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Formulation of Discrete and Continuous Hybrid Methods Using An Orthogonal Polynomial as the Basis Function

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Abstract

This paper presents a framework to introducing hybrid method using multistep collocation with an orthogonal polynomial valid in $[0,1]$ with weight function $w(x) = x^2$ as the basis function. We introduce the hybrid formula as an important step because of the property of utilizing data at points other than the step points. For the step considered, we obtained one off-step and two off-step hybrid schemes at selected grid points for the solution of first order Initial Value Problems (IVPs) in Ordinary Differential Equations (ODEs). The properties of hybrid viz: order, zero-stability, consistency, and convergence are investigated. Numerical examples, each with its own peculiarity, are presented to illustrate the accuracy of the method.

Key words: Collocation, Interpolation, Predictor, Corrector, Orthogonal Polynomial

1.0 Introduction

Several Linear Multistep methods for the solution of initial value problem of the general first order ordinary differential equation of the form

$$y'(x) = f(x, y(x)),\ \ y(a) = « \quad a \le x \le b < \infty \tag{1}$$

have been developed by many researchers [1-5]. Of these methods, the hybrid predictor-corrector methods entail developing separate predictors to implement the corrector while Taylor series expansion of appropriate order is adopted to provide the starting values. Our focus in this paper is the proposition of discrete and continuous hybrid predictor-corrector algorithm with single and double step length for the solution of (1).

2.0 Methodology

3.0 Construction of orthogonal polynomial

We shall try to approximate (1) with

$$y(x_n) = y_n \tag{2}$$

and

$$x_n = x_o + nh, \quad h = \frac{x_n - x_o}{h} \tag{3}$$

using

$$y(x) = \sum_{r=0}^{n} a_r ¤_r(x) \approx y(x) \tag{4}$$

The basis function $¤_r(x)$ in (4) is defined as

$$¤_n(x) = \sum_{r=0}^{n} c_r^{(n)} x^r \tag{5}$$

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which is a polynomial of degree n whose coefficients are determined by the requirement that $\text{¤}_n(x)$ is orthogonal on $[a,b]$ with weight function $w(x) > 0$ that is

$$\langle \text{¤}_m(x), \text{¤}_n(x) \rangle = \int_a^b w(x)\text{¤}_m(x)\text{¤}_n(x)dx = 0, \quad m \neq n \tag{6}$$

cb
$$\text{¤}_n(1) = 1 \tag{7}$$

Using (6) and (7), (5) gives the first five orthogonal polynomials valid in [0,1] with $w(x) = x^2$ as

$$\begin{aligned} \text{¤}_0(x) &= 1 \\ \text{¤}_1(x) &= 4x-3 \\ \text{¤}_2(x) &= 15x^2-20x+6 \\ \text{¤}_3(x) &= 56x^3-105x^2+60x-10 \\ \text{¤}_4(x) &= 210x^4-504x^3+420x^2-140x+15 \end{aligned} \tag{8}$$

4.0 Formulation of Explicit Hybrid Methods

We consider a modification of the general linear multistep method with the introduction of f_{n+v} for the evaluation at off-step points (hybrid).

$$\sum_{j=0}^{k} \sim_k y_{n+j} = h\sum_{j=0}^{k} {}_j f_{n+j} + h \ {}_v f_{n+v} \tag{9}$$

where v is a positive non-integer.

For the explicit when ($_k = 0$) implementation of (9), we require the value of f_{n+v} using

$$y_{n+v} + \sum_{j=0}^{k-1} \bar{\sim}_j y_{n+j} = h\sum_{j=0}^{k-1} \bar{\ }_j f_{n+j} \tag{10}$$

which is the solution evaluated at the off-step point. Equations (9)-(10) form the discrete predictor -corrector algorithm. The number of predictor schemes formed in (10) depends on the number of off-step points (v) considered in (9).

