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New Implicit General Linear Method

[1]M.O. Ibrahim and [2]A.Mustafa

[1]Department of Mathematics Usmanu Danfodiyo University, Sokoto
[2]Department of Mathematics University of Illorin, Illorin.

Abstract

A New implicit general linear method is designed for the numerical solution of stiff differential Equations. The coefficients matrix is derived from the stability function. The method combines the single-implicitness or diagonal implicitness with property that the first two rows are implicit and third and fourth row are explicit. Also the last row of A and U matrix are identical to the first row of B and V of the partitioned block matrix. The method is almost A stable of order four and it has four stages. This has more advantage than the backward differentiation formula in which no A-stable method can be found with order greater than two. This paper also reviews an algorithm for determining the coefficients matrix from the stability function. It is also shown that the new method is less expensive compared to most existing methods.

Keywords: General Linear Methods, stiff differential equations, singly-Implicit Methods.

1.0 Introduction

A general linear method was designed as unifying frame work between the linear multi step and Runge-Kutta methods [1]. General linear method is designed to overcome some of the limitations of both linear multi step and Runge–Kutta, where the advantage of good stability of Runge-Kutta and advantage of multi stage in linear Multi step method are combined and also removed some of the disadvantage of the method and come up with a single method. The general linear method is designed to approximate the solution to stiff differential equations problems of the type.

$$y^1 = f(x) \quad y(x_0) = y_0$$
$$f : R^n \to R^n \quad y(x_0) = y_0 \tag{1}$$

At the start of step number n, r vectors $y_i^{[n-1]}, \quad i = 1,2\ldots, r$, are used as input and at the end, of the step the output $y_i^{[n]}, i = 1,2\ldots, r$. Like Runge–Kutta method, there are S stage values given as $y_i, \quad i = 1,2\ldots, s$ to be computed as well as the corresponding stage derivative $F_i = f(Y_i), \quad i = 1,2,\ldots, s$. These equations are [2]

$$Y_i = h\sum_{j=1}^{s} a_{i,j}F_j + \sum_{j=1}^{r} u_{i,j}y_j^{[n-1]} \quad i = 1,2,\ldots, r$$
$$y_i^{[n]} = h\sum_{j=1}^{s} b_{i,j}F_j + \sum_{j=1}^{r} v_{i,j}y_j^{[n-1]} \quad , i = 1,2,\ldots, r \tag{2}$$

Where the coefficients $a_{i,j}, u_{i,j}, b_{i,j}, v_{i,j}$ comprise the (s + r) X (s + r) partitioned matrix, given as

$$q = \begin{bmatrix} A & U \\ B & V \end{bmatrix} \tag{3}$$

Corresponding author: A. Mustafa, E-mail: mustafa.aminu.udusok.edu.ng, Tel.: +2347033366073, 2348095500073

Now if we write

$$y^{[n-1]} = \begin{bmatrix} y_1^{[n-1]} \\ y_2^{[n-1]} \\ \vdots \\ y_r^{[n-1]} \end{bmatrix}, \quad y^{[n]} = \begin{bmatrix} y_1^{[n]} \\ y_2^{[n]} \\ \vdots \\ y_r^{[n]} \end{bmatrix}, Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_2 \end{bmatrix}, F = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_3 \end{bmatrix} \qquad (4)$$

which can be written in a compact form

$$\begin{aligned} Y &= h\left(A\otimes I_N\right)F + \left(U\otimes I_N\right)y^{[n-1]} \\ y^n &= h\left(B\otimes I_N\right)F + \left(V\otimes I_N\right)y^{[n-1]} \end{aligned} \qquad (5)$$

To overcome some of the limitation of Singly Implicit Runge-Kutta (SIRK) methods, various amendments have been made in this paper, by appending additional diagonally-implicit stages to be appended to the singly-implicit part of A [3,4] and also by allowing free parameters in the newly appended stages for possible choice of methods. Another similar method was developed, where the method is required to have a last stage of A and U to be identical to the output value [2].
The paper is organized as follows, In section 2 we discuss the special general linear method in detail and explain some of the reasons leading to the formulation and the choice of structure. In section 3, we present the derivation of the structure of the new method. Section 4, consists of deriving specific methods based on certain choice of parameters to come up with a new method and stability analysis of the derived method.

