy^{19} & 0 & 0.0614131xy^{17} & 0 \\ 0 & 0 & -0.0677285xy^{19} & 0 & 0.0614131xy^{17} \\ 0 & 0 & 0 & -0.0677285xy^{19} & 0 \\ -0.0066476xy^{23} & 0 & 0 & 0 & -0.0677285xy^{19} \end{vmatrix}$$

$$\Rightarrow p(x,t)=0.0000237433x^5y^{77}-0.0000222671x^5y^{79}+0.0000393979x^5y^{81}-0.0000262833x^5y^{83}$$
$$+0.000024644x^5y^{85}-0.0000148704x^5y^{87}+8.32651\times10^{-6}x^5y^{89}-5.27983\times10^{-6}x^5y^{91}$$
$$+8.76008\times10^{-7}x^5y^{93}-9.61656\times10^{-7}x^5y^{95}$$

$$q(x,t)=\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -0.0677285xy^{19} & 0 & 0.0614131xy^{17} & 0 \\ 0 & 0 & -0.0677285xy^{19} & 0 & 0.0614131xy^{17} \\ 0 & 0 & 0 & -0.0677285xy^{19} & 0 \\ -0.0066476xy^{23} & 0 & 0 & 0 & -0.0677285xy^{19} \end{vmatrix}$$

$$\Rightarrow q(x,t)=0.0000210419x^4y^{76}-1.87273\times10^{-6}x^4y^{78}-2.06531\times10^{-6}x^4y^{80}$$

Where

$$M=1.12838xy-0.957798xy^3-1.90261xy^5-1.17377xy^7-1.25349xy^9-0.710349xy^{11}-0.455522xy^{13}$$
$$-0.2801xy^{15}+0.0614131xy^{17}-0.0677285xy^{19},$$
$$N=1.12838xy-0.957798xy^3-1.90261xy^5-1.17377xy^7-1.25349xy^9-0.710349xy^{11}-0.455522xy^{13}$$
$$-0.2801xy^{15}+0.0614131xy^{17}$$
$$\emptyset=1.12838xy-0.957798xy^3-1.90261xy^5-1.17377xy^7-1.25349xy^9-0.710349xy^{11}-0.455522xy^{13}$$
$$-0.2801xy^{15}$$

$Recalling\ that\ y=t^{0.5}, we\ obtain\ the\ (20,4)\ Multi\ Pade\ approximation\ as;$

$$r(x,t)=\frac{p(x,t)}{q(x,t)}$$

$For\ \alpha=0.75,(26)becomes$

$$u(x,t)=1.08807xt^{0.75}-0.617351xt^{2.25}+0.452183xt^{2.75}+0.485984xt^{3.75}-0.325055xt^{4.25}-0.107914xt^{5.25}$$
$$+0.33527xt^{5.75}-0.0509475xt^{6.25}-0.0896581xt^{7.25}+0.0860988xt^{7.75}-0.0314575xt^{9.25}$$
$$+0.00860988xt^{9.75}-0.00534688xt^{11.25}-0.000371101xt^{13.25}$$

Letting $t^{0.25} = y$, *we obtain*

$$u(x,t) = 1.08807xy^3 - 0.617351xy^9 + 0.452183xy^{11} + 0.485984xy^{15} - 0.325055xy^{17} - 0.107914xy^{21} + 0.33527xy^{23} - 0.0509475xy^{25} - 0.0896581xt^{29} + 0.0860988xt^{31} - 0.0314575xt^{37} + 0.00860988xt^{39} - 0.00534688xt^{45} - 0.000371101xt^{53}$$

Using (20), we obtain the (46, 8) multivariate Pade equations as:

$$r(x,t) = \frac{7.26876 \times 10^{-19}x^9y^{361} - 8.46335 \times 10^{-20}x^9y^{363} - \cdots - 2.11339 \times 10^{-21}x^9y^{405}}{6.68042 \times 10^{-19}x^8y^{360} - 7.77831 \times 10^{-20}x^8y^{361} - \cdots - 4.63655 \times 10^{-20}x^8y^{368}}$$

