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Numerical Performances of Two Orthogonal Polynomials in the Tau Method for Solutions of Ordinary Differential Equations

B.M. Yisa

Department of Mathematics, University of Ilorin, Ilorin, Nigeria.

Abstract

In this work, efforts are made at comparing the numerical effectiveness of the two most accurate orthogonal polynomials; Chebyshev and Legendre polynomials. Although the two have different weight functions, but they most time give close results especially when they are considered in the same interval. This work has therefore used the two polynomials, within the same interval, as bases functions in the Ortiz's Recursive Formulation of Lanczos' Tau method. Numerical experiments show that the two are very accurate.

Keywords: Tau System, Canonical Polynomials, Legendre Polynomials, Tau Approximant.

1.0 Introduction

The usefulness of orthogonal polynomials in computational Mathematics and Mathematical Physics can never be over emphasized. The prominent ones are the Chebyshev Polynomial, Legendre Polynomial, Laguerre Polynomial, and Jacobi Polynomial. Of all these, the two the two leading ones are Chebyshev and Legendre Polynomials due to their even distribution of errors as against the growth in error exponentially (as the number of terms appreciates) in Taylor series and the other polynomials [1 – 4]. In Yisa et al. [5] shifted Chebyshev polynomial of the first kind was used in the generalization of Ortiz's Recursive formulation of the Tau method, but here the second in the category of Chebyshev is used so as to bring to fore the effectiveness of shifted Legendre polynomial of the first kind also. And, eventually compare their accuracy in solving homogeneous and non – homogeneous, constant and variable coefficients linear differential equations.

2.0 Chebyshev and Legendre Polynomials

In this section, a short note is given on the two polynomials for the sake of completeness. Consider the differential equations

$$(1-x^2)y''(x)-xy'(x)+n^2y(x)=0, \qquad n<+\infty \tag{2.1}$$

and

$$(1-x^2)y''(x)-2xy'(x)+n(n+1)y(x)=0, \qquad n<+\infty \tag{2.2}$$

The solution of (2.1) and (2.2) are respectively

$$T_n(x)=Cos\left\{nCos^{-1}\frac{(2x-a-b)}{b-a}\right\}, \quad a\le x\le b \tag{2.3}$$

and

$$P_n^*(x)=\sum_{r=1}^{\frac{n}{2}}(-1)^r\frac{(2n-2r)}{2^r r!(n-r)!(n-2r)!}x^{n-2r} \tag{2.4}$$

(2.3) and (2.4) are in more useful form when expressed in the recurrence forms

$$T_{n+1}^*(x)=2T_1^*(x)T_n^*(x)-T_{n-1}^*(x) \tag{2.5}$$

And

Corresponding author: B.M. Yisa, E-mail: yisa.bm@unilorin.edu.ng, Tel.: +234-8108192432

$$P_{n+1}(x) = \frac{1}{n+1}\left\{(2n+1)xP_n(x) - nP_{n-1}(x)\right\}, \quad n \geq 0 \tag{2.6}$$

respectively.

To obtain $P_0(x)$ in (6) we use the Rodrigue's formula

$$P_{n+1}(x) = \frac{1}{(n+1)!}\frac{d^n}{dx^n}(x^2-1)^n, \quad n \geq 0 \tag{2.7}$$

Using (2.5) and (2.6) we obtain the following first few members of the two orthogonal polynomials as (with the interval shifted from [-1,1] to [0,1]): The first four members of Shifted Chebyshev polynomials are: $T_0^*(x) = 1$, $T_1^*(x) = 2x - 1$, $T_2^*(x) = 8x^2 - 8x + 1$, $T_3^*(x) = 32x^3 - 48x^2 + 18x - 1$, $T_4^*(x) = 128x^4 - 256x^3 + 160x^2 - 32x + 1$. The first four members of the shifted Legendre polynomials are: $P_0^*(x) = 1$, $P_1^*(x) = 2x - 1$, $P_2^*(x) = 6x^2 - 6x + 1$, $P_3^*(x) = 20x^3 - 30x^2 + 12x - 1$, $P_4^*(x) = 70x^4 - 140x^3 + 55x^2 - 20x + 1$.

