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Higher-Step Hybrid Block Methods for the Solution of Initial Value Problems in Ordinary Differential Equations

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Abstract

In this work, hybrid block method of higher step-number, for the solution of initial value problems of first-order ordinary differential equation has been developed by collocation and interpolation technique. For this purpose we constructed a set of orthogonal polynomials valid in the interval [0, 1] with respect to the weight function $w(x) = 1 - x^2$ ***and which serve as the basis function. The derived schemes were implemented on some problems of interest and numerical evidences show that they are effective and accurate.***

Key words: Interpolation, Orthogonal polynomial, Collocation, Continuous scheme

1.0 Introduction

In science and engineering usually mathematical models are developed to help in the understanding of physical phenomena. These model often yield equations that contain some derivatives of an unknown function of one or several variables such equations are called differential equations. Differential equations do not only arise in the physical sciences but also in diverse fields as economics,medicine,psychology, operation research and even in areas such as biology and anthropology.
Interestingly, differential equation arising from the modeling of physical phenomena, often do not have analytic solutions. Hence,the development of numerical method to obtain approximate solutions become necessary. To that extent,several numerical methods such as finite difference methods,among others,have been developed based on the differential equation to be solved.
A differential equation in which the unknown function is a function of two or more independent variable is called a Partial Differential Equations(PDE). Those in which the unknown function is function of only one independent variable are called Ordinary Deferential Equations(ODE).

2.0 Orthogonal Polynomial

The orthogonal polynomial of the first kind of degree n over the interval $[0,1]$ with $w(x) = 1 - x^2$
The polynomial is defined as

$$\ddot{}_n(x) = \sum_{r=0}^{n} c_r^{(n)} x^r$$

The requirements are:

$$< \ddot{}_n \ddot{}_m >= 0 \qquad m = 0,1,2\ldots n-1; \ddot{}_n(1) = 1$$

This yields

$$\ddot{}_0(x) = 1$$

$$\ddot{}_1(x) = \frac{8}{5}x - \frac{3}{5}$$

$$\ddot{}_2(x) = \frac{95}{26}x^2 - \frac{40}{13}x + \frac{11}{26}$$

$$\ddot{}_3(x) = \frac{9755}{55606}x^3 - \frac{15711}{27803}x^2 - \frac{101745}{55606}x + \frac{898509}{27803}$$

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$$\ddot{}_4(x)=\frac{1099681}{11677260}x^4+\frac{2042359}{14012712}x^3+\frac{784069}{3113936}x^2+\frac{101946}{194621}x+\frac{226331}{139015}$$

$$\ddot{}_5(x)=\frac{33643783879}{353120342400}x^5+\frac{7556775767}{58853390400}x^4+\frac{6464947921}{353120342400}x^3$$
$$+\frac{16725127793}{58853390400}x^2+\frac{222380323}{435951040}x+\frac{37297083349}{29426695200}$$

$$\ddot{}_6(x)=\frac{137243066135191}{21208407764544000}x^6+\frac{5298721206533747}{63625223293632000}x^5+\frac{363678610065151}{326283196377630}x^4$$
$$+\frac{385564067540899}{2446123972832000}x^3+\frac{11955236083909}{49436847936000}x^2+\frac{191062218162727}{444931631424000}x+\frac{307581926340607}{296621087616000}$$

3.0 Review of Existing Methods

Consider the class Initial Value Problem (IVPs) in Ordinary Differential Equation :

$$y'(x)=f(x,y)\qquad y(a)=y_0 \tag{1}$$

where a and y_0 are given real constants. Numerical techniques for the solution have been extensively reported in the literature [1-15]. In [8] a continuous five-step block method employing multistep collocation approach which produced a class of eight discrete scheme was derived, while Ibijola et al [9] worked on formation of hybrid block method of higher step-size,through the continuous muti-step collocation. This work has led us to the development of higher step size using orthogonal ploynomial as basis function in our own work.

4.0 Derivation of Method

In this section, we shall be concerned with development of some block methods for the numerical solution of the IVP in ODE;

$$y'(x)=f(x,y)\qquad y(a)=y_0 \tag{2}$$

where a and y_0 are real numbers.

For this purpose we shall construct continuous two-step, three-step and four-step methods, each with one off-step point. These will be used to generate the main method and other methods required to set up the desired block method.

