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On Extended Exponential General Linear Methods PSQ with S>Q

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Abstract

This paper is concerned with the construction and Numerical Analysis of Extended Exponential General Linear Methods. These methods, in contrast to other methods in literatures, consider methods with the step greater than the stage order (S>Q).Numerical experiments in this study, indicate that Extended Exponential General Linear Methods perform better than existing Methods.

Keywords: General Linear Methods, Exponential methods. Order conditions

1.0 Introduction

Our interest in this paper is to consider Extended general linear methods for the problem

$$y'(x) = Wy(x) + M(y(x)),\ 0 \le x \le X, \text{ Given } y(0). \qquad (1.1)$$

This paper dwells on the construction of extended exponential general linear methods (EEGLM) for the autonomous problem (1.1) with S>Q. Butcher and Wright [1], Butcher [2] considered the case of general linear methods with S=Q, while Bazuaye [3], Calvo and Palencia [4] considered general linear methods of S<Q.

For given starting values $y_0, y_1, \cdots y_{q-1}$, the theoretical approximation y_{n+1} at x_{n+1}, $n \le q-1$, is given by the recurrence relation or formula

$$y_{n+1} = e^{hL} y_n + h\sum_{i=1}^{s} B_i(hL)N(Y_{ni}) + h\sum_{k=1}^{q-1} V_k(hL)N(y_{n-k}) \qquad (1.2a)$$

The internal stages Y_{ni}, $1 \le i \le s$, are defined through $Y_{ni} = e^{c_i hL} y_n + h\sum_{j=1}^{i-1} A_{ij}^{(1)}(hL)N(Y_{nj}) + h\sum_{k=1}^{q-1} U_{ik}(hL)N'(y_{n-k})$

$$+ h^2\sum_{j=1}^{i-1} A_{ij}^{(2)}(hL)N'(Y_{nj})$$

$$Y_{n1} = y_{n1} = y_n \qquad (1.2b)$$

we assume throughout this paper that these conditions $U_{1k}(hL) = 0$ which implies $c_1 = 0$ and thus $y_{n1} = y_n$ are satisfied.

The coefficients can be represented in a tablau as shown in Table1

Table1: The table of coefficients of the extended Exponential General Linear methods(EEGLM)

$A_{21}^{(1)}$	$U_{21}\cdots U_{2,q-1}$	$A_{21}^{(2)}$
$\ddots$	$\vdots \quad \vdots$	$\ddots$
$A_{s1}^{(1)}\cdots A_{s,s-1}^{(1)}$	$U_{s1}\cdots U_{s,q-1}$	$A_{s,s-1}^{(2)}\cdots A_{s,s-1}^{(2)}$
$B_1\cdots B_{s-1}\quad B_s$	$V_1\cdots V_{q-1}$	

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2.0 Order Conditions For The Extended Exponential General Linear Methods With S>Q.

Deriving the order conditions for the method (1.1), with **S>Q**, we assume the data to be sufficiently regular. In particular, *we* require that the nonlinearity evaluated at the exact solution $f(t) = M(y(t))$ is sufficiently often differentiable with respect to x for $0 < x < X$.

$$y_{n+1} = e^{hL} y_n + \sum_{i=1}^{s} B_i (hL) N(Y_{ni}) + \sum_{k=1}^{q-1} V_k (hL) f(x_n - kh) \tag{2.1}$$

$$Y_{ni} = e^{c_i hL} y_n + h\sum_{j=1}^{i=1} A_{ij}^{(1)} (hL) f(x_n + c_j h) + h\sum_{k=1}^{q-1} U_{ik} (hL) f(x_n - kh) \tag{2.2}$$

$$+ h^2 \sum_{j=1}^{i=1} A_{ij}^{(2)} f'(x_n + c_j h) + h^2 \sum_{k=1}^{q-1} U_{ik} (hL) f'(x_{n-k}),\ l \le i \le S$$