3.0 Canonical Polynomials

Canonical polynomials play a central role in the Recursive formulation of the Tau method. The generation of these polynomials posed a difficulty and was therefore restricted in application to first order ordinary differential equation (ODE) by Lanczos in [4]. This restriction was however lifted in 1969 when Ortiz [6] generated them recursively. An extension of Ortiz's theory of canonical polynomials in the Tau method was reported by Foes [7]. Application of canonical polynomials in the solution of system of ordinary differential equations was reported in Adeniyi et al. [8]. These polynomials were later generalized in Yisa et al. [8, 9] for both over determined and non – over determined linear ODEs. Consider the initial value problem

$$Ly(x) := \sum_{r=0}^{m}\left(\sum_{k=0}^{N_r}\left(P_{r,k}x^k\right)y^{(r)}(x)\right) = \sum_{r=0}^{n} f_r x^r, \quad n < +\infty \tag{3.1a}$$

with the initial conditions

$$L^* y(x_0) = y^{(k)}(x_0) = \sim_k, \qquad k = 1(1)m-1 \tag{3.1b}$$

where N_r, F are given non – negative integers and x_0, α_k, f_r, $P_{r,k}$ are given real numbers.

Definition 3.1

The ODE in (3.1a) is said to be non – over determined if and only if $m \geq N_r$, otherwise, it is said to be overdetermined. Yisa et al. [10] reported the generalized formula that captures the canonical polynomials for the $m-th$ order non – overdetermined ODEs as

$$Q_r(x) = \frac{x^r - \sum_{k=1}^{m}\sum_{j=k}^{m}\left(j!\binom{r}{j}P_{j,j-k}\right)Q_{r-k}(x)}{\sum_{k=0}^{m}k!\binom{r}{k}P_{k,k}}, \quad r \geq 0, \quad j \leq r \tag{3.2}$$

and the derivatives of these polynomials for the same class of problem were also generalized as

$$Q_r^{(\text{a})}(x) = \frac{\text{a}!\binom{r}{\text{a}}x^{r-\text{a}} - \sum_{k=1}^{m}\sum_{j=k}^{m}\left(j!\binom{r}{j}P_{j,j-k}\right)Q_{r-k}^{(\text{a})}(x)}{\sum_{k=0}^{m}k!\binom{r}{k}P_{k,k}}, \quad r \geq 0, \quad j \leq r \tag{3.3}$$

where r is the order of the canonical polynomials, λ is the order of the derivatives and m is order of the differential equation. (3.2) and (3.3) were validated in [9] using the principles of mathematical induction. (For further reading on the validation and implementation of generalized formulae for the canonical polynomials for both over determined and non – over determined cases, interested readers may check [9], [10] and [5]).

4.0 Recursive Formulation of the Tau Method Using Legendre Polynomial as the Basis Function

Lanczos [4] developed the Tau method for solutions of linear polynomial coefficient ODEs using Chebyshev polynomials as the basis function. Fox and Parker [8] also substantiated Lanczos' position with the same argument that Chebyshev polynomials distribute error evenly, the property that is lacking in Taylor's series. However, in the family of orthogonal polynomials (the class which Chebyshev polynomial belongs), Legendre polynomial also possess similar property of even distribution of errors. This paper is therefore studying the performance of Legendre polynomials and to as well see whether it is as effective as Chebyshev polynomials in the Recursive Formulation of the Tau method.