We approximate the analytical solution of the problem by an approximation with orthogonal polynomial as base function.

We shall seek a k-step multi-step collocation polynomial $y(x)$ of the form

$$\sum_{j=0}^{k}\sim_i y_{n+j}=h\{\sum_{j=0}^{k}\ _j f_{n+j}+\ _v f_{n+\sim}\} \tag{3}$$

where $\sim_j$ and $_j$ are continuous coefficients.

In order to obtain equation (2), we proceed by seeking approximation of the exact solution $y(x)$ of the continuous form:

$$y(x)=\sum_{j=0}^{p+q-1}a_i\ddot{}_n(x) \tag{4}$$

such that $x\in[x_0,b]$, a_i are unknown coefficients and $\sim_n(x)$ are polynomial basis function of degree $p+q-1$, where the number of interpolation point P and the number of distinct collocation point q are respectively chosen to satisfy number $1\le p\le k$ and $q>0$. The integer $k\ge 1$ denotes the step number of the method.

5.0 Two-Step Method with One Off-Step Point

To derive this, one off-step point is introduce. This off-step point carefully chosen to guarantee zero stability condition. For the method, the off-step point $v=\frac{3}{2}$ using (3) with p=1, q=4,

We have a polynomial of degree $p+q-1$ as follows.

$$y(x)=\sum_{j=0}^{4}a_j\sim_n(x) \tag{5}$$

where $t=\dfrac{x-x_n}{h}$.

with the orthogonal polynomial earlier obtained in section2, equation(3) now becomes

$$
\begin{aligned}
Y(x) &= a_0+(\frac{8}{5}t-\frac{11}{5})a_1+(\frac{95}{26}t^2-\frac{135}{13}t+\frac{93}{13})a_2\\
&+(\frac{119793}{27803}-\frac{4818}{27803}t-\frac{60687}{55606}t^2+\frac{9755}{55606}t^3)a_3\\
&+(\frac{5222749}{4003632}+\frac{78618}{973105}t+\frac{2532827}{6672720}t^2-\frac{2311507}{10009080}t^3+\frac{1099681}{11677260}t^4)a_4
\end{aligned}
\tag{6}
$$

Differentiating (5) we have

$$
\begin{aligned}
Y'(x) &= \frac{8}{5}a_1+(\frac{95}{13}t+\frac{135}{13})a_2+(-\frac{4818}{27803}-\frac{60687}{27803}t+\frac{29265}{55606}t^2)a_3\\
&+(\frac{78618}{973105}+\frac{2532827}{3336360}t-\frac{2311507}{3336360}t^2+\frac{1099681}{2919315}t^3)a_4
\end{aligned}
\tag{7}
$$

Interpolating (5) at x_n collocating (6) at x_n, x_{n+1}, $x_{n+\frac{3}{2}}$, $x_{n+\frac{5}{2}}$ yield.

$$
\begin{pmatrix}
1 & \frac{-11}{5} & \frac{93}{13} & \frac{119793}{27803} & \frac{5222749}{4003632}\\
0 & \frac{8}{5} & -\frac{135}{13} & -\frac{4818}{27803} & \frac{78618}{973105}\\
0 & \frac{8}{5} & -\frac{40}{13} & -\frac{101745}{55606} & \frac{1011946}{194621}\\
0 & \frac{8}{5} & \frac{15}{26} & -\frac{503403}{222424} & \frac{29022223}{31139360}\\
0 & \frac{8}{5} & \frac{55}{13} & -\frac{67662}{2780803} & \frac{21501899}{11677260}
\end{pmatrix}
\begin{pmatrix} a_0\\ a_1\\ a_2\\ a_3\\ a_4 \end{pmatrix}
=
\begin{pmatrix} y_1\\ hf_n\\ hf_{n+1}\\ hf_{n+\frac{3}{2}}\\ hf_{n+2} \end{pmatrix}
\tag{8}
$$