Recalling $y = t^{0.25}$, *we obtain*

$$r(x,t) = \frac{7.26876 \times 10^{-19}xt^{0.25} - 8.46335 \times 10^{-20}xt^{0.75} - \cdots - 2.11339 \times 10^{-21}xt^{3.75}}{6.68042 \times 10^{-19} - 7.77831 \times 10^{-20}t^{0.5} - \cdots - 4.63655 \times 10^{-20}t^2}$$

Table 3 shows that the results of the Laplace NIM (LNIM) and Multivariate Pad*e´* approximations are more accurate than the results obtained by the ADM [16]. We also demonstrate the fact that the MultiPad*e´* approximation slightly accelerates the rate of convergence of solutions (at x=t=2.0).

Table 1: Numerical (3rd-order term approximations) when $\alpha = 0.5$ for Example 5.1

x	*t*	ADM	LNIM	MultiPad*e´*
0.01	0.01	0.001118860666	0.001118902868	0.001118991197
0.02	0.02	0.003138022015	0.003139483565	0.003139485921
0.03	0.03	0.005716640278	0.005722623468	0.005722666678
0.04	0.04	0.008727882362	0.008744355088	0.008744322182
0.05	0.05	0.01209607907	0.012131615851	0.012131870921
0.06	0.06	0.01576873408	0.015836287141	0.015836299431
0.07	0.07	0.01970633078	0.019821082174	0.019821923732
0.08	0.08	0.02387754051	0.024061546834	0.024061565828
0.09	0.09	0.02825661342	0.028533899082	0.028533921800
0.10	0.10	0.03252181204	0.033221951795	0.033221978475

Table 2: Numerical (3rd-order term approximations) when $\alpha = 0.75$ for Example 5.1

x	*t*	ADM	LNIM	MultiPad*e´*
0.01	0.01	$0.7125363089 \times 10^{-21}$	0.000343895663	0.000343897165
0.02	0.02	$0.2579676158 \times 10^{-18}$	0.001155669432	0.001155674484
0.03	0.03	$0.8097412617 \times 10^{-17}$	0.002346942115	0.002346952388
0.04	0.04	$0.9339702880 \times 10^{-16}$	0.003877795718	0.003877812718
0.05	0.05	$0.6224118481 \times 10^{-15}$	0.005722203151	0.005722228280
0.06	0.06	$0.2931772549 \times 10^{-14}$	0.007860911381	0.007860945966
0.07	0.07	$0.1086905954 \times 10^{-13}$	0.010278666393	0.010278711704
0.08	0.08	$0.3381735603 \times 10^{-13}$	0.012962849422	0.012962906684
0.09	0.09	$0.9203528751 \times 10^{-13}$	0.015902710710	0.015902981111
0.10	0.10	$0.2253790147 \times 10^{-12}$	0.01908898826	0.019088983522

Example 5.2: Consider the nonlinear time-fractional hyperbolic equation [16].

$$D_t^{\alpha}u(x,t) = \frac{\partial}{\partial x}\left(u(x,t)\frac{\partial u(x,t)}{\partial x}\right), u(x,0) = x^2, u_t(x,0) = -2x^2$$

$t > 0, x \in R, 1 < \alpha \leq 2$

The analytic solution is only known for $\alpha = 2$ and is given by;

$$Exact: u(x,t) = \frac{x^2}{(t+1)^2}$$

Table 3: Numerical (3rd-order term approximations) and Exact values when $\alpha = 1.0$ for Example 5.1

x	t	ADM	LNIM	MultiPade´	Exact
0.01	0.01	0.0000999999987	0.0001000000000	0.0001000000000	0.0001
0.02	0.02	0.0003999999915	0.0004000000000	0.0004000000000	0.0004
0.03	0.03	0.0008999999028	0.0009000000000	0.0009000000000	0.0009
0.04	0.04	0.0015999999454	0.0016000000000	0.0016000000000	0.0016
0.05	0.05	0.002499997917	0.0025000000001	0.0025000000000	0.0025
0.06	0.06	0.00359993779	0.0036000000006	0.0036000000000	0.0036
0.07	0.07	0.004899984313	0.0049000000022	0.0049000000000	0.0049
0.08	0.08	0.006399965047	0.0064000000064	0.0064000000000	0.0064
0.09	0.09	0.008099929141	0.0081000000164	0.0081000000000	0.0081
0.10	0.10	0.00999986667	0.0100000000381	0.0100000000000	0.01
1.00	1.00	0.8666667001	1.0396640161720	1.0396631025362	1
2.00	2.00	-4.53333119840	4.309142333269	4.0229348226823	4