4.1 The Tau Approximant with Legendre Polynomial as Basis Function

In this section, the Tau approximant in [5] for problem (3.1) is reviewed with Legendre polynomial replacing Chebyshev polynomial. We shall as well seek an approximant

$$y_n(x) = \sum_{r=0}^{n} a_r x^r, \quad n < +\infty \tag{4.1}$$

which is the exact solution of the corresponding perturbed problem

$$Ly_n(x) = \sum_{r=0}^{F} f_r x^r + H_n(x) \tag{4.2}$$

where

$$H_n(x) = \sum_{k=1}^{m} \tau_k P_{n-k+1}^{*}(x) \tag{4.3}$$

is the perturbation term. The Tau parameter, $\tau_r, \quad r = 1(1)m$, are to be determined.

$$P_{n+1}^{*}(x) = \frac{1}{n+1}\left\{(2n+1)\frac{(2x-a-b)}{b-a}P_n(x) - nP_{n-1}(x)\right\}, \qquad a \le x \le b \tag{4.4}$$

is the nth degree shifted Legendre polynomial of the first kind with the interval shifted from [-1, 1] to a general interval [a, b] (assuming (3.1) is defined in this interval). We now have from (4.2), with the required substitution made

$$\alpha_\lambda = \sum_{r=0}^{n} f_r Q_r^{(\lambda)}(x) + \sum_{k=1}^{m} \tau_k \sum_{r=0}^{n-k+1} C_r^{(n-k+1)} Q_r^{(\lambda)}(x), \tag{4.5}$$

where λ is the order of the derivatives and it admits the values $\lambda = 0(1)m-1$, m is the order of the given ODE, r is the order of the canonical polynomial and α_k is the set of the initial conditions. It is important to note that the $C_r^{(n-k+1)}$ in (4.5) are the constant coefficients of terms in the Legendre polynomial.

4.3 Tau System Using Legendre Polynomial

The Tau system reported in [5] developed using Chebyshev polynomial as the basis is reviewed in this section with Legendre polynomial performing a function similar to that of Chebyshev. We now put (4.5) in the form

$$\sum_{k=1}^{m} \tau_k \left(\sum_{r=0}^{n-k+1} C_r^{(n-k+1)} Q_r^{(\lambda)}(x) \right) = \alpha_k - \sum_{r=0}^{F} f_r Q_r^{(\lambda)}(x) \tag{4.6}$$

It had, however, been clarified in [5] that F cannot exceed n, to avoid redundant terms $f_{n+1}Q_{n+1}(x)$ on the right of (4.6). We shall now consider some cases for varied values of m, n and λ. $m = 1, n = 3, \lambda = 0$:

$$\tau_1\left(C_0^{(3)}Q_0(x) + C_1^{(3)}Q_1(x) + C_2^{(3)}Q_2(x) + C_3^{(3)}(x)\right)$$

$$= \alpha_0 - f_0Q_0(x) - f_1Q_1(x) - f_2Q_2(x) - f_3Q_3(x) \tag{4.7}$$

$\underline{m = 1, n = 5, \quad \lambda = 0:}$

$$\tau_1\left(C_0^{(5)}Q_0(x) + C_1^{(5)}Q_1(x) + C_2^{(5)}Q_2(x) + C_3^{(5)}(x) + C_4^{(5)}(x) + C_5^{(5)}(x)\right)$$

$$= \alpha_0 - f_0Q_0(x) - f_1Q_1(x) - f_2Q_2(x) - f_3Q_3(x) - f_4Q_4(x) - f_5Q_5(x) \tag{4.8}$$

$\underline{m = 2, n = 3, \lambda = 0,1:}$

$$\begin{pmatrix} C_0^{(3)}Q_0(x) + \ldots + C_3^{(3)}Q_3(x) & C_0^{(2)}Q_0(x) + \ldots + C_2^{(2)}Q_2(x) \\ C_0^{(3)}Q_0'(x) + \ldots + C_3^{(3)}Q_3'(x) & C_0^{(2)}Q_0'(x) + \ldots + C_2^{(2)}Q_2'(x) \end{pmatrix} \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_0 - f_0Q_0(x) - f_1Q_1(x) - f_2Q_2(x) - f_3Q_3(x) \\ \alpha_0 - f_0Q_0'(x) - f_1Q_1'(x) - f_2Q_2'(x) - f_3Q_3'(x) \end{pmatrix} \tag{4.9}$$

