Journal of the Nigerian Association of Mathematical Physics
***Volume 30,* (May, *2015*), pp1 – 4**

On Some New Extensions and Generalizations of Eneström-Kakeya Theorem

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Abstract

In this paper we obtain some new extensions and generalizations of the well-known classical theorem of Eneström and Kakeya.

Keywords and Phrases: Complex number, Polynomial, Zeros, Eneström-Kakeya theorem, Bounds, Modulii, Disk.

1.0 Introduction

The following important result due to Eneström and Kakeya [1] is well known in the theory of the location and distribution of the zeros of polynomials:

Theorem 1.1*:* Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n whose coefficients a_j satisfy

$a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0 > 0.$

Then all the zeros of $P(z)$ lie in the closed unit disk $|z| \leq 1$.

In the literature, there exist some extensions and generalization of Theorem 1.1.

Joyal et al. [2] extended Theorem 1.1 to polynomials whose coefficients are monotonic but not necessarily non-negative by proving the following results:

Theorem 1.2: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that

$a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0 > 0.$

Then all the zeros of $P(z)$ lie in

$|z| \leq \frac{a_n + |a_0| - a_0}{|a_n|}.$

Recently, Aziz and Zagar [3] relaxed the hypothesis of Theorem 1.1 in several ways and they proved the following results:

Theorem1.3*:* Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $k \geq 1$,

$ka_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0 > 0.$

Then all the zeros of $P(z)$ lie in

$|z + k - 1| \leq k.$

Theorem 1.4: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $k \geq 1$

$ka_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0.$

Then all the zeros of $P(z)$ lie in

$|z + k - 1| \leq \frac{ka_n + |a_0| - a_0}{|a_n|}.$

Theorem 1.5: *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n, If for some positive number k and ρ *with* $k \geq 1$ *and* $0 < \rho \leq 1$,

$ka_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0 > 0.$

Then all the zeros of $P(z)$ lie in the closed disk

$|z + k - 1| \leq k + \frac{2a_0}{a_n}(1 - \rho).$

More recently, Gulzar [4] generalized these results to the class of polynomials with complex coefficients by proving the following result.

Theorem 1.6: *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n with complex coefficients such that $\mathrm{Re}\, a_j = \alpha_j$, $\mathrm{Im}\, a_j = \boldsymbol{\beta}_j$, $j = 0, 1, \ldots, n$ and for some real number $\rho \geq 0$

$\rho + \alpha_n \geq \alpha_{n-1} \geq \ldots \geq \alpha_1 \geq \alpha_0.$

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Then $P(z)$ has all its zeros in

$$\left|z+\frac{\rho}{\alpha_n}\right| \le \frac{\rho+\alpha_n+|\alpha_0|+2\sum_{j=o}^{n}|\beta_j|}{|\alpha_n|}.$$

In this paper, we state and prove some new extensions and generalizations of Eneström – Kakeya theorem.

2.0 Main Results

Theorem 2.1: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients, whose coefficients satisfies $\mu \ge 0$, $0 \le \tau < 0$ and $0 \le \lambda \le n-1$ such that

$(\mu+1)a_n \le a_{n-1} \le \ldots \le a_\lambda \ge \ldots \ge a_1 \ge (1-\tau)\, a_0 > 0.$

Then $P(z)$ has all its zeros in

$$|z+\mu| \le \frac{2a_\lambda}{a_n}-\mu-1+\frac{2a_0}{a_n}\tau.$$

Observe that in Theorem 2.1, If we choose μ and τ such that $\mu = k-1$ and $\tau = 1-\rho$, we have the following corollary.

Corollary 2.1: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n, where coefficients satisfies

$$k a_n \le a_{n-1} \le \ldots \le a_\lambda \ge \ldots \ge a_1 \ge \rho a_0 > 0.$$

For $k \ge 1$, $0 < \rho \le 1$ and $0 \le \lambda \le n-1$, then $P(z)$ has all its zeros in

$$|z+k-1| \le \frac{2a_\lambda}{a_n}-k+\frac{2a_0}{a_n}(1-\rho).$$

Remark 2.1. If we set $a_{n-1}=ka_n$ in corollary 2.1, such that $k=1$, $\lambda=n-1$ and then $\rho=1$, we recapture the result of Theorem 1.1 of Eneström-Kakeya [1]

Theorem 2.2: *Let* $P(z)=\sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n* with complex coefficients. If $\mathrm{Re}\,a_j=\alpha_j$, $\mathrm{Im}\,a_j=\beta_j$, $j = 0, 1, \ldots, n$ such that for some real number $k \ge 1$ and $\rho \ge 0$

$\rho+\alpha_n \ge \alpha_{n-1} \ge \ldots \ge \alpha_1 \ge k\alpha_0.$

Then all the zeros of $P(z)$ lie in the closed disk

$$\left|z+\frac{\rho}{\alpha_n}\right| \le \frac{\rho+\alpha_n+|\alpha_0|-k\alpha_0+2\sum_{j=0}^{n}|\beta_j|}{|\alpha_n|}.$$

Remark 2.2. *In* Theorem 2.2, if $k = 1$, we recapture the result of Theorem 1.6 of Gulzar[4]

3.0 Proofs of Theorem

Proof of Theorem 2.1.