Solving (7) by Gaussian elimination method, with the aid of Maple yields

$$
\begin{aligned}
a_0 &= -\frac{18763180648541}{10298292628800}hf_{n+1}+y_n-\frac{64968272586163}{30894877886400}hf_n\\
&+\frac{34301939202631}{5149146314400}hf_{n+\frac{3}{2}}-\frac{210366817951}{154474389432}hf_{n+2}\\
a_1 &= \frac{12163466752075}{978337799736}hf_{n+\frac{3}{2}}-\frac{14919812779851}{1304450399648}hf_{n+1}\\
&-\frac{8248562803559}{2608900799296}hf_{n+2}+\frac{21848520071863}{7826702397888}hf_n\\
a_2 &= -\frac{598719908891}{370582499900}hf_{n+1}-\frac{330212598599}{741164999800}hf_{n+2}\\
&+\frac{664382061343}{2223494999400}hf_n+\frac{19592875939}{11117474997}hf_{n+\frac{3}{2}}\\
a_3 &= \frac{10324310414}{1755390789}hf_{n+\frac{3}{2}}-\frac{12309583629}{1950434210}hf_{n+1}\\
&+\frac{14692801183}{11702605260}hf_{n+2}+\frac{59164700591}{35107815780}hf_n\\
a_4 &= -\frac{7784840}{1099681}hf_{n+\frac{3}{2}}+\frac{5838630}{1099681}hf_{n+1}\\
&+\frac{2919315}{1099681}hf_{n+2}-\frac{973105}{973105}hf_n
\end{aligned}
\tag{9}
$$

Substituting (8) into (5) yields

$$y(x) = \sim_0(x)_{y_n} + h\{\sum_{j=1}^{2} {}_i(x) f_{n+j} + {}_{n+\frac{3}{2}} f_{n+\frac{3}{2}}\} \quad (10)$$

where $\sim_0(x)$ and ${}_i(x)$ are continuous coefficients. Equation (9) yields the parameter $\sim_i$ and ${}_i$ as the following continuous function of t

$$\left.\begin{aligned} &\sim(t) = t - \frac{13}{12}t^2 + \frac{1}{2}t^3 - \frac{1}{12}t^4 \\ &{}_1(t) = 3t^2 - \frac{7}{3}t^3 + \frac{1}{2}t^4 \\ &{}_{\frac{2}{3}}(t) = -\frac{8}{3}t^2 + \frac{8}{3}t^3 - \frac{2}{3}t^4 \\ &{}_2(t) = -\frac{3}{4}t^2 - \frac{5}{6}t^3 + \frac{1}{4}t^4 \end{aligned}\right\} \quad (11)$$

where $t = \dfrac{x - x_n}{h}$.

We evaluate (10) at $x_{n+1}, x_{n+\frac{3}{2}}$ and x_{n+2} in order to derive the block method

$$\left.\begin{aligned} &y_{n+1} = y_n + \frac{h}{3}(2fn + 7f_{n+1} - 4f_{n+\frac{3}{2}} + f_{n+2}) \\ &y_{n+\frac{3}{2}} = y_n + \frac{3h}{64}(7f_n + 30f_{n+1} - 8f_{n\frac{3}{2}} + 3f_{n+2}) \\ &y_{n+2} = y_n + \frac{h}{3}(f_n + 4f_{n+1} + f_{n+2}) \end{aligned}\right\} \quad (12a)$$

which is in Block form.

6.0 Three-Step Method With One Off-Step Point

Following the same procedure for the off-step point $v = \dfrac{5}{2}$ using (2) with p=1, q=5 with the orthogonal polynomial earlier determined in section2, and finding the first derivative of the polynomial, we obtain the Block as

$$\left.\begin{aligned} &y_{n+1} = y_n + h(\frac{599}{1800}f_n + \frac{361}{360}f_{n+1} - \frac{101}{120}f_{n+2} + \frac{152}{225}f_{n+\frac{5}{2}} - \frac{61}{360}f_{n+3}) \\ &y_{n+2} = y_n + h(\frac{71}{225}f_n + \frac{64}{45}f_{n+1} + \frac{1}{15}f_{n+2} + \frac{64}{225}f_{n+\frac{5}{2}} - \frac{4}{45}f_{n+3}) \\ &y_{n+\frac{5}{2}} = y_n + h(\frac{365}{1152}f_n + \frac{1625}{1152}f_{n+1} + \frac{125}{384}f_{n+2} + \frac{5}{9}f_{n+\frac{5}{2}} - \frac{128}{1152}f_{n+3}) \\ &y_{n+3} = y_n + h(\frac{63}{20}f_n + \frac{57}{40}f_{n+1} + \frac{9}{40}f_{n+2} + \frac{24}{25}f_{n+\frac{5}{2}} + \frac{3}{40}f_{n+3}) \end{aligned}\right\} \quad (12b)$$

System (12b) is hybrid block method.