Expanding the functions in (2.1)

$$f(x_n + c_j h) = f(x_n) + c_j hf'(x_n) + \frac{(c_j h)^2}{2!} f''(x_n)$$

$$+ \frac{(c_j h)^3}{3!} f'''(x_n) + \cdots + \frac{(c_j h)^{m-1}}{(m-1)!} f^{(m-1)}(x_n) + U(s,m)$$

where $$U(s,\ m) = \int_0^{\pm} \frac{(\pm - \P)}{(m-1)!} f^{(m)}(x_n + \P)d\P \tag{2.3}$$

$$f(x_n - kh) = f(x_n) + (-kh) f'(x_n) + \frac{(-kh)^2}{2!} f''(x_n)$$

$$\frac{(-kh)^3}{3!} f'''(x_n) + \cdots + \frac{(-kh)^{m-1}}{(m-1)!} f''(x_n) + U(s,\ r) \tag{2.4}$$

$$f'(x_n + c_j h) = f'(x_n) + c_j hf''(x_n) + \frac{(c_j h)^2}{2!} f'''(x_n)$$

$$+ \frac{(c_j h)^3}{3!} f^{(iv)}(x_n) + \cdots + \frac{(c_j h)^{m-1}}{(m-1)!} f^{(m)}(x_n) + U(s,\ m) \tag{2.5}$$

$$\frac{(-kh)^3}{3!} f^{(iv)}(x_n) + \cdots + \frac{(-kh)^{m-1}}{(m-1)!} f^{(m)}(x_n) + U(s,\ m) \tag{2.6}$$

$$f(x_n + c_i h) = f(x_n) + c_i hf'(x_n) + \frac{(c_i h)^2}{2!} f''(x_n)$$

$$+ \frac{(c_i h)^3}{3!} f'''(x_n) + \cdots + \frac{(c_i h)^{m-1}}{(m-1)!} f^{(m-1)}(x_n) + U(s,\ m) \tag{2.7}$$

Substituting where appropriate into (2.1) we have

$$y_{n+1} = e^{hL} y_n + h\sum_{i=1}^{s} B_i N(Y_{ni}) + h\sum_{k=1}^{q-1} V_k \Bigg[f(x_n - kh) = f(x_n) + (-kh) f'(x_n) + \frac{(-kh)^2}{2!}$$

$$\frac{(-kh)^3}{3!} f'''(x_n) + \cdots + \frac{(-kh)^{m-1}}{(m-1)!} f^{(m-1)}(x_n) + U(s,\ r) \Bigg]$$

$$Y_{ni} = e^{c_i hL} y_n + h\sum_{j=1}^{i=1} A_{ij}^{(1)}\left[f(x_n) + c_j hf'(x_n) + \frac{(c_j h)^2}{2!} f''(x_n) + \frac{(c_j h)^3}{3!} f'''(x_n) + \cdots + \frac{(c_j h)^{m-1}}{(m-1)!} f^{(m-1)}(x_n) + U(s,\, m)\right]$$

$$+h\sum_{k=1}^{q-1} U_{ik}\left[f(x_n) + (-kh) f'(x_n) + \frac{(-kh)^2}{2!} f''(x_n)\frac{(-kh)^3}{3!} f'''(x_n) + \cdots + \frac{(-kh)^{m-1}}{(m-1)!} f^{(m-1)}(x_n) + U(s,\, r)\right]$$

$$+h^2\sum_{j=1}^{i-1} A_{ij}^{(2)}\left[f'(x_n) + c_j hf''(x_n) + \frac{(c_j h)^2}{2!} f''''(x_n)\right.$$

$$\left. + \frac{(c_j h)^3}{3!} f^{(iv)}(x_n) + \cdots + \frac{(c_j h)^{m-2}}{(m-2)!} f^{(m-1)}(x_n) + U(s,m)\right]$$

Substituting the exact solution values

$$\bar{y}_n = y_n,\, \bar{Y}_{ni} = y(x_n + c_i h),\, 1 \le i \le s,\, n \ge 0 \tag{2.8}$$

$$y(t_n + c_i h) = e^{c_i hL} y_n + c_i h\int_0^1 e^{(1-²)c_i hL} f(x_n + c_i h), \tag{2.9}$$

$$f(x_n + c_i h) = f(x_n) + (c_i h) f'(x_n) + \frac{(c_i h)^2}{2!} f''(x_n) + \frac{(c_i h)^3}{3!} f'''(x_n) + \cdots$$