2.0 The formulation of the new Method

The formulation used in this paper follows Burrage and Butcher [5].They used a partitioned (s + r) x(s + r) matrix to represent a general linear method. This matrix contains four matrices, A, B, U, and Vas can be seen in equation (3).The coefficients of those matrices indicate the relationships among various numerical quantities that arises in the computation. The structure of leading coefficient matrix A determines the implementation costs of these methods, In this formulation of the new Special General Linear Method (SGLM) a certain restriction and condition are imposed to have low costs. The formulations are based on the following restrictions:

(i) The Matrix A is required to have all eigen values equal to one.
(ii) Two additional diagonal stage with 1 at diagonal
(iii) Use parameter for value of c_3
(iv) Use $c_4 = \bar{}$ with free parameter u_{43}
(v) First row of [BV] same as last row of [AU].
(vi) second row of [BV] should be [0,0,0,1,0,0,0]
(vii) the final row of [BV] is computed with v_{32} and v_{33} as free parameters
(viii) $v_{31} = 0$, $v_{33} = 0$

The main aim of the first restriction will be to select the coefficient of A-matrix so that it will have one point spectrum with single eigen values equal to a i.e a zeros of Laguerre polynomials [1]. In our case the eigen values of A-matrix should be exactly equal to one. This can be achieved by using the stability function, given as

$$\begin{aligned} \exp(z\bar{}) &= \frac{a_0 + a_1 z + \ldots\ldots a_s z^s}{(1-z\bar{})^s} a_0 + a_1 z + \ldots\ldots a_s z^s \\ &= \exp(z\bar{})(1-z\bar{})^s \end{aligned} \qquad (6)$$

then A-matrix satisfies the conditions of single-implicitness $\circ(A) = 1$ and stage order s

$$C(s): \sum_{j=1}^{s} a_{i,j} c_j^{k-1} = \frac{1}{k} c_i^k, i = 1,2,\ldots s \qquad (7)$$

Together with following ideas since a_s are related to the abscissas of A-matrix, then it can be obtained from the formula outline:

$$\left.\begin{aligned} a_s &= L_s(\,)(-1)^s a_{s-1} \\ &= \frac{d}{d}\, a_s(\,)\, a_{s-2} \\ &= \frac{d}{d}\, a_{s-1}(\,)\, a_0 \\ &= a_{s+1} \\ &= \int_0^z a_s(z)dz \end{aligned}\right\} \tag{8}$$

where L_s is Laguerre Polynomials.

This requirement is imposed to have the order of the method to be equal to the stage order, i.e p=s. This requirement will be useful when it comes to implementation issues, where we need a matrix T, for transformation. It increases the accuracy of method, since the order of our proposed GLM is n and the stage is n-1. Therefore to satisfy the stage order, n, an additional stage order is needed to make the order of method equal to the stage order i.e. $(I-A)^{n+2}=0$. This can be seen from the given lemma:

Lemma 1.0

A Runge-Kutta method with an invertible matrix A satisfies C(n) if and only if $A^{-1}\ \ (c) = \ '(c) + \ \ (0)A^{-1}e$ for all polynomials $\ (x)$ of degree not exceeding n

Proof. 1.0 [6]

Another fact is from the theorem which states that:

Theorem 1.0

A DIMSIM has order and stage order p if and only if

$$\left.\begin{aligned} \exp(cz) &= zA\exp(cz) + UZ + O(z^{p+1}) \\ \exp(\,z)Z &= zB\exp(cz) + VZ + O(z^{p+1}) \end{aligned}\right\} \tag{10}$$

where $\exp(cz)$ denotes the vector with components given by $\exp(c_i z), i=1,2,..s$ and Z denotes the vector with elements given by

$$Z_i = \sum_{k=0}^{p} a_{ik} z^k, i=1,2,\dots r \tag{11}$$

Proof.[7]

It is also clear from the above theorem the need for two additional free stages.