$m = 2, n = 4, \lambda = 0,1$:

$$\begin{pmatrix} C_0^{(4)}Q_0(x)+\ldots+C_4^{(4)}Q_4(x) & C_0^{(3)}Q_0(x)+\ldots+C_3^{(3)}Q_3(x) \\ C_0^{(4)}Q_0^{'}(x)+\ldots+C_4^{(4)}Q_4^{'}(x) & C_0^{(3)}Q_0^{'}(x)+\ldots+C_3^{(3)}Q_3^{'}(x) \end{pmatrix}\begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_0 - f_0Q_0(x) - f_1Q_1(x) - f_2Q_2(x) - f_3Q_3(x) - f_4Q_4(x) \\ \alpha_0 - f_0Q_0^{'}(x) - f_1Q_1^{'}(x) - f_2Q_2^{'}(x) - f_3Q_3^{'}(x) - f_4Q_4^{'}(x) \end{pmatrix} \tag{4.10}$$

The values of the tau parameters obtain in the tau system above are resubstituted in (4.11) to get the corresponding values of a_r, $r = 0(1)n$.

$$\sum_{r=0}^{n} a_r Q_r^{(\lambda)}(x) = \sum_{r=0}^{n} f_r Q_r^{(\alpha)}(x) + \sum_{k=1}^{m} \tau_k \sum_{r=0}^{n-k+1} C_r^{(n-k+1)} Q_r^{(\alpha)}(x), \quad \alpha = 0(1)m-1 \tag{4.11}$$

For $m = 1$ and $n = 4$, we have upon expansion

$$a_0Q_0(x)+a_1Q_1(x)+a_2Q_2(x)+a_3Q_3(x)+a_4Q_4(x)$$
$$= f_0Q_0(x)+f_1Q_1(x)+f_2Q_2(x)+f_3Q_3(x)+f_4Q_4(x)$$
$$+\tau_1\left(C_0^{(4)}Q_0(x)+C_1^{(4)}Q_1(x)+C_2^{(4)}Q_2(x)+C_3^{(4)}Q_3(x)+C_4^{(4)}Q_4(x)\right) \tag{4.12}$$

which can be further simplified to get

$$a_0Q_0(x)+a_1Q_1(x)+a_2Q_2(x)+a_3Q_3(x)+a_4Q_4(x)$$
$$=(f_0+\tau_1C_0^{(4)})Q_0(x)+(f_1+\tau_1C_1^{(4)})Q_1(x)+(f_2+\tau_1C_2^{(4)})Q_2(x)$$
$$+(f_3+\tau_1C_3^{(4)})Q_3(x)+(f_4+\tau_1C_4^{(4)})Q_4(x) \tag{4.13}$$

Comparing the coefficients of $Q_r(x), r = 0(1)n$ in (4.13) gives $a_0 = f_0 + \tau_1 C_0^{(4)}$, $a_1 = f_1 + \tau_1 C_1^{(4)}$, $a_2 = f_2 + \tau_1 C_2^{(4)}$, $a_3 = f_3 + \tau_1 C_3^{(4)}$, and $a_4 = f_4 + \tau_1 C_4^{(4)}$. The values of τ parameters and those of a_r's are to be substituted in the equation

$$y_n(x) = \sum_{r=0}^{n} a_r Q_r(x) \tag{4.14}$$

to get the desired approximate solution.

5.0 Numerical Experiments

In this section, the results presented above are applied to some selected problems that include, constant and variable coefficients homogeneous ODEs, and constant and variable coefficients non – homogeneous ODEs. The tables of results show the results obtained from both Legendre and Chebyshev polynomials. The Mathematica – Tau – Program (MTP) developed to handle the Numerical Problems in [5] is again modified here to handle the problems solved in the present work. Meanwhile, the errors presented in the tables of results below for the respective problems are the absolute values of errors obtained from the equation

$$Error = \max : 0 \le x \le 1 \left| y(x) - y_n(x) \right| \tag{5.1}$$

where $y(x)$ is the exact solution and $y_n(x)$ is the approximate solution obtained through the use of Chebyshev polynomial or Legendre polynomial as basis function. The interval [0, 1] is partitioned in each case into 100 points; 0.001, 0.002, … , 1.0 and the maximum error is selected.

