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A New Method for Calculating the Energy of the Bound States in a Non-Symmetrical Potential Well

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Abstract

In this paper simple method for calculating the energy of the bound states of a nonrelativistic particle in a potential well is presented. The particle in one-dimensional rectangular potential well was considered first in which the solution of Schrödinger equation is used to find its energy of bound states. We then generalized our method for non -symmetrical potential well. This method is a good tool that can be used to modify the generalized Bohr- sommerfeld quantization Rule.

Keywords: Potential well, Bohr- sommerfeld quantization, bound states energy, wave function.

1.0 Introduction

In this paper we present a simple method for calculating the energies of the bound states of a non-relativistic particle in a potential well [1,2,3,4,5]. To demonstrate the method, we consider a one dimensional rectangular potential as seen below and a particle coming fromx $< -\frac{L}{2}$ to x $> \frac{L}{2}$ of the potential well.

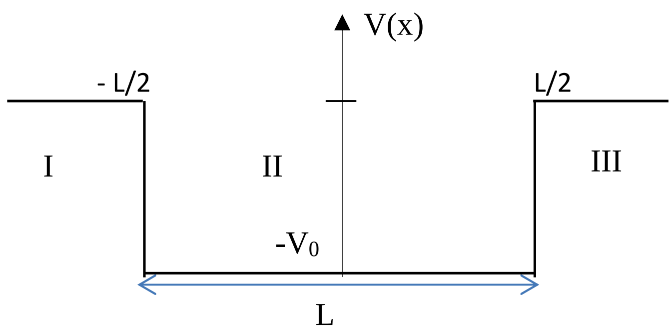

Fig.1: One dimensional rectangular potential well of width L and depth V_0

$$V(x) = \begin{cases} 0 \quad f \quad |x| > \frac{L}{2} \\ -V_0 f \quad |x| \le \frac{L}{2} \end{cases} \quad (1)$$

Where V_0 is the potential depth, and L is its width.

As usual, for the bound states we need the solutions of the Schrödinger equation which varnishes at $x \to \infty$, this gives the wave functions

$\psi_1(x) = Ae^{K'_{II}}$, $\psi_2 = Be^{i l} + Ce^{-i l} \alpha$ $\psi_3(x) = De^{-K'_{II}x}$ for region I, II and III respectively i.e

$$\psi_I(x) = Ae^{K'_{II}x}\alpha \quad < -\frac{1}{2}$$

$$\psi_I\,(x) = Be^{i l} + Ce^{-i l} \quad \alpha \; -\frac{1}{2} \le x \le \frac{1}{2}$$

$$\psi_I\,(x) = De^{-K'_{II}x}\alpha \quad > \frac{1}{2}$$

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For finiteness of the wave function at $x \to \infty$ we have the above equations.

By definition $k = \sqrt{\frac{2m}{\hbar}}$, E is the particle's energy with reference to the bottom of the well, m is the mass of the particle $k' = \sqrt{U - k^2}a \quad = 2mv_0/\hbar^2$

Taking the constant $\frac{\hbar^2}{2m} = 1$, it means that all the energies are measured in a system of units in which $\frac{\hbar^2}{2m} = 1$.

Matching the wave functions and their derivatives, we obtain a homogenous system of equations given as:

$$\psi_I(x) = \psi_I(x)a \quad \frac{d\psi_I}{d} = \frac{d\psi_I}{d} \tag{2a}$$

$$\psi_I(x) = \psi_I(x)a \quad \frac{d\psi_I}{d} = \frac{d\psi_I}{d} \tag{2b}$$

Applying boundary conditions $\left(x = \frac{-L}{2}\right)$ on equation (2a) above we have:

$$Ae^{-k'_u\frac{L}{2}} = Be^{-U\frac{L}{2}} + Ce^{-U\frac{L}{2}} \tag{3}$$

$$Ak'_u e^{-k'_u\frac{L}{2}} = U\left(Be^{-U\frac{L}{2}} - Ce^{-U\frac{L}{2}}\right) \tag{4}$$

Also, applying the same boundary conditions $\left(x = \frac{L}{2}\right)$ on equation (30b) above we have:

$$Be^{-U\frac{L}{2}} + Ce^{U\frac{L}{2}} = De^{-k'_u\frac{L}{2}} \tag{5}$$

$$U\left[Be^{U\frac{L}{2}} - Ce^{U\frac{L}{2}}\right] = -k'_u De^{-k'_u\frac{L}{2}} \tag{6}$$

These equations can be solved simultaneously.