$$+ \frac{(c_i h)^{m-1}}{(m-1)!} f^{(m-1)}(x_n) + U(s,\, m) \tag{2.10}$$

Inserting (2.10) into (2.9)

$$y(t_n + c_i h) = e^{c_j hL} y_n = h\int_0^1 e^{(1-²)c_i hL}\left[f(t_n) + c_i hf'(t_n) + \frac{(c_i h)^2}{2!} f''(t_n)\right.$$

$$\left. + \frac{(c_i h)^3}{3!} f'''(x_n) + \cdots + \frac{(c_i h)^{m-1}}{(m-1)!} f^{(m-1)}(x_n) + U(s,m)\right] \tag{2.11}$$

Similarly,

$$y'(x_n + c_i h) = e^{c_j hL} y_n + h\int_0^1 e^{(1-²)c_j hL} f'(x_n + c_i h), \tag{2.12}$$

$$f'(x_n + c_i h) = f'(x_n) + c_i hf''(x_n) + \frac{(c_j h)^2}{2!} f'''(x_n)$$

$$+ \frac{(c_i h)^3}{3!} f^{(iv)}(x_n) + \cdots + \frac{(c_i h)^{m-2}}{(m-2)!} f^{(m-1)}(x_n) + U(s,\, m) \tag{2.13}$$

inserting (2.13) into (2.12) gives

$$y'(x_n + c_i h) = e^{c_j hL} y_n + h\int_0^1 e^{(1-²)c_i hL}\left[f'(x_n) + c_i hf''(x_n) + \frac{(c_i h)^2}{2!} f'''(x_n)\right.$$

$$\left. + \frac{(c_i h)^3}{3!} f^{(iv)}(x_n) + \cdots + \frac{(c_i h)^{m-2}}{(m-2)!} f^{(m-1)}(x_n) + U(s.m)\right] \tag{2.14}$$

Subtracting the exact solution from the numerical solution which denotes the defect (Z_{ni}) of the stages and expressing them in terms of h we have

$$h^1 := c_i \int_0^1 e^{(1-²)c_i hL} f(x_n) d² - \sum_{j=1}^{i-1} A_{ij}^{(1)} f(x_n) - \sum_{k=1}^{q-1} U_{ik}^{(1)} f(x_n)$$

$$h^2 := \frac{c_1^2}{2!}\int_0^1 e^{(1-\tau)c_i hL}\ f'(x_n)d\tau \ - c_j\sum_{j=1}^{i-1} A_{ij}^{(1)} f'(x_n) - (-k)\sum_{k=1}^{q-1} U_{ik}\ f'(x_n)$$

$$-\sum_{j=1}^{i-1} A_{ij}^{(2)} f'(x_n)$$

$$h^3 := \frac{c_1^3}{3!}\int_0^1 e^{(1-\tau)c_i hL}\ f''(x_n)d\tau \ - \frac{c_j^2}{2!}\sum_{j=1}^{i-1} A_{ij}^{(1)} f''(x_n) - \frac{(-k)^2}{2!}\sum_{k=1}^{q-1} U_{ik}\ f''(x_n) - \ c_j\sum_{j=1}^{i-1} A_{ij}^{(2)} f''(t_n)$$

$$\vdots$$

$$h^m := \frac{c_1^m}{m!}\int_0^1 e^{(1-\tau)c_i hL}\ f^{(m-1)}(t_n)d\tau \ - \frac{c_j^{m-1}}{(m-1)!}\sum_{j=1}^{i-1} A_{ij}^{(1)} f^{(m-1)}(t_n) - \frac{(-k)^{m-1}}{(m-1)!}\sum_{k=1}^{q-1} U_{ik}^{(1)}\ f^{(m-1)}(t_n)$$

$$-\ \frac{c_j^{m-2}}{(m-2)!}\sum_{j=1}^{i-1} A_{ij}^{(2)} f^{(m-1)}(t_n) \qquad (2.15a)$$

$$Z_{ni} \ = hC_i^1\lambda_1(c_i hl) \ + \ h^2 c_i^2\lambda_2(c_i hl) \ + h^3 c^3\lambda_3(c_i hl) + \cdots + h^Q c_i^Q \lambda_Q(c_i hl)$$