In this formulation, the abscissas c_3 will be set as free parameters, because of the stability criterion. We don't know which value of abscissas c_3 will produces a good stability, since our aim is to apply the method for stiff problem, and it is generally accepted that for a method to solve stiff problem at reasonable steps size, the method has to be A-stable or Almost A-stable at infinity. Many lemmas and theorem exists for this requirement, the same reason for $c_4 =$ and u_{43} as free parameter.

Another restriction also imposed in the formulation of the new method is that the First row of [BV], should be the same as last row of [AU]. This is known as FSA and it is also defined as F-property in [7]. This property is needed in the practical implementation of the method as stated in [7] and it is also used to widen the options for error estimates in GLM.

Definition 1.0

A general linear method with IRKS property has property F if

(i) $c_s = 1$

(ii) $b_{1,j} = a_{s,j}$ for, $j=1,2,\dots,s$

(iii) $v_{1,j} = v_{s,j}$

(iv) $b_{2,j} = ¢_{s,j}$ for, $j=1,2,\dots,s$

(v) $v_{2,j} = 0,\ for,\ j=1,2,\dots,r$

Also the third row of [BV] is computed v_{32} and v_{33} as free parameter, also with $v_{33}=0$.

Also, the second row of [BV], should be [0,0,0,1,0,0,0] ,this restriction will be imposed into the formulation of the method due to requirement of V to be bounded, because the stability of the method depends on the V-matrix and it is always required that V be power bounded not only power-bounded, but also the V-matrix should be of rank 1,which not only guarantees power-bounded but also the stronger condition of zero-stability. This is backed-up by the definition in [7], which states as follows.

Definition 1.1

A GLM (A, U, B, V) is "stable" if there exists a constant *k* such that

$$\|V^n\| \le K, \quad for \quad n = 1, 2, \ldots \tag{11}$$

3.0 Derivation and the structure of the New (SGLM) Method

Now, having outlined the requirements and restriction on the new method, we will now outline steps on how to derive the new method in line with the requirements and restriction.

i) Obtaining the abscissas that will insure all spectrum of A-matrix to be one:
ii) Appending two additional stages to meet stage order condition
iii) Obtaining coefficients of A and U-matrices, and using the idea of stage-order and
iv) order conditions that was highlighted in the formulation stages
v) Obtaining coefficients of B and V-matrices, and using the same idea in (iii)
vi) Subjecting the derived method to test whether the requirements mentioned in (ii) are met.

Using the formulation insection (2), the two zeros of Laguerre polynomial were obtained as follows:
For given n-degree polynomials in (4) and setting s=2, we have

$$a_2 = (-1)^2 L_2(\bar{\ }) \tag{12}$$

$$L_2(\bar{\ }) = 1 - 2\bar{\ } + \frac{1}{2}\bar{\ }^2 \tag{13}$$

Where

$$a_1 = \frac{d}{d\bar{\ }}(a_2) \tag{14}$$

$$= -2 + \bar{\ }$$

$$a_0 = 1 \tag{15}$$

Where the values a_s are found by integrating a_{s+1} from 0 to z

$$\left.\begin{aligned} a_2 &= 1 - 2\bar{\ } + \frac{1}{2}\bar{\ }^2 \\ a_3 &= \int_0^z a_2\,dz \\ &= z - z^2 + \frac{1}{6}z^3 \\ a_4 &= \int_0^z a_3\,dz \\ &= \frac{1}{2}z^2 - \frac{1}{3}z^3 + \frac{1}{24}z^4 \\ &= (z-2)(z-6) \\ & z_1 = 2 \quad, z_2 = 6 \end{aligned}\right\} \tag{16}$$

where z_1 ,and z_2 are terms as the c_1, and c_2. Therefore, the two zeroes are used as abscissas for the new method.