However, when we consider the particle within the region

$x = \frac{-L}{2}a \quad = \frac{L}{2}$ then we consider this particle as bound and we find a general equation for a bound system as:

$$\psi(x) = A \quad + B \tag{7}$$

At the boundary of the well, the particle is not there.

Therefore $\psi(x) = 0 \; a \quad = \pm\frac{L}{2}$

i.e $0 = A \quad \left(\frac{L}{2}\right) + B \quad \left(\frac{L}{2}\right), a \quad = \frac{L}{2}$

$$\Rightarrow A \quad \frac{L}{2} + B \quad \frac{L}{2} = 0 \tag{8}$$

Also

$0 = A \quad \left(-k\frac{L}{2}\right) + B \quad \left(-k\frac{L}{2}\right)$

$$\Rightarrow -A \quad \frac{L}{2} + B \quad \frac{L}{2} = 0 \tag{9}$$

Where sine is an odd function and cosine is an even function of x. Adding (8) and (9) gives

$$2B \quad \frac{L}{2} = 0 \tag{10}$$

Subtracting (9) from (8) gives

$$2A \quad \frac{k}{2} = 0 \tag{11}$$

There is no value of k for which both $c \quad \left(\frac{K}{2}\right)$ and $s \quad \left(\frac{K}{2}\right)$ are simultaneously zero. And we should not set A and B equal to zero.

We satisfy the equation by either choosing k such that $c \quad \left(\frac{K}{2}\right)$ is zero and A is zero (1st class solution) or by choosing k such that $s \quad \left(\frac{K}{2}\right) = 0$ and B is zero (2nd class solution).

Hence, $\psi(x) = B \quad , whe \quad \sin\frac{K}{2} = 0$

The allowed values of k for the 1st class solution are

$$\frac{k}{2} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots \dots \dots \tag{12}$$

$\Rightarrow k_n = \frac{n}{L}$

Where $n = 1,3,5,7, \dots \dots \dots$

Also the allowed values of k for the 2nd class solution are

$$\frac{k}{2} = \pi, 2\pi, 3\pi, \dots \dots \dots \dots \tag{13}$$

$$\text{Since } \sin\pi = \sin 2\pi = s \quad 3\pi = \cdots = 0 \tag{14}$$

$$\Rightarrow k_n = \frac{n}{L} \tag{15}$$

Where $n = 2,4,6,8, \dots \dots \dots \dots$
Hence

$$\psi_n(x) = B_n C \quad k_n x, k_n = \frac{n}{L}, n = 1,3,5, \dots \dots \dots \qquad (16)$$

$$\psi_n(x) = A_n S \quad k_n x, k_n = \frac{n}{L}, n = 2,4,6, \dots \dots \dots \qquad (17)$$

The number of energy level n is used to label the corresponding eigen values E_n. This is obtained from the relation

$$E_n = \frac{(\hbar k_n)^2}{2m} \qquad (18)$$

Or $\qquad k = \sqrt{2mE_n} / \hbar$

$$whe \quad _n = \frac{n}{L}$$

$$\frac{k^2{}_n \hbar^2}{2m} = E_n \quad o \quad E_n = k^2{}_n$$

$$\Rightarrow E_n = \frac{n^2 \pi^2 \hbar}{2mL^2} \qquad (19)$$

Equation (19) shows that only certain values of the total energy E are allowed. The total energy of the particle in the box is quantized.

2.0 A New Approach to Bound States

The aim and objective is to show a different way to derive the equation for the eigen-energies E_n which may be generalized to potentials of arbitrary form.

3.0 Self Consistent Reflections In The Potential Well

From the self-consistent reflection in the potential well Fig.1, let us consider a particle in the well which starts propagation from the left wall $x = -L/2$ towards the right wall at $x = L/2$. The propagation is described by the wave function

$$\psi_R(x) = e^{i\,(x+L/2)}. \qquad (20)$$

After reaching the right wall, the particle acquires the phase factor e^{L}, is reflected from the right with reflection amplitude $\rho(k)$, and propogates towards the left wall, which is described by the wave function

$$\psi_L(x) = \rho(x) e^{L} \quad e^{-i\,(x-L/2)} \qquad (21)$$

After reaching the left wall, the particle acquires and additional phase factor e^{L}. Repeating the same process, a point is reached where the wave function is

$$\psi_R(x) = \rho^2(k) e^{2L} \quad . e^{i\,(x+L/2)} \qquad (22)$$

which is identical to the initial wave function except for the amplitude. For this to be identical to the former, we have:

$$\rho^2(k) e^{2L} \quad = 1 \qquad (23)$$

The solution of this equation gives all the eigen values k_n and therefore

$$E_n = k_n^2 \qquad (24)$$

The next step is to find $\rho(k)$ and substitute it into (23)
From the relation:

$$\rho^2(k) e^{2L} \quad = 1 \qquad (25)$$

We recall here how to find the amplitude $\rho(k, u)$ for reflection of a particle with wave number k from the potential step of height U. let suppose that the step is at $x \geq 0$. The wave function of the particle is

$$\psi(x) = \begin{cases} e^{i} \quad + \rho(k,U) e^{-i} \quad , \; x < 0 \\ \tau(k,U) e^{-k'_U x}, \qquad x > 0 \end{cases} \qquad (26)$$

Applying boundary condition

$$\psi_1(x)|_{x=0} = \psi_2|_{x=0} \text{ and } \frac{d\psi_1(x)}{d}\bigg|_{x=0} = \frac{d\psi_2(x)}{d}\bigg|_{x=0}$$

$$\Rightarrow e^{i\,(0)} + \rho(k,U) e^{-i\,(0)} = \tau(k,U) e^{-k'_U(0)}$$

$$1 + \rho(k,U) = \tau(k,U) \qquad (27)$$

and

$$ike^{i\,(0)} - i \quad (k,U) e^{-i\,(0)} = -k'_U \tau(k,U) e^{-k'_U(0)}$$

$$i \; - i \quad (k,U) = -k'_U \tau(k,U)$$

or

$$i\,[1 - \rho(k,U)] = -k'_U \tau(k,U) \qquad (28)$$

A New Method for Calculating… ***Y.Y. Ibrahim, Hamisu Ado, Idris Ahmad J of NAMP***

Solution of these equations can be found from the following:
From (27)
$\rho(k,U) = \tau(k,U) - 1$ (29)
Put (29) into (28) to get;
$\Rightarrow ik\,[1 - (\tau(k,U) - 1)] = -k'_U\tau(k,U)$
After expanding and collecting like terms we get;
$\tau(k,U) = \frac{-2ik}{(k'_U - ik\ )}$ (30)
Multiplying numerator and denominator of (30) by i we have:
$\tau(k,u) = \frac{2k}{k + ik'_u}$ (31)
Substitute (31) into (29) to get
$$\rho(k,u) = \frac{2k}{k + ik'_u} - 1$$
i.e $\rho(k,u) = \frac{k - ik'_u}{k + ik'_u}$ (32)

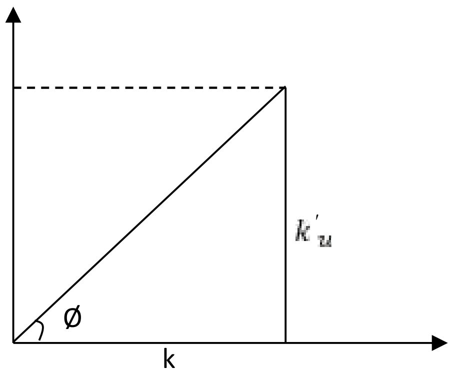

Fig.2: Argand diagram

From Fig.2 we have [6]
$Z = k + ik'_U,\ |Z| = \sqrt{k^2 + k'^2_U},$ (33)
$$\cos\emptyset = \frac{k}{\sqrt{k^2 + k'^2_U}}$$
$\sin\emptyset = \frac{k'_U}{\sqrt{k^2 + k'^2_U}},$
$$\frac{\sin\emptyset}{\cos\emptyset} = \tan\emptyset = \frac{k'_U}{k}$$
$\Rightarrow \sin\emptyset = \tan\emptyset\cos\emptyset = \frac{k'_U}{k}\cos\emptyset$ (34)
Where
$k'_U = \sqrt{U - k^2}$
$\Rightarrow k'^2_U = U - k^2$
$$\cos\emptyset = \frac{k}{\sqrt{k^2 + U - k^2}} = \frac{k}{\sqrt{U}}$$
$\Rightarrow k = \sqrt{U}\cos\emptyset$ $\sin\emptyset = \frac{k'_U}{k}.\frac{k}{\sqrt{U}} = \frac{k'_U}{\sqrt{U}}$
$\tan\emptyset = \frac{k'_U}{k} \Rightarrow \emptyset = \tan^{-1}\left(\frac{k'_U}{k}\right)$ $\cos\emptyset = \frac{k}{\sqrt{U}} \Rightarrow \emptyset = \cos^{-1}\left(\frac{k}{\sqrt{U}}\right)$ (35)
Substitution gives us:
$\rho(k,U) = \frac{\sqrt{U}(\cos\emptyset - i\sin\emptyset)}{\sqrt{U}(\cos\emptyset + i\sin\emptyset)} = \frac{\cos\emptyset - i\sin\emptyset}{\cos\emptyset + i\sin\emptyset}$ (36)
From the Euler formula
$\cos\emptyset + i\sin\emptyset = e^{i\emptyset}$
$\Rightarrow \cos\emptyset - i\sin\emptyset = e^{-i\emptyset}$