7.0 Four- Step Method With One Off-Step Point

Following the same procedure for the off-Step point $v = \dfrac{7}{2}$, we set p=1, q=6 in (2). Together with the orthogonal polynomial and finding the first derivative of the polynomial we obtain the block method.

$$y_{n+1} = y_n + h(\frac{811}{2520}f_n + \frac{377}{360}f_{n+1} - \frac{89}{120}f_{n+2} + \frac{323}{360}f_{n+3} - \frac{24}{35}f_{n+\frac{7}{2}} + \frac{29}{180}f_{n+4})$$

$$y_{n+2} = y_n + h(\frac{193}{636}f_n + \frac{22}{15}f_{n+1} + \frac{2}{45}f_{n+2} + \frac{22}{45}f_{n+3} - \frac{128}{315}f_{n+\frac{7}{2}} + \frac{1}{10}f_{n+4})$$

$$y_{n+3} = y_n + h(\frac{87}{280} f_n + \frac{57}{40} f_{n+1} + \frac{21}{40} f_{n+2} + \frac{51}{40} f_{n+3} - \frac{24}{35} f_{n+\frac{7}{2}} + \frac{3}{20} f_{n+4}) \quad (13)$$

$$y_{n+\frac{7}{2}} = y_n + h(\frac{7147}{23040} f_n + \frac{343}{240} f_{n+1} + \frac{5831}{11520} f_{n+2} + \frac{4459}{2880} f_{n+3} - \frac{77}{180} f_{n+\frac{7}{2}} + \frac{343}{2560} f_{n+4})$$

$$y_{n+4} = y_n + h(\frac{14}{45} f_n + \frac{64}{45} f_{n+1} + \frac{8}{15} f_{n+2} + \frac{64}{45} f_{n+3} + \frac{14}{45} f_{n+4})$$

System(13) is a hybrid block method.

8.0 Analysis of The Methods

9.0 Order of the Methods

The Linear Multistep Methods, with the associate operator L defined by:

$$L[y(x):h] = \sum_{j=0}^{k}[\sim_j y(x_n + jh) - h \quad _j y'(x_n + jh)] \quad (14)$$

where y(x) is an arbitrary test function that is continuously differentiable in the interval [a,b]. Expanding $y(x_n + jh)$ and $y'(x_n + jh)$ in Taylor series about x_n and collecting like terms in h and y gives:

$$L[y(x):h] = C_0 y(x) + C_1 h y'(x) + C_2 h^{(2)} y''(x) + ... + C_p h^p y^p(x) \quad (15)$$

The differential operator in (15) and the associated Linear Multistep Method are said to be of order p if :

$$C_0 = C_1 = C_2 = ...C_p = 0, C_{p+1} \neq 0 \quad (16)$$

The term C_{p+1} is called error constant and it implies that the local truncation error is given by

$$E_{n+k} = C_{p+1} h^{p+1} y^{p+1}(x_n) + 0(h^{p+2}) \quad (17)$$

10.0 Consistency and zero-Stability

Definition 4.3 The linear Multistep Method is said to be consistent if it has order $p \geq 1$.

Definition 4.4 The linear Multistep Method is said to be zero-stable if no root of the first characteristic polynomial has modulus greater than one and if every root with modulus 1 is simple.

Definition 4.5 The hybrid block method is said to be zero stable if the roots R of the characteristic polynomial $\overline{p}(R)$, defined by:

$$^\circ(R) = det[RA - A']$$

satisfy $|R| \leq 1$ and every root with $|R_0| = 1$ has multiplicity not exceeding two in the limit as $n \to 0$.

11.0 Convergence

The convergence of the continuous hybrid two step method is considered in the light of the basic properties discussed earlier in conjunction with the fundamental theorem of Dahlquist[10] for linear multistep method. We state Dahlquist theorem without proof.

Theorem 4.1

The necessary and sufficient condition for a multistep method to be convergent is for it to be consistent and zero stable.