Following the procedures discussed in the previous section, we can obtain the following successive approximations

$$u_0(x,t) = x^2(1-2t)$$

$$u_1(x,t) = \frac{18x^2t^\alpha}{(\alpha+1)(\alpha+2)\Gamma(\alpha)} - \frac{24x^2t^{\alpha+1}}{(\alpha+1)(\alpha+2)\Gamma(\alpha)} + \frac{12x^2t^\alpha}{\Gamma(\alpha+3)} - \frac{48x^2t^{\alpha+1}}{\Gamma(\alpha+3)} + \frac{48x^2t^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{6x^2t^\alpha}{\Gamma(\alpha+3)}$$

$$u_2(x,t) = \frac{27\times 2^{5-2\alpha}\Gamma\left(\frac{1}{2}\right)x^2t^{2\alpha}}{(\alpha+1)^2(\alpha+2)^2\Gamma(\alpha)\Gamma\left(\alpha+\frac{1}{2}\right)} + \frac{9\times 2^{5-2\alpha}\Gamma\left(\frac{1}{2}\right)x^2t^{2\alpha}}{(\alpha+1)^2(\alpha+2)^2\Gamma(\alpha+1)\Gamma\left(\alpha+\frac{1}{2}\right)} + \cdots$$

$$\vdots$$

The first three term approximation is

$$u(x,t) = x^2(1-2t) + \frac{18x^2t^\alpha}{(\alpha+1)(\alpha+2)\Gamma(\alpha)} - \frac{24x^2t^{\alpha+1}}{(\alpha+1)(\alpha+2)\Gamma(\alpha)} + \frac{12x^2t^\alpha}{\Gamma(\alpha+3)} - \frac{48x^2t^{\alpha+1}}{\Gamma(\alpha+3)} + \frac{48x^2t^{\alpha+2}}{\Gamma(\alpha+3)} + \cdots \qquad (29)$$

For $\alpha = 2$, (29) *becomes*

$$u(u,t) = x^2\left(1 - 2t + 3t^2 - 4t^3 + 5t^4 - 6t^5 + \frac{29t^6}{5} - \frac{32t^7}{7} + 3t^8 - \frac{4t^9}{3} + \frac{4t^{10}}{15}\right) \qquad (30)$$

From (20), we obtain the Pade equations of order (10, 2) as follows:

$$p(x,t) = \begin{vmatrix} P & Q & R \\ \frac{-4x^2t^9}{15} & 3x^2t^8 & \frac{-32x^2t^7}{7} \\ \frac{4x^2t^{10}}{15} & \frac{-4x^2t^9}{15} & 3x^2t^8 \end{vmatrix}$$

$$= x^6\left(\frac{817t^{16}}{105} - \frac{1678t^{17}}{105} + \frac{36937t^{18}}{1575} - \frac{48704t^{19}}{1575} + \frac{6719t^{20}}{175} - \frac{72238t^{21}}{1575} + \frac{69299t^{22}}{1575} - \frac{24716t^{23}}{735} + \frac{1159331t^{24}}{55125}\right)$$

Where

$$P = x^2 - 2x^2t + 3x^2t^2 - 4x^2t^3 + 5x^2t^4 - 6x^2t^5 + \frac{29x^2t^6}{5} - \frac{32x^2t^7}{7} + 3x^2t^8$$

$$Q = x^2 - 2x^2t + 3x^2t^2 - 4x^2t^3 + 5x^2t^4 - 6x^2t^5 + \frac{29x^2t^6}{5} - \frac{32x^2t^7}{7}$$

$$R = x^2 - 2x^2t + 3x^2t^2 - 4x^2t^3 + 5x^2t^4 - 6x^2t^5 + \frac{29x^2t^6}{5}$$

$$q(x,t) = \begin{vmatrix} 1 & 1 & 1 \\ \frac{-4x^2t^9}{15} & 3x^2t^8 & \frac{-32x^2t^7}{7} \\ \frac{4x^2t^{10}}{15} & \frac{-4x^2t^9}{15} & 3x^2t^8 \end{vmatrix}$$

$$q(x,t) = x^4\left(\frac{817t^{16}}{105} - \frac{44t^{17}}{105} - \frac{164t^{18}}{225}\right)$$