$$-\sum_{j=1}^{i-1}\frac{C_j^{l-1}}{(l-1)!}A_{ij(hL)} \ -\sum_{k=1}^{q-1}\frac{(-k)^{l-1}}{(l-1)!}U_{ik}(hL) f^{(l-1)}(x_n)$$

$$-\sum_{j=1}^{i=1}\frac{C_j^{l-2}}{(l-2)!}A_{ij}^{(2)}(hL) f^{(l-1)}(x_n) \qquad (2.15b)$$

$$Z_{ni} \ = \sum_{l=1}^{Q} h^l T_{li}(hl) f^{(l-1)}(x_n) \qquad (2.16)$$

where

$$T_{li}(hl) \ = \ C_i^l\lambda_l(ch_i hl) - \sum_{j=1}^{i-1}\frac{C_j^{l-1}}{(j-1)!}A_{ij}^{(1)} \ - \left\{\sum_{k-1}^{q-1}\frac{(-k)^{l-1}}{(l-1)!}U_{ik}(hL)\right\} \ - \sum_{j=1}^{i-1}\frac{C_j^{l-1}}{(l-1)!}A_{ij}^{(2)} \qquad (2.17)$$

Likewise, the numerical solution defects equals

$$z_{n+1} \ = \ \sum_{l=1}^{p} h^l w_l(hL) f^{(l-1)}(x_n) + \cdots$$

where

$$w_l(hl) = \lambda_l(hL) \ - \ \sum_{l=1}^{s}\frac{c_i^{l-1}}{(l-1)!}B_i(hL) \ - \ \sum_{k=1}^{q-1}\frac{(k-1)^{l-1}}{(l-1)!}V_k(hL) \qquad (2.18)$$

Definition

Our numerical scheme (1.2) is said to be of stage order Q and order P if $Z_{mi} = 0(h^{Q+1})$ *and* $z_{n+1} = 0(h^{P+1})$. That is, requiring $T_{li}(hL) = 0$ *and* $w_l(hL) = 0$ [5-6], we obtain the order conditions

$$c_i^l\lambda_l(c_i hL) \ = \ \sum_{j=1}^{i-1}\frac{c_j^{l-1}}{(l-1)!}A_{ij}^{(1)}(hL) \ + \ \sum_{k=1}^{q-1}\frac{(-k)^{l-1}}{(l-1)!}U_{ik}(hL) \qquad (2.19)$$

$$+\sum_{j=1}^{i-1}\frac{c_j^{l-2}}{(l-2)!}A_{ij}^{(2)}(hL)$$

$$\lambda_l(hL) = \sum_{i=1}^{s}\frac{c_j^{l-1}}{(l-1)!}B_i(hL) + \sum_{k=1}^{q-1}\frac{(-k)^{l-1}}{(l-1)!}V_k(hL) \qquad (2.20)$$

and so by definition $c_1 = 1$ for all $1 \leq i \leq S$.

3.0 Construction of Extended Exponential General Linear Methods Oredr Five Step Two Stage Order One

The extended exponential general linear methods order five step two stage order one (known as methods 521) is given as

$$y_{n+1} = e^{hl} y_n + hB_1(hL)N(y_n) + hB_2(hL)N(Y_{n2}) \quad (3.1)$$

$$Y_{n2} = e^{c_2 hl} y_n + hA^{(1)}{}_{21}N(y_n) + h^2 A^{(2)}{}_{21}N(y_n) \quad (3.2)$$

using the order conditions (2.19) and (2.20) to determine the coefficient matrix in (3.1) and (3.2)

$$c_1^1 A_{21}^{(1)} + c_1^0 A_{21}^{(2)} + = \hat{\lambda}_2$$

$$\therefore \hat{\lambda}_2 = A_{21}^{(1)} + A_{21}^{(2)}$$

$$\frac{c_1^2 A_{21}^{(1)}}{2!} + c_1^1 A_{21}^{(2)} = c_2^3 \hat{\lambda}_3 (c_2 hl)$$

$$\frac{A_{21}^{(1)}}{2!} + A_{21}^{(2)} = \hat{\lambda}_3$$

$$\frac{c_1^3 A_{21}^{(1)}}{3!} + \frac{c2 A_{21}^{(2)}}{2!} = c_2^4 \hat{\lambda}_4 (hl)$$