$\rho(k,U) = \frac{e^{-i\emptyset}}{e^{i}} = e^{-i\emptyset}.e^{-i\emptyset} = e^{-2i\emptyset}$ (37)

$\Rightarrow \rho^2(k,U) = e^{-4i\emptyset}$ (38)

Substitution gives us:

$e^{-4i\emptyset(k,U)}.e^{2i} \quad = 1$

i.e $\qquad e^{2i} \quad = e^{4i\emptyset(k,U)}$ (39)

Taking the natural log of both sides of equation (39) we get:

$2i \quad = 4i\emptyset(k,U)$

$k \quad = 2\emptyset(k,U)$

$k \quad - 2\emptyset(k,U) = 0$ (40)

Put equation (35) in equation (40) to get

$k \quad - 2c \quad ^{-1}\left(\frac{k}{\sqrt{U}}\right) = n$ (41)

where n is an integer which can be -1 and 0. The value n=-1 corresponds to k=0, this means a constant wave function for a particle on the bottom of the well, however this constant should be 0, because the wave function should vanish at $|x| = \infty$. Thus, the lowest acceptable eigenvalue is n=0.

4.0 Non-Symmetrical Well

If the two walls of the potential are somehow different, as seen in Fig. 3,

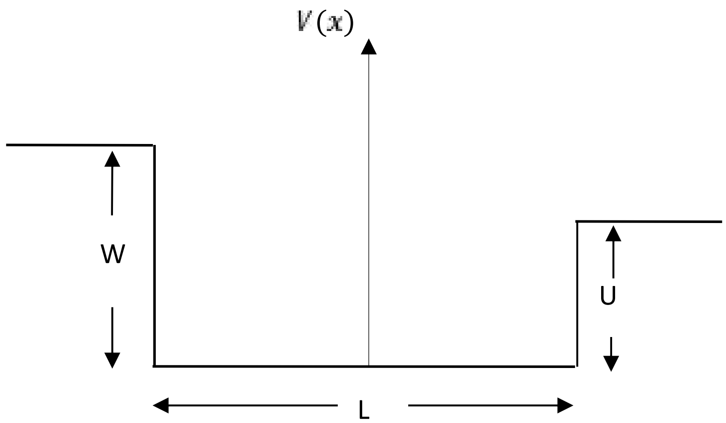

Fig.3. The square potential well of width L with two different levels W at the left and U at the right wall

eqn. (23) becomes

$\rho_l(k)\rho_r(k)e^{2i} \quad = 1$ (42)

where $\rho_l(k)a \quad \rho_r(k)$ are the reflection amplitude from the left and right wall respectively. Also, eqn. (42) can be generalized to

$k \quad - \emptyset_l(k) - \emptyset_r(k) = n$ (43)

Where $\emptyset_l(k)a \quad \emptyset_r(k)$are the phase factors for the reflection from the left and right walls respectively.

ConsideringFig.3, the reflection amplitude at the left wall is now

$\rho_l(k) = \rho(k,W) = \frac{k - ik'_W}{k + ik'_W} = e^{-2i\emptyset(k,W)}$ (44)

Where

$k'_W = \sqrt{w - k^2}a \quad \emptyset(k,W) = \arctan\left(\frac{k'_W}{k}\right) = \arccos\left(\frac{k}{\sqrt{W}}\right)$

Thus, eqn. (43) for eigen -levels is

$k \quad - \arccos\left(\frac{k}{\sqrt{U}}\right) - \arccos\left(\frac{k}{\sqrt{W}}\right) = n$ (45)

From the identity [7],

$\arccos(x) + \arccos(y) = \arccos\left[x \quad - \sqrt{(1-x^2)(1-y^2)}\right]$ (46)