Thus, the (10,2) Multivariate Pade approximation is;

$$r(x,t)=\frac{x^6\left(\frac{817t^{16}}{105}-\frac{1678t^{17}}{105}+\frac{36937t^{18}}{1575}-\frac{48704t^{19}}{1575}+\frac{6719t^{20}}{175}-\frac{72238t^{21}}{1575}+\frac{69299t^{22}}{1575}-\frac{24716t^{23}}{735}+\frac{1159331t^{24}}{55125}\right)}{x^4\left(\frac{817t^{16}}{105}-\frac{44t^{17}}{105}-\frac{164t^{18}}{225}\right)}$$

For $\alpha = 1.5$ (29) $becomes$

$$u(x,t)=x^2(1-4.51352t^{1.5}-7.22163t^{2.5}+12t^3+4.12664t^{3.5}-27t^4+\cdots-18.3464t^{7.5}+4.31681t^{8.5}) \quad (31)$$

$Letting\ t^{0.5}=y, we\ obtain\ Pade\ equations$

$$p(x,y)=\begin{vmatrix} S & T & T \\ 0 & -18.3464x^2y^{15} & 0 \\ 4.31681x^2y^{17} & 0 & -18.3464x^2y^{15}\end{vmatrix}$$

$$= x^6(336.59y^{30}-593.983y^{32}-1519.21y^{33}-\cdots-3450.85y^{45})$$

Where

$$S=x^2-4.51352x^2y^3-7.22163x^2y^5+12x^2y^6+4.12664x^2y^7-27x^2y^8+14.0112x^2y^9+21.6x^2y^{10} - 32.6079x^2y^{11}-7.2x^2y^{12}+34.3996x^2y^{13}-18.3464x^2t^{15}$$

$$T=x^2-4.51352x^2y^3-7.22163x^2y^5+12x^2y^6+4.12664x^2y^7-27x^2y^8+14.0112x^2y^9+21.6x^2y^{10} - 32.6079x^2y^{11}-7.2x^2y^{12}+34.3996x^2y^{13}$$

$$q(x,y)=\begin{vmatrix} 1 & 1 & 1 \\ 0 & -18.3464x^2y^{15} & 0 \\ 4.31681x^2y^{17} & 0 & -18.3464x^2y^{15}\end{vmatrix}$$

$$q(x,y)=336.59x^4y^{30}-79.1979x^4y^{32}$$

$Recalling\ that\ t^{0.5}=y, we\ obtain\ the\ (17,2)\ Multivariate\ Pade\ approximation\ as$

$$r(x,t)=\frac{x^6(336.59t^{15}-593.983t^{16}-1519.21t^{16.5}-\cdots-3450.85t^{22.5})}{336.59x^4t^{15}-79.1979x^4t^{16}}$$

$For\ \alpha=1.75$, (29) $becomes$

$$u(x,t)=x^2-2x^2t+3.73051x^2t^{1.75}-5.4262t^{2.75}+6.18997x^2t^{3.5}+2.89397x^2t^{3.75}-13.0677x^2t^{4.5} - 5.25396x^2t^{5.25}+9.50379x^2t^{5.5}-11.0047x^2t^{6.25}-2.92424x^2t^{6.5}+10.524x^2t^{7.25} - 5.10254x^2t^{8.25}+1.10325x^2t^{9.25}$$

In a similar manner as above and letting $t^{0.25}=y$, we obtain the (31, 8) multivate Pade approximation as

$$r(x,t)=\frac{8.84567\times10^6x^{18}t^{58}+889378.11751x^{18}t^{58.25}+\cdots-3.26799\times10^6x^{18}t^{65.75}}{8.84567\times10^6x^{18}t^{58}+889378.11751x^{18}t^{58.25}+\cdots+2.3591\times10^6x^{16}t^{60}}$$

Tables 4, 5 and 6 show the computational results for the ADM, the LNIM and the Multivariate Pade approximation for different values of α.