Two additional stages were also appended, with two arbitrary abscissas as free parameters, $c3 = a$ and $c4 = \bar{\ }$. This ensures the full block A matrix, which also ensures that s=p, but also improves the performance of the methods.

3.1 Structure of the A-U-B-V-matrix

$$\begin{bmatrix} A & U \\ B & V \end{bmatrix} = \begin{bmatrix} a_{1,1} & a_{1,2} & 0 & 0 & u_{1,1} & u_{1,2} & u_{1,3} \\ a_{2,1} & a_{2,2} & 0 & 0 & u_{2,1} & u_{2,2} & u_{2,3} \\ a_{3,1} & a_{3,2} & 1 & 0 & u_{3,1} & u_{3,2} & u_{3,3} \\ a_{4,1} & a_{4,2} & a_{4,3} & 1 & u_{4,1} & u_{4,2} & u_{4,3} \\ b_{1,1} & b_{1,2} & b_{1,3} & 1 & v_{1,1} & v_{1,2} & v_{1,3} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ b_{3,1} & b_{3,2} & b_{3,3} & 1 & v_{3,1} & v_{3,2} & v_{3,3} \end{bmatrix} \qquad (17)$$

3.2 Obtaining coefficients of A-U-B-V-matrices

To obtain the coefficients, considering conditions and requirements with available information and the abscissas, we use the idea stated in theorem 1.0.

Now to obtain the coefficients of block A-B-matrix we consider equations (10) and (11). But here we will only derive for illustrations purposes, for the first row, since other rows of A and U follow the same procedure;

$$\left.\begin{aligned} \exp(2z) &= za_{11}\exp(2z) + za_{12}\exp(6z) + 1 + u_{12}z + u_{22}z^2 + O(z^5) \\ \exp(6z) &= za_{21}\exp(2z) + za_{22}\exp(6z) + 1 + u_{22}z + u_{23}z^2 + O(z^5) \end{aligned}\right\} \qquad (18)$$

Then solving the above equations simultaneously, we now obtain the coefficient of the first row, second row, third row and fourth row ofA-U-Matrix as follows;

$$\begin{bmatrix} A & U \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{-1}{108} & 0 & 0 & 1 & \frac{34}{27} & \frac{5}{9} \\ \frac{27}{4} & \frac{5}{4} & 0 & 0 & 1 & -2 & -3 \\ a_{3,1} & a_{3,2} & 1 & 0 & 1 & u_{3,2} & u_{3,3} \\ a_{4,1} & a_{4,2} & a_{4,3} & 1 & 1 & u_{4,1} & u_{4,2} \end{bmatrix} \qquad (19)$$

Where

$$\left.\begin{aligned} a_{3,1} &= \frac{3}{16}\text{a}^3 - \frac{1}{64}\text{a}^4 - \frac{3}{8}\text{a}^2 \\ a_{3,2} &= -\frac{5}{432}\text{a}^3 + \frac{1}{576}\text{a}^4 + \frac{1}{72}\text{a}^2 \\ u_{3,2} &= \text{a} - \frac{19}{108}\text{a}^3 + \frac{1}{72}\text{a}^4 + \frac{13}{36}\text{a}^2 - 1 \\ u_{3,3} &= -\text{a} - \frac{11}{36}\text{a}^3 + \frac{1}{48}\text{a}^4 + \frac{76}{}\text{a}^2 \\ a_{4,1} &= -\frac{1}{96}\,\frac{72\text{a}\text{¯} - 48\text{a}\text{¯}^2 + 72\text{a}u_{4,3} + 4\text{a}\text{¯}^3 - 72\text{¯}^2 + 36\text{¯}^3 - 3\text{¯}^4}{\text{a} - 2} \\ a_{4,2} &= \frac{1}{288}\,\frac{4\text{a}\text{¯}^3 + 24\text{a}\text{¯} - 24\text{a}\text{¯}^2 + 24\text{a}u_{4,3} - 24\text{¯}^2 + 20\text{¯}^3 - 3\text{¯}^4}{\text{a} - 6} \\ u_{4,2} &= \frac{1}{144}\,\frac{-144\text{a} + 240\text{a}\text{¯} - 60\text{a}\text{¯}^2 + 96\text{a}u_{4,3} + 4\text{a}\text{¯}^3 + 144\text{¯} + 144u_{4,3} + 44\text{¯}^3 - 3\text{¯}^4 - 168\text{¯}^2}{\text{a}} \end{aligned}\right\} \qquad \{20\}$$