$$\frac{A_{21}^{(1)}}{6} + \frac{A_{21}^{(2)}}{2} = \hat{\lambda}_4$$

$$\frac{c_1^4 A_{21}^{(1)}}{4!} + \frac{c_1^3 A_{21}^{(2)}}{3!} = c_2^5 \hat{\lambda}_5 (hl)$$

$$\frac{A_{21}^{(1)}}{24} + \frac{A_{21}^{(2)}}{6} = \hat{\lambda}_5$$

We have:

$$\left.\begin{aligned} \hat{\lambda}_2 &= A_{21}^{(1)} + A_{21}^{(2)} \\ \frac{A_{21}^{(1)}}{2!} + A_{21}^{(2)} &= \hat{\lambda}_3 \\ \frac{A_{21}^{(1)}}{6} + \frac{A_{21}^{(2)}}{2} &= \hat{\lambda}_4 \\ \frac{A_{21}^{(1)}}{24} + \frac{A_{21}^{(2)}}{6} &= \hat{\lambda}_5 \end{aligned}\right\} \quad (3.3)$$

Solving the above equations (3.3) simultaneously we have:

$$A_{21}^{(1)} = \hat{\lambda}_2 - 2\hat{\lambda}_3$$

$$A_{21}^{(2)} = 2\hat{\lambda}_3 - \hat{\lambda}_2$$

Similarly,

$$\left.\begin{aligned} B_1 &= \hat{\lambda}_1 \\ B_2 &= \hat{\lambda}_2 \end{aligned}\right\} \quad (3.4)$$

With

$\hat{\lambda}_l$ -function defined below:

For integers $l \geq 0$ we define $\hat{\lambda}_l$ as

$$\lambda_l(z) = \int_0^1 e^{(1-2)z} \frac{2}{(l-1)!}^{l-1} d2, l \geq 1, \qquad \lambda_0(z) = e^z$$

Consequently, the recurrence relation

$$\lambda_l(z) = \frac{1}{l!} + z\lambda_{l+1}(z), \qquad , l \geq 0$$

Is valid. With

$$\lambda_1(z) = \frac{e^z - 1}{z},$$

$$\lambda_2(z) = \frac{e^z - z - 1}{z^2},$$

$$\lambda_3(z) = \frac{e^z - \frac{z^2}{2} - z - 1}{z^2},$$

4.0 Numerical Experiments

In this section, we compare the accuracy of methods **PSQ** with **S>Q** with other general linear methods.

Problem1. $y' = -10(y-1)^2$, $y(0) = 2$ $\quad with \;\; x£(0,1)$ with theoretical solution given as $y = \dfrac{2+10x}{1+10x}$

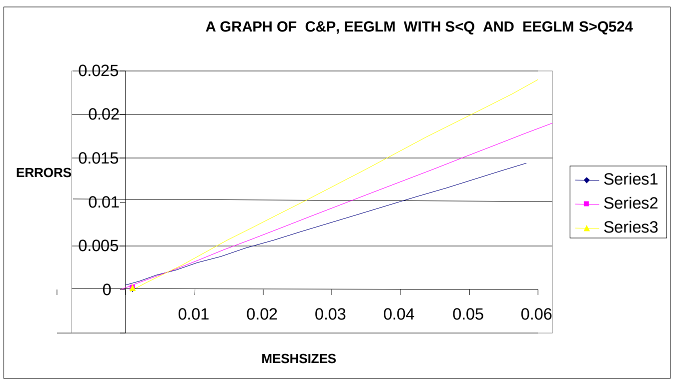

Fig 1: Comparing the accuracies of our method with EEGLM [3] and Calvo and Palencia [4]

Our new scheme 521 exhibits remarkable improvement over the scheme of EEGLM 523 [3] and Calvo and Palencia [4]

5.0 Conclusion

This paper has investigated Extended Exponential General linear methods with the step greater than the stage order, using variation of constant formula. Its order conditions were derived. These order conditions form the basis for the construction of methods S>Q. Experimental experience carried out in this study reveals that our Methods perform better than the scheme EEGLM 523 [3], Calvo and Palencia [4]

6.0 References

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