5.0 Two-step method

We consider here the transformation $X = ax + b$ which shifts our interval [0,1] to $[x_n, x_{n+2}]$ where

$$x_{n+2} = x_n + 2h \quad \text{then} \quad X = \frac{x - x_n}{2h}$$

6.0 Two-step Method with one Off-step Point

Considering (4) with n=3 gives

$$Y(X) = a_0 + a_1(4X-3) + a_2(15X^2 - 20X + 6) + a_3(56X^3 - 105X^2 + 60X - 10)$$

$$Y(x) = \left\{ \begin{array}{l} a_0 + a_1\left[\dfrac{4(x-x_n)}{2h} - 3\right] + a_2\left[\dfrac{15(x-x_n)^2}{4h^2} - \dfrac{20(x-x_n)}{2h} + 6\right] \\ + a_3\left[\dfrac{56(x-x_n)^3}{8h^3} - \dfrac{105(x-x_n)^2}{4h^2} + \dfrac{60(x-x_n)}{2h} - 10\right] \end{array} \right\} \tag{11}$$

Interpolate (11) at x_{n+2} and collocate at x_{n+1}, x_{n+2} and $x_{n+\frac{7}{3}}$ to have

$$\left.\begin{aligned} y_{n+2} &= a_0 + a_1 + a_2 + a_3 \\ 2hf_{n+1} &= 4a_1 - 5a_2 - 3a_3 \\ hf_{n+2} &= 2a_1 + 5a_2 + 9a_3 \\ 6hf_{n+\frac{7}{3}} &= 12a_1 + 45a_2 + 131a_3 \end{aligned}\right\} \tag{12}$$

Solving (12) gives

$$\left.\begin{array}{lcl} a_0 & = & y_{n+2} - \frac{3h}{20} f_{n+1} - \frac{h}{2} f_{n+2} + \frac{3}{20} hf_{n+\frac{7}{3}} \\ a_1 & = & \frac{25}{84} hf_{n+1} + \frac{13}{42} hf_{n+2} - \frac{3}{28} hf_{n+\frac{7}{3}} \\ a_2 & = & -\frac{11}{60} hf_{n+1} + \frac{1}{3} hf_{n+2} - \frac{3}{20} hf_{n+\frac{7}{3}} \\ a_3 & = & \frac{h}{28} f_{n+1} - \frac{h}{7} hf_{n+2} + \frac{3h}{28} hf_{n+\frac{7}{3}} \end{array}\right\} \quad (13)$$

Substituting (13) in (11) gives the continuous scheme

$$Y(x) = y_{n+2} + h\left[{}_{1}(x) f_{n+1} + {}_{2}(x) f_{n+2} + {}_{\frac{7}{3}}(x) f_{n+\frac{7}{3}} \right]$$

$$\left.\begin{array}{lcl} {}_{1}(x) & = & \frac{-5}{2} + \frac{7}{2}\frac{(x-x_n)}{h} - \frac{13(x-x_n)^2}{8h^2} + \frac{(x-x_n)^3}{h^3} \\ {}_{2}(x) & = & 2 - \frac{7(x-x_n)}{h} + \frac{7(x-x_n)}{h} + \frac{5(x-x_n)}{h^2} - \frac{(x-x_n)^3}{h^3} \\ {}_{\frac{7}{3}}(x) & = & \frac{-3}{2} + \frac{9}{2}\frac{(x-x_n)}{h} + \frac{9}{2}\frac{(x-x_n)}{h} - \frac{27(x-x_n)^2}{8h^2} + \frac{3(x-x_n)^3}{4h^3} \end{array}\right\} \quad (14)$$