Consider $F(z)=(1-z)P(z)$

$=(1-z)(a_n z^n + a_{n-1}z^{n-1}+\ldots+a_\lambda z^\lambda + a_1 z + a_0)$

$= a_n z^n + a_{n-1}z^{n-1}+\ldots+a_{\lambda+1}z^{\lambda+1}+a_\lambda z^\lambda+\ldots+a_{\lambda-1}z^{\lambda-1}+\ldots+$
$a_1 z + a_0 - a_n z^{n+1} - a_{n-1}z^n - \ldots - a_{\lambda+1}z^{\lambda+2} - a_\lambda z^{\lambda+1} - a_{\lambda-1}z^\lambda - \ldots - a_1 z^2 - a_0 z$

$= -a_n z^{n+1} + a_n z^n - (\mu+1)a_n z^n + (\mu+1)a_n z^n - a_{n-1}z^n + \ldots + (a_{\lambda+1}-a_\lambda)z^{\lambda+1} + \ldots +$
$(a_\lambda - a_{\lambda-1})z^\lambda + \ldots + \ldots + a_1 z - (1-\tau)a_0 z + (1-\tau)a_0 z - a_0 z + a_0$

$= -a_n z^{n+1} + a_n z^n - (\mu+1)a_n z^n + ((\mu+1)a_n - a_{n-1})z^n + \cdots + (a_{\lambda+1}-a_\lambda)z^\lambda + \ldots +$
$(a_1-(1-\tau)a_0)z + ((1-\tau)-1)a_0 z + a_0.$

Therefore, for $|z| \ge 1$, $0 \le \lambda \le n-1$, $0 \le \tau < 1$ and $\mu \ge 0$, we have

$|F(z)| = |-a_n z^{n+1} + a_n z^n - (\mu+1)a_n z^n + (k(\mu+1) - a_{n-1})z^n + \ldots$
$+ (a_{\lambda-1} - a_\lambda)z^{\lambda+1} + (a_\lambda - a_{\lambda-1})z^\lambda + \ldots + \ldots + (a_1-(1-\tau)a_0)z +$
$((1-\tau)-1)a_0 z + a_0|$

$\ge |a_n z^{n+1} - a_n z^n + (\mu+1)a_n z^n| - |((\mu+1)a_n - a_{n-1})z^n + \ldots +$
$(a_{\lambda+1} - a_\lambda)z^{\lambda+1} + (a_\lambda - a_{\lambda-1})z^\lambda + \ldots + \ldots + (a_1 - (1-\tau)a_0)z +$
$((1-\tau)-1)a_0 z + a_0|$

$\ge |a_n||z|^n|z+(\mu+1)-1| - |z|^n\{((\mu+1)a_n - a_{n-1}) + (a_{n-1}-a_{n-2})1/z$
$+\ldots+|a_{\lambda+1}-a_\lambda|\frac{1}{|z|^{n-\lambda-1}} + |a_{\lambda-1}-a_\lambda|\frac{1}{|z|^{n-\lambda}} + \ldots + |a_1-(1-\tau)a_0|\frac{1}{|z|^{n-1}} +$
$|\tau||a_0||\frac{1}{|z|^{n-1}}| + |a_0|\frac{1}{|z|^n}.$

Now let $|z| \ge 1$, so that $\frac{1}{|z|^{n-j}} \le 1$, $0 \le j \le n$. Then we get

$F(z) \geq |a_n||z|^n (|z+\mu| - \frac{1}{|a_n|}(a_{n-1} - (\mu+1)a_n) + (a_{n-2} - a_{n-1}) + \ldots +$
$(a_\lambda - a_{\lambda+1}) + (a_\lambda - a_{\lambda-1}) + \ldots + (a_1 - (1-\tau)a_0) + \tau|a_0| + |a_0|)$
$= |a_n||z|^n \{|z+\mu| - \frac{1}{|a_n|}\{2a_\lambda - (\mu+1)a_n + 2\tau a_0\}) > 0.$

If $|z+\mu| > \frac{1}{a_n}\{2a_\lambda - (\mu+1)a_n + 2\tau a_0\}$, Then all the zeros of F (z) whose modulus is less than 1 lie in the closed disk

$|z+\mu| \leq \frac{1}{a_n}\{2a_\lambda - (\mu+1)a_n + 2\tau a_0\}.$

This completes the proof.

Proof of Theorem 2.2.