Table 1: Features of Two-Step Method

k=2	**Evaluating Point**	**order**	**Error Constant**
K=2	$y(x = x_{n+2})$	4	$-\frac{1}{90}$
K=2	$y(x = x_{n+\frac{3}{2}})$	4	$-\frac{31}{2880}$
K=2	$y(x = x_{n+k})$	4	$-\frac{51}{5120}$

Table 2: Features of Three-Step Method

k =3	**Evaluating Point**	**order**	**Error Constant**
k =3	$y(x = x_{n+3})$	5	$\frac{13}{1200}$
k =3	$y(x = x_{n+\frac{5}{2}})$	5	$\frac{7}{900}$
k =3	$y(x = x_{n+2})$	5	$\frac{25}{3072}$
k =3	$y(x = x_{n+1})$	5	$\frac{3}{400}$

Table3: Features of Four-Step Method

k =4	**Evaluating Point**	**order**	**Error Constant**
k =4	$y(x = x_{n+4})$	6	$-\frac{1159}{120960}$
k =4	$y(x = x_{n+\frac{7}{2}})$	6	$-\frac{53}{7560}$
k =4	$y(x = x_{n+3})$	6	$-\frac{37}{4480}$
k =4	$y(x = x_{n+2})$	6	$-\frac{4459}{552960}$
k =4	$y(x = x_{n+1})$	6	$-\frac{8}{945}$

12.0 Stability of Block Method

13.0 Stability Analysis for Method (9)

The equation (9) When put together formed the block as

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}y_{n+1}\\y_{n+\frac{3}{2}}\\y_{n+2}\end{bmatrix}=\begin{bmatrix}0&0&1\\0&0&1\\0&0&1\end{bmatrix}\begin{bmatrix}y_{n+2}\\y_{n+1}\\y_{n}\end{bmatrix}+h\begin{bmatrix}\frac{7}{6}&-\frac{2}{3}&\frac{1}{6}\\\frac{90}{24}&-\frac{24}{64}&\frac{9}{64}\\\frac{4}{3}&0&\frac{1}{3}\end{bmatrix}\begin{bmatrix}f_{n+1}\\f_{n+\frac{3}{2}}\\f_{n+2}\end{bmatrix}$$

$$+h\begin{bmatrix}0&0&\frac{1}{3}\\0&0&\frac{24}{64}\\0&0&\frac{1}{3}\end{bmatrix}\begin{bmatrix}f_{n+2}\\f_{n+\frac{3}{2}}\\f_{n}\end{bmatrix}$$

Normalizing this by multiplying with the inverse of A^0 we obtain

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}y_{n+1}\\y_{n+\frac{3}{2}}\\y_{n+2}\end{bmatrix}=\begin{bmatrix}0&0&1\\0&0&1\\0&0&1\end{bmatrix}\begin{bmatrix}y_{n+2}\\y_{n+1}\\y_{n}\end{bmatrix}+h\begin{bmatrix}\frac{7}{6}&-\frac{2}{3}&\frac{1}{6}\\\frac{90}{24}&-\frac{24}{64}&\frac{9}{64}\\\frac{4}{3}&0&\frac{1}{3}\end{bmatrix}\begin{bmatrix}f_{n+1}\\f_{n+\frac{3}{2}}\\f_{n+2}\end{bmatrix}$$

$$+h\begin{bmatrix}0&0&\frac{1}{3}\\0&0&\frac{24}{64}\\0&0&\frac{1}{3}\end{bmatrix}\begin{bmatrix}f_{n+2}\\f_{n+\frac{3}{2}}\\f_{n}\end{bmatrix}$$

The first given characteristic polynomial of the hybrid block method is given as

$$\rho(R) = det[RA^0 - A'] \quad \text{where} \quad A^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A' = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\rho(R) = det\left(R\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$= det\left(\begin{bmatrix} R & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & R \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$= det\begin{bmatrix} R & 0 & -1 \\ 0 & R & -1 \\ 0 & 0 & R-1 \end{bmatrix}$$

$$= R(R(R-1)) = R_1 = 0, R_2 = 0, R_3 = 1$$

Since $|R_j| \leq 1 \quad j \in \{1,2,3\}$ then the method as a block is zero stable on its own. The hybrid block method is also consistent as its order $p > 1$.

From [10], we can safely assert the convergence of the block method.