At the step-point x_{n+3}, we obtain the discrete scheme

$$y_{n+3} = y_{n+2} + \frac{h}{8}\left[f_{n+1} - 8 f_{n+2} + 15 f_{n+\frac{7}{3}} \right] \quad (15)$$

which is of order three with error constant $c_4 = \frac{11}{216}$. To determine $f_{n+\frac{7}{3}}$ in (15), we interpolate (11) at x_{n+2} and collocate at x_n, x_{n+1} and x_{n+2} to have

$$\left.\begin{array}{lcl} y_{n+2} & = & a_0 + a_1 + a_2 + a_3 \\ hf_n & = & 2a_1 - 10a_2 + 30a_3 \\ 2hf_{n+1} & = & 4a_1 - 5a_2 - 3a_3 \\ hf_{n+2} & = & 2a_1 + 5a_2 + 9a_3 \end{array}\right\} \quad (16)$$

which when solved and substituted back into (11) gives

$$Y(X) = y_{n+2} + h\left[{}_{0}(x) f_n + {}_{1}(x) f_{n+1} + {}_{2}(x) f_{n+2} \right]$$

$$\left.\begin{array}{lcl} {}_{0}(x) & = & \frac{-1}{3} + \frac{(x-x_n)}{h} - \frac{3(x-x_n)^2}{4h^2} + \frac{(x-x_n)^3}{6h^3} \\ {}_{1}(x) & = & \frac{-4}{3} + \frac{(x-x_n)^2}{h^2} - \frac{(x-x_n)^3}{3h^3} \\ {}_{2}(x) & = & \frac{-1}{3} - \frac{(x-x_n)^2}{4h^2} + \frac{(x-x_n)^3}{6h^3} \end{array}\right\} \quad (17)$$

At the off-step point $x_{n+\frac{7}{3}}$, we have

$$y_{n+\frac{7}{3}} = y_{n+2} + \frac{h}{324}\left[11 f_n - 40 f_{n+1} + 137 f_{n+2} \right] \quad (18)$$

which is of order 3 and error constant $c_4 = \frac{49}{1944}$. The scheme (18) is used as an accurate predictor for the scheme (15).

7.0 Two Step Method with Two Off-step Points

Now let us consider (4) with n=4 which gives

$$Y(x) = \left. \begin{array}{l} a_0 + a_1\left[\frac{4(x-x_n)}{2h} - 3\right] + a_2\left[\frac{15(x-x_n)^2}{4h^2} - \frac{20(x-x_n)}{2h} + 6\right] \\ + a_3\left[\frac{56(x-x_n)^3}{8h^3} - \frac{105(x-x_n)^2}{4h^2} + \frac{60(x-x_n)}{2h} - 10\right] \\ + a_4\left[\frac{210(x-x_n)^4}{16h^4} - \frac{504(x-x_n)^3}{8h^3} + \frac{420(x-x_n)^2}{4h^2} - \frac{140(x-x_n)}{2h} + 15\right] \end{array} \right\} \quad (19)$$

Interpolating (19) at x_{n+2} and collocating at x_{n+1}, x_{n+2}, $x_{n+\frac{8}{3}}$ and $x_{n+\frac{9}{4}}$ to have

$$\left. \begin{array}{rcl} y_{n+2} & = & a_0 + a_1 + a_2 + a_3 + a_4 \\ 2hf_{n+1} & = & 4a_1 - 5a_2 - 3a_3 + 7a_4 \\ hf_{n+2} & = & 2a_1 + 5a_2 + 9a_3 + 14a_4 \\ 9hf_{n+\frac{8}{3}} & = & 18a_1 + 90a_2 + 354a_3 + 127a_4 \\ 128hf_{n+\frac{9}{4}} & = & 256a_1 + 880a_2 + 2328a_3 + 5593a_4 \end{array} \right\} \quad (20)$$

Solve (20) and substitute the values into (19) to have the two-step continuous scheme

$$Y(x) = y_{n+2} + h\left[{}_{1}(x) f_{n+1} + {}_{2}(x) f_{n+2} + {}_{\frac{8}{3}}(x) f_{n+\frac{8}{3}} + {}_{\frac{9}{4}}(x) f_{n+\frac{9}{4}} \right]$$