Consider $F(z) = (1-z)P(z)$

$= (1-z)(a_n z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0)$
$= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \ldots + (a_1 - a_0)z + a_0$
$= -\alpha_n z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + \ldots + (\alpha_1 - \alpha_0)z + \alpha_0 - i\beta_n z^{n+1} +$
$i(\beta_n - \beta_{n-1})z^n + \ldots + i(\beta_1 - \beta_0)z + i\beta_0$
$= -\alpha_n z^{n+1} - \rho z^n + (\rho + \alpha_n - \alpha_{n-1})z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1} + \ldots +$
$(\alpha_1 - \alpha_0)z - \rho\alpha_0 z + \rho\alpha_0 z + \alpha_0 - i\beta_n z^{n+1} + i(\beta_n - \beta_{n-1})z^n + \ldots + i(\beta_1 - \beta_0)z + i\beta_0$
$= -\alpha_n z^{n+1} - \rho z^n + (\rho + \alpha_n - \alpha_{n-1})z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1} + \ldots +$
$(\alpha_1 - k\alpha_0)z - (k-1)\alpha_0 z + \alpha_0 + \alpha_0 - i\beta_n z^{n+1} + i(\beta_n - \beta_{n-1})z^n + \ldots +$
$i(\beta_1 - \beta_0)z + i\beta_0$
$= -\alpha_n z^{n+1} - \rho z^n + (\rho + \alpha_n - \alpha_{n-1})z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1} + \ldots +$
$(\alpha_1 - k\alpha_0)z - (k-1)\alpha_0 z + \alpha_0 + \alpha_0 - i\{\beta_n z^{n+1} + (\beta_n - \beta_{n-1})z^n + \ldots +$
$(\beta_1 - \beta_0)z + \beta_0\}.$

We have

$|F(z)| = |-\alpha_n z^{n+1} - \rho z^n + (\rho + \alpha_n - \alpha_{n-1})z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1} + \ldots + (\alpha_1 - k\alpha_0)z - (k-1)\alpha_0 z + \alpha_0 + \alpha_0 -$
$i[\beta_n z^{n+1} + (\beta_n - \beta_{n-1})z^n + \ldots +$
$(\beta_1 - \beta_0)z + \beta_0|].$
$\geq |z^n|[\{|\alpha_n z + \rho| - |\rho + \alpha_n - \alpha_{n-1}| - |\alpha_0| \frac{1}{|z|^n} - |\alpha_{n-1} - \alpha_{n-2}| \frac{1}{|z|^n} - + \ldots$
$|\alpha_1 - k\alpha_0|\frac{1}{|z|^{n-1}} - |k-1||\alpha_0|\frac{1}{|z|^{n-1}} - [-\beta_n z^{n+1} + |\beta_n - \beta_{n-1}|z^n + \ldots +$
$|\beta_1 - \beta_0|z + \beta_0|]\}.$

Now, let $|z| \geq 1$, to have $\frac{1}{|z|^{n-j}} \leq 1, 0 \leq j \leq n$. Thus, we get

$|F(z)| \geq |z|^n\{|\alpha_n z + \rho| - (\rho + \alpha_n - \alpha_{n-1}) - |\alpha_0| - (\alpha_{n-1} - \alpha_{n-2}) + \ldots +$
$|\alpha_1 - k\alpha_0| - |k-1||\alpha_0| - [|\beta_n| - |\beta_0| + \sum_{j=1}^{n}(|\beta_j| + |\beta_{j-1}|)]\}$
$= |z^n|\{|\alpha_n z + \rho| - (\rho - \alpha_n + 2|\alpha_0| - 2\alpha_0 k) - 2\sum_{j=1}^{n}|\beta_j|\} > 0$

The above inequality holds if

$|\alpha_n z + \rho| > (\rho + \alpha_n + 2|\alpha_0| - 2\alpha_0 k) + 2\sum_{j=1}^{n}|\beta_j|$
$> (\rho + \alpha_n + |\alpha_0| - \alpha_0 k) + 2\sum_{j=1}^{n}|\beta_j|,$

i.e.

$$\left|z + \frac{\rho}{\alpha_n}\right| \leq \frac{\rho + \alpha_n + |\alpha_0| - k\alpha_0 + 2\sum_{j=0}^{n}|\beta_j|}{|\alpha_n|}.$$

This shows that all the zeros of F (z) whose modulus is greater than or equal to 1 lie in the disk.

$$\left|z + \frac{\rho}{\alpha_n}\right| \leq \frac{\rho + \alpha_n + |\alpha_0| - k\alpha_0 + 2\sum_{j=0}^{n}|\beta_j|}{|\alpha_n|}.$$

Since all the zeros of F(z) are also zeros of $P(z)$, it follows that all the zeros of $P(z)$ lie in the disk

$$\left|z + \frac{\rho}{\alpha_n}\right| \leq \frac{\rho + \alpha_n + |\alpha_0| - k\alpha_0 + 2\sum_{j=0}^{n}|\beta_j|}{|\alpha_n|}.$$

This completes the proof.

4.0 Conclusion

We stated and proved some new extensions and generalizations of Eneström – Kakeya theorem.

4.0 References

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