14.0 Stability Analysis for Method (10)

The equation (10) when put together formed the block as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+\frac{5}{2}} \\ y_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} y_{n+3} \\ y_{n+2} \\ y_{n+1} \\ y_n \end{bmatrix} + \begin{bmatrix} \frac{361}{360} & \frac{101}{120} & \frac{152}{225} & -\frac{61}{360} \\ \frac{64}{45} & \frac{1}{15} & \frac{64}{225} & -\frac{4}{45} \\ \frac{1625}{1152} & \frac{125}{384} & \frac{5}{9} & \frac{125}{1152} \\ \frac{57}{40} & \frac{9}{40} & \frac{24}{25} & \frac{3}{40} \end{bmatrix}\begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+\frac{5}{2}} \\ f_{n+3} \end{bmatrix}$$

$$+h\begin{bmatrix} 0 & 0 & 0 & \frac{599}{1800} \\ 0 & 0 & 0 & \frac{71}{225} \\ 0 & 0 & 0 & \frac{365}{1152} \\ 0 & 0 & 0 & \frac{63}{200} \end{bmatrix}$$

Proceeding in a similar manner as in the case of method(9)above yields the values:

$R_1 = 0, R_2 = 0, R_3 = 0, R_4 = 1$ Since $|R_j| \leq 1,\ j \in \{1,2,3,4\}$, the method as a block is zero stable on its own.

15.0 Stability Analysis for Method (11)

The equation (11) when put together formed the block as further analysis give the values the R's in the case as: $R_1 = R_2 = R_3 = R_4 = 0, \quad R_5 = 1$ Since $|R_j| \leq 1,\ j \in \{1,2,3,4,5\}$, then the method as a block is zero stable on its own.

16.0 Numerical Examples

The following notation are adopted in the tables of numerical results of this section.

2S1HBM Two step one hybrid method

3S1HBM Three step one hybrid method

4S1HBM Four step one hybrid method

Example 5.1

A constant-coefficient non-homogeneous problem

$$y'(x) + y(x) = x \quad y(0) = 0 \quad h = 0.1 \quad 0 \le x \le 1$$

Analytical solution: $y(x) = x + e^{-x} - 1$

Example 5.2

A constant-coefficient homogenous problem

$$y'(x) + y^2(x) = 0 \quad y(0) = 1 \quad h = 0.1 \quad 0 \le x \le 1$$

Analytical solution: y(x) = $\dfrac{1}{x+1}$

Example 5.3

Constant coefficient non-homogeneous problem.

$$y' + 4y = x^2, \quad y(0) = 1, \quad h = 0.2, \quad 0 \le x \le 2 .$$

Block method are implemented on all the examples using Maple.
Results generated are as presented in Tables 4 - 9.

Table 4: Numerical Results for Example 5.1

x	Exact	2S1HBM	3S1HBM	4S1HBM
0.1	0.004837418	0.004837327	0.004837409	0.004837417
0.2	0.018730753	0.018730667	0.018730747	0.018374173
0.3	0.040818220	0.040818068	0.040818216	0.040818220
0.4	0.070320046	0.070319905	0.070315622	0.070320045
0.5	0.106530659	0.106530472	0.010652928	0.106530658
0.6	0.148811636	0.148811463	0.148805738	0.148811635
0.7	0.196585303	0.196850981	0.196572279	0.196585303
0.8	0.249328964	0.249328776	0.249321748	0.249328963
0.9	0.306569659	0.306569448	0.306550329	0.367879440
1.0	0.367879441	0.367879248	0.367856097	0.367879440

Table 5: Error of the Methods for Example 5.1

x	2S1HBM	3S1HBM	4S1HBM
0.1	9.09×10^{-08}	8.64×10^{-09}	7.23×10^{-10}
0.2	8.55×10^{-08}	5.32×10^{-09}	3.57×10^{-10}
0.3	1.51×10^{-07}	3.94×10^{-09}	1.90×10^{-10}
0.4	1.40×10^{-07}	4.42×10^{-06}	4.50×10^{-10}
0.5	1.87×10^{-07}	1.38×10^{-06}	2.00×10^{-10}
0.6	1.72×10^{-07}	1.38×10^{-06}	6.00×10^{-10}
0.7	2.05×10^{-07}	5.90×10^{-06}	3.00×10^{-10}
0.8	1.88×10^{-07}	1.30×10^{-06}	6.00×10^{-10}
0.9	2.10×10^{-07}	1.93×10^{-06}	2.00×10^{-10}
1.0	1.97×10^{-07}	2.33×10^{-05}	5.00×10^{-10}