$$\left. \begin{array}{rcl} {}_{1}(x) & = & \frac{-244}{75} + \frac{144(x-x_n)}{25h} - \frac{19(x-x_n)^2}{5h^2} + \frac{83(x-x_n)^3}{75h^3} - \frac{3(x-x_n)^4}{25h^4} \\ {}_{2}(x) & = & \frac{35}{3} - \frac{36(x-x_n)}{h} + \frac{131(x-x_n)^2}{4h^2} - \frac{71(x-x_n)^3}{6h^3} + \frac{3(x-x_n)^4}{2h^4} \\ {}_{\frac{8}{3}}(x) & = & \frac{81}{25} - \frac{243(x-x_n)}{25h} + \frac{189(x-x_n)^2}{20h^2} - \frac{189(x-x_n)^3}{50h^3} + \frac{27(x-x_n)^4}{50h^4} \\ {}_{\frac{9}{4}}(x) & = & \frac{-1024}{75} + \frac{1024(x-x_n)}{25h} - \frac{192(x-x_n)^2}{5h^2} + \frac{1088(x-x_n)^3}{75h^3} - \frac{48(x-x_n)^4}{25h^4} \end{array} \right\} \quad (21)$$

And at the step-point x_{n+3}, it gives the discrete scheme

$$y_{n+3} = y_{n+2} + \frac{h}{300}\left[-4f_{n+1} + 125f_{n+2} + 243f_{n+\frac{8}{3}} - 64f_{n+\frac{9}{4}}\right] \quad (22)$$

which is of order four with error $c_5 = \frac{13}{5760}$.

Similarly, to obtain $f_{n+\frac{8}{3}}$ and $f_{n+\frac{9}{4}}$ in (22), we interpolate (19) at x_{n+1} and x_{n+2} and also collocate at x_n, x_{n+1} and x_{n+2} to have

$$\left. \begin{array}{rcl} 8y_{n+1} & = & 8a_0 - 8a_1 - 2a_2 + 6a_3 + a_4 \\ y_{n+2} & = & a_0 + a_1 + a_2 + a_3 + a_4 \\ 2hf_{n+1} & = & 4a_1 - 5a_2 - 3a_3 + 7a_4 \\ hf_{n+2} & = & 2a_1 + 5a_2 + 9a_3 + 14a_4 \end{array} \right\} \quad (23)$$

Solve for a_o, a_1, a_2, a_3, a_4 in (23) and substitute the values into (19) gives

$$Y(x) = \sim_1 y_{n+1} + \sim_2 y_{n+2} + h[\ {}_{0} f_n + {}_{1} f_{n+1} + {}_{2} f_{n+2}]$$

$$\left.\begin{aligned}
\sim_1(x) &= \frac{4(x-x_n)^2}{h^2}-\frac{4(x-x_n)^3}{h^3}+\frac{(x-x_n)^4}{h^4}\\
\sim_2(x) &= 1-\frac{4(x-x_n)^2}{h^2}+\frac{4(x-x_n)^3}{h^3}\,\frac{4(x-x_n)^4}{h^4}\\
{}_0(x) &= -\frac{1}{3}+\frac{(x-x_n)}{h}-\frac{13(x-x_n)^2}{12h^2}+\frac{(x-x_n)^3}{2h^3}-\frac{(x-x_n)^4}{12h^4}\\
{}_1(x) &= \frac{-4}{3}+\frac{11(x-x_n)^2}{3h^2}-\frac{3(x-x_n)^3}{h^3}+\frac{2(x-x_n)^4}{3h^4}\\
{}_2(x) &= \frac{-1}{3}+\frac{17(x-x_n)^2}{12h^2}-\frac{3(x-x_n)^3}{2h^3}+\frac{5(x-x_n)^4}{12h^4}
\end{aligned}\right\} \tag{24}$$

At the step points $x_{n+\frac{8}{3}}$ and $x_{n+\frac{9}{4}}$ (24) gives

$$y_{n+\frac{8}{3}}=\frac{256}{81}y_{n+1}-\frac{175}{81}y_{n+2}+\frac{h}{243}\left[-25f_n+380f_{n+1}+575f_{n+2}\right] \tag{25}$$

$$y_{n+\frac{9}{4}}=\frac{81}{256}y_{n+1}+\frac{175}{256}y_{n+2}+\frac{h}{3072}\left[-25f_n+440f_{n+1}+1325f_{n+2}\right] \tag{26}$$

respectively. We note here that the corrector schemes (15) and (22) are in form of (9) while the predictor schemes (18),(25) and(26) are in form of (10).