3.3 Coefficients of the A, U, B and V Matrices

Now, the new method and the five free parameters will be chosen to have a good stability. The five free parameter and complete block method are given below;

The free parameters are, $c_3 = \text{a}, c_4 = \text{¯}, v_{3,2}, v_{3,3}, and\ u_{4,3}$

$$\begin{bmatrix} A & U \\ B & V \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{-1}{108} & 0 & 0 & 1 & \frac{34}{27} & \frac{5}{9} \\ \frac{27}{4} & \frac{5}{4} & 0 & 0 & 1 & -2 & -3 \\ a_{3,1} & a_{3,2} & 1 & 0 & 1 & u_{3,2} & u_{3,3} \\ a_{4,1} & a_{4,2} & a_{4,3} & 1 & 1 & u_{4,2} & u_{4,3} \\ a_{4,1} & a_{4,2} & a_{4,3} & 1 & 1 & u_{4,2} & u_{4,3} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ b_{3,1} & b_{3,2} & b_{3,3} & 1 & 1 & v_{3,2} & v_{3,3} \end{bmatrix} \quad (21)$$

4.0 Stability Analysis of the New SGLM

Now, before analyzing stability of the new method, some definitions are required.

Definition 2.1 (Stability matrix)

For a general linear method (A, U, B, V), the stability matrix M (z) is defined

$$M(z) = V + zB(I - zA)^{-1}U \quad (22)$$

Definition 2.2 (Order and stage order)

A method with stability function $\Phi(w,z)$, has stability order *p* if

$$\Phi(\exp(z), z) = O(z^{p+1}) \quad (23)$$

Definition 2.3 (Characteristic polynomials)

If stability matrix for GLM has the form in (22), then the characteristic polynomial is

$$\Phi(w,z) = \det(wI - M(z)). \quad (24)$$

But it was stated in (8), that going by this will only complicate the analysis, instead a method was used for which *P(w,z)* is factorized.

Definition 2.4 (Runge-Kutta stability)

A general linear method (A, U, B, V) has 'Runge-Kutta stability' if the characteristic polynomials given by definition (2.3) has the form,

$$\Phi(w,z) = w^{r-1}(w - R(z)) \quad (25)$$

For a method with Runge-Kutta stability, the rational function *R (z)* is known as the stability *function* of the method.

For easy stability analysis of the New method, a simple modification, of U, B, V-matrix is required which does not change the meaning and structure of method, since characteristic polynomial of M1 and $\tilde{M}1$ are same except for factor w, therefore is almost similar to use $\tilde{M}1$ for the analysis of the method.

This modification is implemented with simple assumptions that, the inputs to the first step are not $y_1^{[0]}, y_2^{[0]}$, and $y_3^{[0]}$ as in the usual formulation but instead

$y_1^{[0]}, zy_1^{[0]}$, and $y_2^{[0]}$ where we use the fact that the second input is always exactly the derivative, calculated from the first input. The vector of stages values and the output values satisfy:

$$Y = zAY + U\begin{bmatrix} 1 & 0 \\ z & 0 \\ 0 & 1 \end{bmatrix} y^{[0]} \quad (26)$$

$$y^{[1]} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\left(BY + V\begin{bmatrix} 1 & 0 \\ z & 0 \\ 0 & 1 \end{bmatrix}\right) y^{[0]} \quad (27)$$