PROBLEM 5.1 Consider the first order variable coefficient homogeneous IVP

$$(1+x)y'(x)+y(x)=0, \qquad 0 \le x \le 1$$

with the initial conditions $y(0) = 1$, and analytic solution $y(x) = \frac{1}{1+x}$. Table 5.1 shows the actual errors for both Chebyshev and Legendre polynomials for Problem 5.1.

Table 5.1: Actual Errors for Problem 5.1

n	Chebyshev Error	Legendre Error
5	1.1195E-4	5.92361E-5
6	1.67144E-5	9.95122E-6
7	2.77898E-6	1.68028E-6
8	4.83153E-7	2.88852E-7
9	8.15791E-8	4.94033E-8
10	1.36163E-8	8.38007E-9
11	2.2492E-9	1.44438E-9

Problem 5.2 Considerthe second order variable coefficient homogeneous IVP

$$y''(x)-(1-x)y'(x)+y(x)=0, \qquad 0\le x\le 1$$

with the initial condition $y(0)=1, y'(0)=1$ and analytic solution $y(x)=e^{x-x^2/2}$ .The extracted values are $m=2, P_{0,0}=1, P_{1,0}=-1, P_{1,1}=1, \alpha_0=1, \alpha_1=1$ and $f_0=0$. The results obtained from MTP are shown in Table 5.2.

Table 5.2:Actual Errors for Problem 5.2

n	Chebyshev Error	Legendre Error
5	9.41617E-5	2.03411E-5
6	2.61675E-5	5.03838E-6
7	1.81163E-7	6.73136E-8
8	1.44989E-7	3.31498E-8
9	1.07751E-9	2.13588E-10
10	7.67658E-10	1.78849E-10
11	1.20995E-11	2.53395E-12

Problem 5.3

Consider the third order constant coefficient non – homogeneous IVP

$$y'''(x)+2y''(x)-9y'(x)-18y(x)=-18x^2-18x+22, \qquad 0\le x\le 1$$

with the initial conditions $y(0)=-2,\ y'(0)=-8,\ y''(0)=-12$ and the exact solution $y(x)=e^{-2x}-1-2e^{3x}+x^2$. The extracted values for this problem are: $m=3,\ P_{0,0}=-18,\ P_{1,0}=-9,\ P_{2,0}=2, P_{3,0}=1, \alpha_0=-2, \alpha_1=-8, \alpha_2=-12, f_0=22, f_1=-18$ and $f_2=-18$. The results are as shown in Table 5.3.

Table 5.3: Actual Errors for Problem 5.3

n	Chebyshev Error	Legendre Error
5	1.75173E0	3.03812E-1
6	2.06632E-1	1.36600E-2
7	1.71935E-2	6.03022E-4
8	1.33989E-3	4.49930E-5
9	9.55160E-5	3.03548E-6
10	6.59794E-6	1.89227E-7
11	4.20684E-7	1.12886E-8

The results presented above show a superiority in performance of Legendre polynomials over Chebyshev, when the two were used within the same interval. This has really shown that Legendre polynomial can be more accurate than Chebyshev, despite the minimax property of the latter. Thus, most computational problems where Chebyshev polynomial performs well can be carried out using Legendre polynomial too.

6.0 Conclusion:

The Recursive Formulation of the Tau method using the Chebyshev and Legendre polynomials have been presented. Unlike in [5], where the generalization and automation were carried out using the later, here the latter has been incorporated and the results correct the notion that only Chebyshev polynomial can give reliable results. Legendre polynomial has performed excellently well too, has shown in this work.

7.0 References

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