8.0 Analysis of the Properties of the Methods

In this section, we shall discuss the consistence, zero-stability and convergence of schemes earlier derived.

9.0 Consistency

A LMM of the form (9) is said to be consistent if it has order $p \geq 1$.

(i) method (15) has order $p = 3 > 1$

(ii) method (22) has order $p = 4 > 1$

Hence,the two schemes are consistent.

10.0 Zero-stability

The LMM of the form (9) is said to be zero-stable if no root of the first characteristic
Polynomial has a modulus greater than one.

(i) method (15) has $p(¶) = r^3 - r^2$

$\Rightarrow\ r^3 - r^2 = 0,\ r = 0\ or\ 1\ \ therefore\ |r| = 0\ 0r\ 1$

(ii) method (22) has $p(¶) = r^3 - r^2 \Rightarrow\ r^3 - r^2 = 0,\ r = 0\ or\ 1\ \ therefore\ |r| = 0\ or\ 1$

This shows that the two schemes are zero-stable.

11.0 Convergence

The necessary and sufficient condition for a LMM of the form (9) to be convergent is for it to be consistent and zero-stable. Consequently, the two schemes are convergent.

12.0 Numerical Examples

We consider here two selected problems, each with its own peculiarity,for experimentation with schemes derived in the preceding section.

Problem 1 (A nonlinear, variable coefficient, homogeneous D.E)

$$y' + xy^2 = 0:\ y(0) = 1\ \ with\quad y(x) = \frac{2}{x^2+2}$$

Problem 2 (A linear, constant coefficient, non-homogeneous D.E)

$$y' - y = x : y(0) = 1 \quad with \quad y(x) = 2e^{x} - x - 1$$

and the resulting errors compare favorably with [5].

Table 1: Errors for Problem 4.1 with h=0.1 , $¢x = \frac{h}{10}$ and $y(x) = -1 - x + 2e^{x}$

X	Discrete method method (15)	Continuous method (14)	Discrete method method (22)	Continuous method (21)
.01		$1.44779E-4$		$1.13404E-5$
.02		$1.15247E-4$		$7.02610E-6$
.03		$9.18252E-5$		$3.69770E-6$
.04		$7.36799E-5$		$1.19400E-6$
.05		$6.00383E-5$		$6.29100E-7$
.06		$5.01861E-5$		$1.90100E-6$
.07		$4.34689E-5$		$2.73620E-6$
.08		$3.92894E-5$		$3.23440E-6$
.09		$3.71096E-5$		$3.48080E-6$
.10	$2.48756E-5$	$2.48754E-5$	$1.24378E-6$	$1.24376E-6$
.11		$3.68781E-5$		$3.49850E-6$
.12		$3.80325E-5$		$3.37840E-6$
.13		$3.95935E-5$		$3.22460E-6$
.14		$4.13019E-5$		$3.06610E-6$
.15		$4.29486E-5$		$2.92180E-6$
.16		$4.43763E-5$		$2.80240E-6$
.17		$4.54788E-5$		$2.71380E-6$
.18		$4.61985E-5$		$2.65400E-6$
.19		$4.65260E-5$		$2.61940E-6$
.20	$4.64986E-5$	$4.64983E-5$	$2.60314E-6$	$2.60309E-6$
21		$4.61960E-5$		$2.59640E-6$
.22		$4.57458E-5$		$2.58950E-6$
.23		$4.53150E-5$		$2.58290E-6$
.24		$4.51114E-5$		$2.56560E-6$
.25		$4.53819E-5$		$2.54410E-6$
.26		$4.64110E-5$		$2.53090E-6$
.27		$4.85182E-5$		$2.53510E-6$
.28		$5.20574E-5$		$2.58770E-6$
.29		$5.74143E-5$		$2.73210E-6$
.30	$6.50060E-5$	$6.50056E-5$	$3.01296E-6$	$3.01290E-6$
.31		$7.52768E-5$		$3.49840E-6$
.32		$8.86986E-5$		$4.28040E-6$
.33		$1.05770E-4$		$5.45250E-6$
.34		$1.27011E-4$		$7.14200E-6$
.35		$1.52964E-4$		$9.49890E-6$
.36		$1.84191E-4$		$1.26842E-5$
.37		$2.21273E-4$		$1.68979E-5$
.38		$2.64806E-4$		$2.23627E-5$
.39		$3.15402E-4$		$2.93350E-5$
.40	$7.31579E-5$	$7.31572E-5$	$3.03807E-6$	$3.03806E-6$