Therefore, if the stability matrix of the usual formulation is

$$M(z) = V + zB(I - zA)^{-1}U$$

Then the modified BUV-matrix will be

$$\tilde{M} = \tilde{V}1 + z\tilde{B}1\left(I - zA\right)^{-1} \tilde{U}1. \quad (28)$$

If

$$\tilde{U}1 = \tilde{U}T, \tilde{B}1 = \tilde{S}B, \tilde{V}1 = SVT \quad (29)$$

Where

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, T = \begin{bmatrix} 1 & 0 \\ z & 0 \\ 0 & 1 \end{bmatrix} \quad (30)$$

Now having our New method which depend on five parameters, then we check for the stage order and order condition. This is achieved by finding the stability matrix of the method as the characteristic polynomial of the matrix, which is exactly the same as taking the Taylor expansion of characteristic polynomials in terms of w and z about z=0, which we obtained. This is simply obtained by using Maple code as follows.

The stability matrix M, given by;

$$M = V + zB.(I - zA)U\} \quad (31)$$

Is said to be A-stable if M is power-bounded for z in the left-plane

Equation (31) confirmed that the method is of order 4, since all the terms contain zero coefficient up to z^4.Then to check for stage order , we use C(s) condition, which satisfies the following;

$$\left.\begin{aligned} A1 &= c \\ Ac &= \frac{1}{2}c^2 \\ Ac^2 &= \frac{1}{3}c^3 \\ Ac^3 &= \frac{1}{4}c^4 \end{aligned}\right\} \quad (32)$$

This can be proved by using Cayley-Hamilton theorem, which states that:

Theorem: 2.1

$$(A - \alpha I)^s c^s = 0 \quad (33)$$

By letting s=2 and $\alpha = 1$,we've

$$\left.\begin{aligned} Ac^2 &= \frac{1}{3}c^3 \\ A^2c^2 &= \frac{1}{3}Ac^3 \\ A^2c^2 &= \frac{1}{12}c^4 \end{aligned}\right\} \quad (34)$$

$$\left.\begin{aligned} & A^2c^2 - 2Ac^2 + c^2 = 0 \\ & \frac{c^4}{12} - \frac{2c^3}{3} + c^2 = 0 \\ & \frac{c^2}{12}(c^2 - 8c + 12) = 0 \\ & c^2 - 8c + 12 = 0 \\ & c_1 = 2, c_2 = 0 \end{aligned}\right\} \quad (35)$$

Here, we analyze whether the new method satisfies the single-implicitness, since for method to satisfy single-implicitness, the matrix A must have one-point spectrum with single eigenvalues, i.e. $\sigma(A) = \{\alpha\}$. This was achieved by finding the eigen values of the derived method in terms of its free parameters, which shows that the method has single eigenvalues all equal to one. Now, having checked the stage order of the derived method, we now search for good stability. To perform this task a powerful Matlab and Maple code was developed, to search for best values for each parameter in which A-stable methods are sought. After rigorous search we were able to have a convenient choice of these parameters as follows:

The abscissas are $[2,6,4,9/2], v_{32} = \frac{1901}{59762}, u_{43} = \frac{7576}{6627}, v_{33} = 0$,so that the tableau defining the method is found to be:

$$\begin{bmatrix} A & U \\ B & V \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{-1}{108} & 0 & 0 & 1 & \frac{34}{27} & \frac{5}{9} \\ \frac{27}{4} & \frac{5}{4} & 0 & 0 & 1 & -2 & -3 \\ a_{3,1} & a_{3,2} & 1 & 0 & 1 & u_{3,2} & u_{3,3} \\ a_{4,1} & a_{4,2} & a_{4,3} & 1 & 1 & u_{4,2} & u_{4,3} \\ a_{4,1} & a_{4,2} & a_{4,3} & 1 & 1 & u_{4,2} & u_{4,3} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ b_{3,1} & b_{3,2} & b_{3,3} & b_{3,4} & 1 & v_{3,2} & v_{3,3} \end{bmatrix} \tag{36}$$