Table 4: Numerical (3rd-order term approximations) when $\alpha = 1.5$ for Example 5.2

x	t	ADM	LNIM	MultiPade′
0.01	0.01	0.000098445371	0.000098445346	0.000098445346
0.02	0.02	0.000388983322	0.000388981742	0.000388981746
0.03	0.03	0.000866403431	0.000866385865	0.000866385880
0.04	0.04	0.001527388854	0.001527292208	0.001527292248
0.05	0.05	0.002370102454	0.002369740722	0.002369740810
0.06	0.06	0.003393997434	0.003392936115	0.003392936280
0.07	0.07	0.004599726730	0.004597094073	0.004597094352
0.08	0.08	0.005989109024	0.005983333132	0.005983333571
0.09	0.09	0.007565131844	0.007553593353	0.007553594006
0.10	0.10	0.009331981000	0.009310571698	0.009310572627

Example 5.3: Lastly, we consider the nonlinear time-fractional Fisher's equation [16].

$$D_t^\alpha u(x,t)=u_{xx}(x,t)+6u(x,t)\big(1-u(x,t)\big), u(x,t)=\frac{1}{(1+e^x)^2}$$

$t>0, x\in R, 0<\alpha\leq 1$

The exact solution is only known for $\alpha = 1$ $and\ is\ given\ by$;

$$Exact: u(x,t)=\frac{1}{(1+e^{x-5t})^2}$$

Table 5: Numerical (3rd-order term approximations) when $\alpha = 1.75$ for Example 5.2

x	t	ADM	LNIM	MultiPad*e´*
0.01	0.01	0.000098116324	0.000098116323	0.000098116323
0.02	0.02	0.000385544317	0.000385544203	0.000385544203
0.03	0.03	0.000852974901	0.000852973332	0.000852973333
0.04	0.04	0.001492265504	0.001492255460	0.001492255463
0.05	0.05	0.002296248891	0.002296206616	0.002296206620
0.06	0.06	0.003258613123	0.003258476592	0.003258476598
0.07	0.07	0.004373819216	0.004373451923	0.004373451922
0.08	0.08	0.005637043845	0.005636179420	0.005636179389
0.09	0.09	0.007044140677	0.007042303687	0.007042303558
0.10	0.10	0.008591616573	0.008588014696	0.008588014311

Table 6: Numerical (3rd-order term approximations) and Exact values when $\alpha = 2.0$ for Example 5.2

x	t	ADM	LNIM	MultiPad*e´*	Exact
0.01	0.01	0.0000980296050	0.00009802960494	0.00009802960494	0.00009802960494
0.02	0.02	0.0003844675200	0.0003844675125	0.0003844675125	0.0003844675125
0.03	0.03	0.0008483364450	0.0008483363175	0.0008483363175	0.0008483363182
0.04	0.04	0.001479290880	0.0014792899340	0.0014792899340	0.001479289941
0.05	0.05	0.002267578126	0.002267573655	0.002267573655	0.002267573696
0.06	0.06	0.003204002880	0.0032039870137	0.0032039870138	0.003203987184
0.07	0.07	0.004279895445	0.0042798491997	0.0042798492000	0.0042798492769
0.08	0.08	0.005487083520	0.0054869668392	0.0054869668401	0.0054869696845
0.09	0.09	0.006817867605	0.0068176039233	0.0068176039267	0.006817607945
0.10	0.10	0.008265000000	0.0082644537155	0.0082644537262	0.008264462810