Table 2 : Errors for Problem 4.2 with h=0.1 , $\text{¢}x = \dfrac{h}{10}$ and $y(x) = e^{2x}$

X	Discrete method method (15)	Continuous method (14)	Discrete method method (22)	Continuous method (21)
.01		$6.01380E-5$		$2.22000E-6$
.02		$4.70930E-5$		$1.42200E-6$
.03		$3.66880E-5$		$8.11000E-7$
.04		$2.85870E-5$		$3.57000E-7$
.05		$2.24730E-5$		$2.80000E-8$
.06		$1.80500E-5$		$1.99000E-7$
.07		$1.50390E-5$		$3.46000E-7$
.08		$1.31900E-5$		$4.36000E-7$
.09		$1.22660E-5$		$4.80000E-7$
.10	$8.503151E-6$	$8.50310E-6$	$1.69151E-7$	$4.91000E-7$
.11		$1.23750E-5$		$4.86000E-7$
.12		$1.30540E-5$		$4.68000E-7$
.13		$1.39440E-5$		$4.43000E-7$
.14		$1.49280E-5$		$4.24000E-7$
.15		$1.59030E-5$		$4.02000E-7$
.16		$1.67930E-5$		$3.85000E-7$
.17		$1.75450E-5$		$3.77000E-7$
.18		$1.81300E-5$		$3.71000E-7$
.19		$1.85420E-5$		$3.70000E-7$
.20	$1.87953E-5$	$1.87950E-5$	$3.74320E-7$	$3.74316E-7$
21		$1.89350E-5$		$3.78000E-7$
.22		$1.90280E-5$		$3.68000E-7$
.23		$1.91620E-5$		$3.87000E-7$
.24		$1.94550E-5$		$3.79000E-7$
.25		$2.00500E-5$		$3.70000E-7$
.26		$2.11100E-5$		$3.94000E-7$
.27		$2.28280E-5$		$3.87000E-7$
.28		$2.54250E-5$		$3.88000E-7$
.29		$2.91420E-5$		$4.35000E-7$
.30	$3.42531E-5$	$3.42530E-5$	$4.82152E-7$	$4.82148E-7$
.31		$4.10520E-5$		$5.42000E-7$
.32		$4.98660E-5$		$6.87000E-7$
.33		$6.10460E-5$		$8.63000E-7$
.34		$7.49740E-5$		$1.12400E-6$
.35		$9.20520E-5$		$1.51500E-6$
.36		$1.12719E-4$		$2.00700E-6$
.37		$1.37438E-4$		$2.65900E-6$
.38		$1.66699E-4$		$3.50600E-6$
.39		$2.01027E-4$		$4.62200E-6$
.40	$5.27233E-5$	$5.2730E-5$	$6.41282E-7$	$6.41282E-7$

13.0 Conclusion

A method for the derivation of the discrete and continuous hybrid schemes for the solution of Initial Value Problems in Ordinary Differential Equations has been presented. For this purpose, an orthogonal polynomial has been employed as the basis function and a collocation approach was adopted. Taylor series of appropriate order were used to obtain the starting values. The continuous schemes reproduce their corresponding discrete equivalents at grid points. The continuous schemes produces several output of solutions at the off-step points without additional interpolation and at no extra cost. These numerical evidences show that the methods (proposed herein) are accurate and effective.

14.0 References

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