Table 6: Numerical Results for Example 5.2

x	Exact	2S1HBM	3S1HBM	4S1HBM
0.1	0.904837418	0.9048373271	0.9048544369	0.9048374176
0.2	0.818730753	0.8187306675	0.8187360313	0.8187307529
0.3	0.740818221	0.7408080688	0.7408409447	0.7408182205
0.4	0.670320046	0.6703199059	0.6703532159	0.6703200455
0.5	0.606530659	0.6065304720	0.6065531749	0.6065306589
0.6	0.548811636	0.5488114642	0.5488453053	0.548116355
0.7	0.496585303	0.4965850982	0.4966251096	0.496583032
0.8	0.449328964	0.4493287763	0.4493594271	0.4493289635
0.9	0.406569659	0.4065694490	0.4066070744	0.4065696590
1.0	0.367879441	0.3678792489	0.3678792412	0.3698784407

Table 7: Error of Methods for Example 5.2

X	2S1HBM	3S1HBM	4S1HBM
0.1	9.09×10^{-08}	1.70×10^{-09}	4.00×10^{-10}
0.2	8.55×10^{-08}	5.28×10^{-09}	1.00×10^{-10}
0.3	1.51×10^{-07}	2.27×10^{-09}	5.00×10^{-10}
0.4	1.40×10^{-07}	1.40×10^{-06}	5.00×10^{-10}
0.5	1.87×10^{-07}	1.87×10^{-06}	1.00×10^{-10}
0.6	1.72×10^{-07}	3.37×10^{-06}	5.00×10^{-10}
0.7	2.05×10^{-07}	3.98×10^{-06}	2.50×10^{-10}
0.8	1.88×10^{-07}	3.05×10^{-06}	5.00×10^{-10}
0.9	2.10×10^{-07}	3.75×10^{-06}	0.00×10^{-10}
1.0	1.97×10^{-07}	3.70×10^{-05}	2.00×10^{-10}

Table 8: Numerical Results for Example 5.3

x	Exact	2S1HBM	3S1HBM	4S1HBM
0.1	0.451537434	0.4505943450	0.4516128504	0.4513030279
0.2	0.216837251	0.2163416043	0.2166022719	0.2167967912
0.3	0.134330170	0.1337203846	0.1349133567	0.1340830071
0.4	0.130738385	0.1305384997	0.1310891337	0.1307070573
0.5	0.173993275	0.1738652131	0.1741365212	0.1739696514
0.6	0.249222567	0.2829076511	0.2493278552	0.2492145946
0.7	0.349832305	0.3649610942	0.3498118578	0.3498274266
0.8	0.472859633	0.4796392211	0.4789732650	0.4728570806
0.9	0.616973225	0.6200181314	0.6168950386	0.6169717190
1.0	0.781574979	0.7829394623	0.7814285410	1.7815743969

Table 9: Errors of Method for Example 5.3

x	2S1HBM	3S1HBM	4S1HBM
0.1	9.43×10^{-04}	7.54×10^{-05}	2.34×10^{-04}
0.2	4.96×10^{-04}	2.35×10^{-04}	4.03×10^{-05}
0.3	6.10×10^{-04}	5.83×10^{-04}	2.47×10^{-04}
0.4	2.00×10^{-04}	3.31×10^{-04}	3.13×10^{-05}
0.5	1.28×10^{-04}	1.43×10^{-04}	2.36×10^{-05}
0.6	3.36×10^{-02}	1.05×10^{-04}	7.97×10^{-06}
0.7	1.51×10^{-02}	1.51×10^{-02}	4.87×10^{-06}
0.8	6.77×10^{-03}	3.77×10^{-05}	2.55×10^{-06}
0.9	3.04×10^{-03}	7.82×10^{-05}	1.54×10^{-06}
1.0	1.36×10^{-03}	1.46×10^{-04}	5.82×10^{-07}

17.0 Summary and Conclusion

A class of hybrid collocation methods for direct solution of initial value problems of general first order differential equations have been developed in this research. The collocation technique yielded very consistent and zero stable implicit hybrid methods. The methods are implemented without the need for development of predictor nor requiring any other method to generate starting values. The derived integration scheme were applied to some selected problem of varied peculiarities and numerical evidences show that the methods are accurate and effective.

18.0 References

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