With the approach discussed in the previous section, we can obtain the following successive approximation of the LNIM

$$u_0(x,t) = \frac{1}{(1+e^x)^2}$$

$$u_1(x,t) = \frac{10e^x t^\alpha}{(1+e^x)^3 \Gamma(\alpha+1)}$$

$$u_2(x,t) = \frac{25\times 2^{1-2\alpha}\Gamma\left(\frac{1}{2}\right)e^x t^{2\alpha}}{(1+e^x)^6\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma(\alpha+1)} + \frac{75\times 2^{1-2\alpha}\Gamma\left(\frac{1}{2}\right)e^{3x} t^{2\alpha}}{(1+e^x)^6\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma(\alpha+1)} + \frac{25\times 2^{2-2\alpha}\Gamma\left(\frac{1}{2}\right)e^{4x} t^{2\alpha}}{(1+e^x)^6\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma(\alpha+1)} - \frac{600\Gamma(2\alpha+1)e^{2x}t^{3\alpha}}{(1+e^x)^6\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma(3\alpha+1)}$$

The first three term approximation of the LNIM solution

$$u(x,t) = \frac{1}{(1+e^x)^2} + \frac{10e^x t^\alpha}{(1+e^x)^3 \Gamma(\alpha+1)} + \frac{25\times 2^{1-2\alpha}\Gamma\left(\frac{1}{2}\right)e^x t^{2\alpha}}{(1+e^x)^6\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma(\alpha+1)} + \frac{75\times 2^{1-2\alpha}\Gamma\left(\frac{1}{2}\right)e^{3x} t^{2\alpha}}{(1+e^x)^6\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma(\alpha+1)} + \cdots \quad (32)$$

For $\alpha = 1$, (32) *yields*

$$u(x,t) = \frac{1}{(1+e^x)^2} + \frac{10te^x}{(1+e^x)^3} - \frac{25t^2e^x}{(1+e^x)^6} + \frac{75t^2e^{3x}}{(1+e^x)^6} + \frac{50t^2e^{4x}}{(1+e^x)^6} - \frac{200t^3e^{2x}}{(1+e^x)^6} \quad (33)$$

To obtain the MultiPade approximation, we obtain the Taylor's expansion of (33) about the origin to obtain

$$u(x,t) = \frac{1}{4} - \frac{x}{4} + \frac{x^2}{16} + \frac{x^3}{48} - \frac{x^4}{96} - \frac{x^5}{480} + \frac{17x^6}{11520} + \frac{71x^7}{80640} - \frac{31x^8}{161280} - \frac{31x^9}{1451520} + \frac{691x^{10}}{29030400} +$$
$$t\left(\frac{5}{4} - \frac{5x}{8} - \frac{5x^2}{16} + \frac{5x^3}{24} + \frac{5x^4}{96} - \frac{17x^5}{384} - \frac{17x^6}{2304} + \frac{31x^7}{4032} + \frac{31x^8}{32256} + \frac{691x^9}{580608} + \frac{691x^{10}}{5806080}\right) +$$

$$t^2\left(\frac{25}{16}+\frac{25x}{16}-\frac{25x^2}{16}-\frac{25x^3}{48}+\frac{425x^4}{768}+\frac{85x^5}{768}-\frac{155x^6}{1152}-\frac{155x^7}{8064}+\frac{3455x^8}{129024}+\frac{3455x^9}{1161216}-\frac{5461x^{10}}{1161216}\right)+$$
$$t^3\left(\frac{-25}{8}+\frac{25x}{8}+\frac{25x^2}{32}-\frac{175x^3}{96}+\frac{25x^4}{384}+\frac{235x^5}{384}-\frac{55x^6}{576}-\frac{1255x^7}{8064}+\frac{2365x^8}{64512}+\frac{2765x^9}{82944}-\frac{46303x^{10}}{4644864}\right)$$

Thus, we obtain the (10.3) Multivariate Pade approximation as;

$$r(x,t)=\frac{p(x,t)}{q(x,t)}$$

Where we have decided not to write the results of $r(x,t)$ because of their voluminousity. However, we give the computational results of these approximations in Table 9.