$$A = \begin{bmatrix} \frac{3}{4} & -\frac{1}{108} & 0 & 0 \\ \frac{27}{4} & \frac{5}{4} & 0 & 0 \\ 2 & -\frac{2}{27} & 1 & 0 \\ \frac{2502889}{2262016} & -\frac{5727845}{20358144} & \frac{2237671}{2262016} & 1 \end{bmatrix}, U = \begin{bmatrix} 1 & \frac{34}{27} & \frac{5}{9} \\ 1 & -2 & -3 \\ 1 & \frac{29}{27} & \frac{4}{9} \\ 1 & \frac{34316309}{20358144} & \frac{7576}{6627} \end{bmatrix} \tag{37}$$

$$B = \begin{bmatrix} \frac{2502889}{2262016} & -\frac{5727845}{20358144} & \frac{2237671}{2262016} & 1 \\ 0 & 0 & 0 & 1 \\ -\frac{320973}{2390480} & \frac{286277}{1434288} & -\frac{242907}{239048} & \frac{411914}{448215} \end{bmatrix}, B = \begin{bmatrix} 1 & \frac{34316309}{20358144} & \frac{7576}{6627} \\ 0 & & 0 \\ 0 & \frac{1901}{59762} & 0 \end{bmatrix} \tag{38}$$

4.1 Stability Region

The stability region of the derived method was also drawn to ascertain the A-stability with chosen parameter. The region has shown us that the method is almost A-stable with real minimum error of -0.001, which means the method is $A(\sim)$-stable. But we believe that with more rigorous search the A-stable or even L-stable method can emerge from this formulation.
The region is shown in Figure 1.

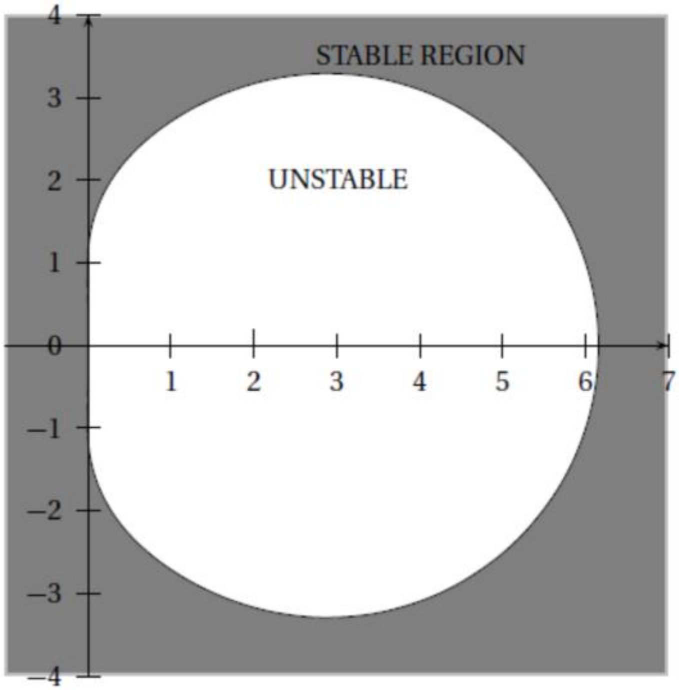

Figure 1: Region of A-stability of the new method

5.0 Conclusion

This paper derives a new implicit general linear method (NIMGLM), with real and exactly equal one point spectrum, which will reduce the implementation cost of method. The method has a wide region of A-stability on the left half plane which is a requirement for a method to be good candidate in solving stiff differential equation, as well as the oscillatory differential equations. Considering the algebraic analysis of the method, it is shown that the new special type of general linear method has the potential of solving stiff ordinary differential equations, with low cost of implementation, which possesses the same stage order as well as the overall order of the method which is $p=s$. The method has four stages with two implicit stages and two explicit stages, This advantage lowers the cost of implementation compared to higher order BDF methods. Future work will address the implementation issues of the new (NIMGLM) method.

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