For $\alpha = 0.5$ (32) *becomes*

$$u(x,t)=\frac{1}{(1+e^x)^2}+\frac{11.283t^{0.5}e^x}{(1+e^x)^3}-\frac{50t^2te^x}{(1+e^x)^6}+\frac{150te^{3x}}{(1+e^x)^6}+\frac{100te^{4x}}{(1+e^x)^6}-\frac{574.679t^{1.5}e^{2x}}{(1+e^x)^6} \quad (34)$$

By obtaining the Taylors expansion of (34) about the origin and substituting $y = t^{0.5}$, we obtain the (10, 3) Pade approximation of (34). We omit the results of these approximations because of their voluminousity and give their computational results in Table 7.

For $\alpha = 0.75$, (32) *becomes*

$$u(x,t)=\frac{1}{(1+e^x)^2}+\frac{10.8807t^{0.75}e^x}{(1+e^x)^3}-\frac{37.6126t^2t^{1.5}e^x}{(1+e^x)^6}+\frac{112.838t^{1.5}e^{3x}}{(1+e^x)^6}+\frac{75.2253t^{1.5}e^{4x}}{(1+e^x)^6}-\frac{370.411t^{2.25}e^{2x}}{(1+e^x)^6} \quad (35)$$

By obtaining the Taylors expansion of (35) about the origin and substituting $y = t^{0.5}$, we obtain the (16, 3) Pade approximations of (35). We also omit the results of these approximations because of the voluminousity and give their computational results in Table 8.

Table 7: Numerical (3rd-order term approximations) when α = 0.5 for Example 5.3

x	*t*	ADM	MultiPad*e´*
0.01	0.01	0.419502917120	0.410514948415
0.02	0.02	0.506206750836	0.481319666874
0.03	0.03	0.579616989032	0.534368393352
0.04	0.04	0.646241850082	0.577306452553
0.05	0.05	0.708640807796	0.613323611273
0.06	0.06	0.768122728648	0.644174026779
0.07	0.07	0.825460345010	0.670971923738
0.08	0.08	0.881152204499	0.694491646667
0.09	0.09	0.935539818929	0.715305494214
0.10	0.10	0.988867304140	0.733855728432

Table 8: Numerical (3rd-order term approximations) when α = 0.75 for Example 5.3

x	*t*	ADM	MultiPad*e´*
0.01	0.01	0.292673772370	0.292492584873
0.02	0.02	0.323407200849	0.322554090799
0.03	0.03	0.351675580491	0.349573232547
0.04	0.04	0.378798460444	0.374824296067
0.05	0.05	0.405317795783	0.398821626825
0.06	0.06	0.431520097705	0.421834777828
0.07	0.07	0.457578193467	0.444026985405
0.08	0.08	0.483605151222	0.465507233983
0.09	0.09	0.509679025540	0.486353822256
0.10	0.10	0.535855700853	0.506626398951

Table 9: Numerical (3rd-order term approximations) when $\alpha = 1.0$ for Example 5.3

x	t	ADM	Multipade´	Exact
0.01	0.01	0.260102571491	0.260098163467	0.260098640219
0.02	0.02	0.270409946856	0.270385449243	0.270388913875
0.03	0.03	0.280933196121	0.280851370028	0.280868961627
0.04	0.04	0.291677793894	0.291485866442	0.291508082565
0.05	0.05	0.302650186439	0.302279308421	0.302317424601
0.06	0.06	0.313856472970	0.313222495704	0.313279369176
0.07	0.07	0.325302401707	0.324306657424	0.324382901203
0.08	0.08	0.336993366310	0.335523450821	0.335616589128
0.09	0.09	0.348934402710	0.348649590965	0.346968632956
0.10	0.10	0.361101863217	0.358323688425	0.358426914371

6.0 Conclusion

The New Iteration Method has been used to solve many functional equations. This paper presents a general framework of the NIM by incorporating the Laplace transform and the Multi- variate Pade´ approximation technique to solve nonlinear partial differential equations of fractional order. The LNIM produces series solution and we form the Multivariate Pade´ approximants of the LNIM solution to accelerate the rate of convergence of solutions. Numerical examples given in the previous section shows that this approach produces results that are more accurate than those produced by the ADM. The approach employed in this paper is expected to be further used to solve other functional equations.

References

[1] V. Daftardar-Gejji and H. Jafari, An iterative method for solving nonlinear functional equations, Journal of Mathematical Analysis and Applications, 36 (2) (2006) 753-763.

[2] A. A. Hemeda, New iterative method: application to nth order integro-differential equations, Information B. 16 (6) (2013) 3841-3852.

[3] A. A. Hemeda, Formulation and solution of nth-order derivative fuzzy integrodifferential equation using new iterative method with a reliable algorithm, Journal of Applied Mathematics, (2012) Article ID 325473, 17 pages, 2012.

[4] A. A. Hemeda, New Iterative Method: An application for solving Fractional Physical Differential Equations, Abstract and Applied Analysis, (2013) Article ID 617010, 9 pages.

[5] S. Bhalekar and V. Daftardar Gejji, New iterative method: application to partial differential equations, Applied Mathematics and Computation, 203 (2) (2008) 778 - 783.

[6] S. Bhalekar and V. Daftardar Gejji, Convergence of the new iterative method, International Journal of Differential Equations, (2011) Article ID 989065, 10 pages.

[7] S. Bhalekar and V. Daftardar Gejji, Solving evolution equations using a new iterative method, Numerical Methods for Partial Differential Equations, 26 (4) (2010) 906 - 916.

[8] J. H. He, Approximate analytical solution for seepage flow with fractional derivatives in porous media, Computer Methods in Applied Mechanics and Engineering 167 (1-2) (1998) 57-68

[9] J. H. He, Variational iteration method- a kind of nonlinear analytical techniques: some examples, International Journal of Nonlinear Mechanics 34 (4) (1999) 699-708

[10] J. H. He, Variational iteration method- some recent results and new interpretations, Journal of Computational and Applied Mathematics 207 (1) (2007) 3-17

[11] J. H. He and X. H. Wu, Variational iteration method- new development and applications, Comp. Appl. Math 54 (7-8) (2007) 881-894

[12] T. A. Biala, O. O. Asim and Y. O. Afolabi, A combination of the Laplace transform and the variational iteration method for the analytical treatment of Delay Differential Equations, Int. J. of Diff. Equations and Appl., 13 (3) (2014), 164-175.

[13] T. A. Biala, Y. O. Afolabi, P. L. Ndukum, O. Abdulhakeem , An efficient algorithm via the Laplace transform and variational iteration method for solving fractional partial differential equations, Int. J. of Comp. and Appl. Math., 9 (2) (2014), 71-87.

[14] G. Adomian, A review of the decomposition method in applied mathematics, J, Math. Anal. Appl., 135 (1988) 501 - 544

[15] G. Adomian, Solving Frontier Problems of Physics: The Decomposition Method, Kluwer Academic Publishers, Boston, 1994.

[16] Z. Odibat and S. Momani, Numerical methods for nonlinear partial differential equations of fractional order, Appl. Math. Modelling, 32 (2008) 28 - 39.

[17] S. Momani and Z. Odibat, Homotopy perturbation method for nonlinear partial differential equations of fractional order. Phys. Lett. A 365(5-6),(2007) 345-350

[18] S. Momani and Z. Odibat, Numerical comparison of methods for solving linear differential equations of fractional order, Chaos, Solitons and Fractals 31 (5) (2007) 1248-1255

[19] M. Javidi andd M. A. Raji, Combination of Laplace transform and homotopy perturbation method to solve the parabolic partial differential equations, Commun. Fract. Calc. 3(1) (2012) 10-19

[20] Ph. Guillaume and A. Huard, Multivariate Pade´ Approximations, Journal of Computational and Applied Mathematics, 121 (1-2) 2000, 197-219.

[21] V. Turut, Numerical Approximations for Solving Partial Differential Equations with Variable Coefficients, Applied and Computational Mathematics, 2 (1) 2013, 19-23

[22] V. Turut and N. Guzel, Multivariate Pade´ Approximation for Solving Nonlinear Partial Differential Equations of Fractional Order, Abstract and Applied Analysis, 2013. ArticleID: 746401, http://dx.doi.org?10.1155/2013/746401

[23] A. A. Kilbas, H. M. Srvastav and J. J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, New York (2006).