Equation (45) becomes

$k \quad - a \qquad \left(\frac{k^2 - k'_U k'_W}{\sqrt{U}}\right) = n$ (47)

5.0 Complex Non-Symmetrical Potential Well

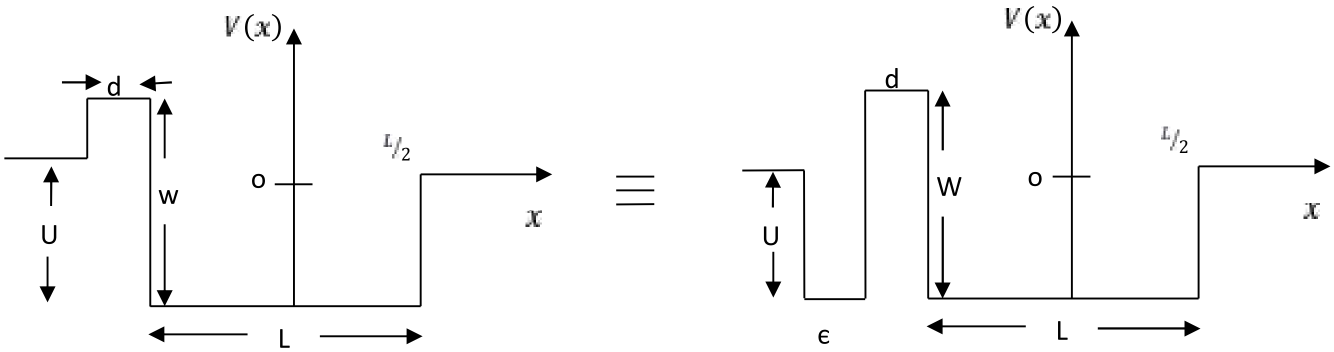

Fig. 4: The square potential well of width L and depth U with a barrier W-U of width d at the left wall.

A more complicated nonsymmetrical potential well with the same level to the right and left is shown in Fig. 4. The potential on the left is equivalent to the potential at the right with the barrier separated from the well by an infinitesimal gapϵ [8-14]. The reflection amplitude from the right wall is the same $\rho_r(k) = \rho(k, U)$, as above, but $\rho_l(k)$requires some calculation. This can be done by splitting the potential with an infinitesimal gap ϵ as shown inFig.5.

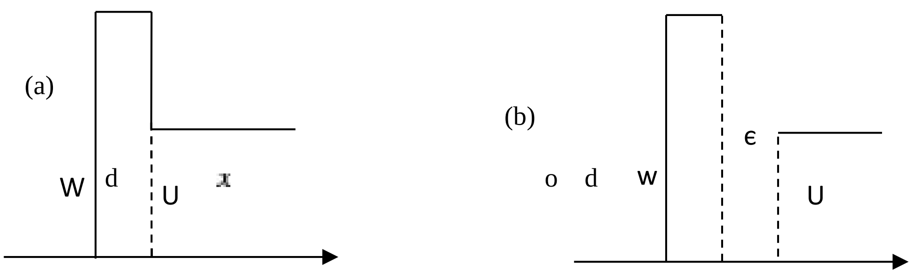

Fig. 5: (a & b): Reflection from the potential of the wall in Fig.4.

The potential in Fig.4 can be split by an infinitesimal gap ϵ as seen in Fig.5 (b).

Now let us denote reflection and transmission amplitudes of the first rectangular barrier as ρ_1 and $\tau_1$and reflections amplitude of the second step asρ_2. Then, taking into account multiple reflections inside the gap of infinitesimal width ϵ, where the phase acquired between the walls can be neglected, we obtain for the reflection ρ_l of the whole system the infinite sum.

$$\rho_l = \rho_1 + \sum_{n=0}^{\infty} \tau_1 \rho_2 (\rho_1 \rho_2)^n \tau_1 = \rho_1 + \frac{\tau_1^2 \rho_2}{1-\rho_1 \rho_2} \tag{48}$$

Comparing (48) with infinite geometric series

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$$

$\Rightarrow a = \tau_1^2 \rho_2, r = \rho_1 \rho_2, k = n$

$\rho_l\ (k) cc$

$$\rho_l\ (k) = \rho(k, W, d) + \frac{\tau_1^2(k, W, d)\rho(k, U)}{1 - \rho(k, W, d)\rho(k, U)}$$

$$\rho_l\ (k) = \frac{\rho(k,W,d)+\rho(k,U)[\tau_1^2(k,W,d)-\rho^2(k,W,d)]}{[1-\rho(k,W,d)\rho(k,U)]} \tag{49}$$

where we have introduced notations for reflection $\rho(k, W, d)$ and transmission $\tau(k, W, d)$ amplitudes of the rectangular potential barrier of height W and width d, these amplitudes can be calculated as follows:

Let us look at the rectangular barrier of height W and width d in fig.4 and denote reflection and transmission amplitudes from vacuum into the barrier as ρ_1, τ_1, and reflection and transmission amplitudes from inside the barrier into vacuum asρ_2, τ_2, obtained above as:

$$\tau_1 = \frac{2k}{k+ikr_W}, \tau_2 = \frac{2ikr_W}{k+ikr_W}, \rho_1 = -\rho_2 = \frac{k-ikr_W}{k+ikr_W} \tag{50}$$

And

$$\tau_1 \tau_2 = 1 - \rho^2 \tag{51}$$

We can obtain formulae for reflection, ρ,and transmission τ,of the whole barrier. Indeed, taking into account multiple reflections at the two edges at$x = 0$, and $x = d$, and also extinction $e = \exp(-k'_W d)$ for the propagation between the edges, we get ρ and τ as sums of infinite geometrical progressions thus:

$$\rho = \rho_1 + \sum_{n=0}^{\infty} \tau_2 \rho_2 (\rho_2 e)^{2n} \tau_1 \quad (52)$$

Comparing with infinite geometric series as above we get:

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

$\Rightarrow \tau_1 \tau_2 \rho_2 = a,\ \rho_2{}^2 e^2 = r$

Equation (52) can be written as

$$\rho = \rho_1 + \frac{\tau_1 \tau_2 \rho_2}{1-\rho_2{}^2 e^2} \quad (53)$$

But from eqn. (51), $\tau_1 \tau_2 = 1 - \rho_1{}^2$ a f e (50), $\rho_1 = -\rho_2$ thus equation (53) becomes

$$\rho = \rho_1 + \frac{(1-\rho_1{}^2)(-\rho_1)}{1-(-\rho_1)^2 e^2} = \rho_1 - \frac{\rho_1 + \rho_1{}^3}{1-\rho_1{}^2 e^2} \quad (54)$$

Given that

$$\rho_1{}^2 = |\rho_1 \rho_1{}^*| = 1, \rho_1{}^3 = \rho_1 |\rho_1 \rho_1{}^*| = \rho_1 \quad (55)$$

Substituting (55) in (54) we get

$$\therefore \rho = \rho_1 - \frac{\rho_1 + \rho_1{}^3}{1-\rho_1{}^2 e^2} = \frac{\rho_1}{1} - \frac{\rho_1 + \rho_1{}^3}{1-\rho_1{}^2 e^2} = \frac{\rho_1(1-\rho_1{}^2 e^2) - \rho_1 + \rho_1{}^3}{1-\rho_1{}^2 e^2}$$

$$= \frac{\rho_1(1-\rho_1{}^2 e^2) - \rho_1 + \rho_1{}^3}{1-\rho_1{}^2 e^2} = \frac{\rho_1(1-e^2)}{1-\rho_1{}^2 e^2} \quad (56)$$

And

$$\tau = e \sum_{n=0}^{\infty} \tau_2 (\rho_2 e)^{2n} \tau_1 = \frac{e \tau_1 \tau_2}{1-\rho_1{}^2 e^2} = \frac{e(1-\rho_1{}^2)}{1-\rho_1{}^2 e^2} \quad (57)$$

S (53) i (56) a (57), a $\rho_1 = \rho(k, W), \rho = \rho(k, W, d)$,

$\tau = \tau(k, W, d)$ w :

$$\rho(k, W, d) = \rho(k, W).\frac{1-e^{-2krw}}{1-\rho^2(k,W)e^{-2krw}} \quad (58)$$

$$\tau(k, W, d) = e^{-krw} . \frac{1-\rho^2(k,W)}{1-\rho^2(k,w)e^{-2krw}} \quad (59)$$

From eqn. (36) $\rho(k, W) = e^{-2i\emptyset(k,W)}$

For $d \to 0$

i. $\rho(k, W, d) = \rho(k, W).\frac{1-e^{2krw}}{1-\rho^2(k,W)e^{-2krw}}$ $a - i$

$$\therefore \rho(k, W, d) \to 0 \quad (60)$$

ii. $\tau(k, W, d) = e^{-2krw} . \frac{1-e^{-4i\emptyset(k,W)}}{1-e^{-4i\emptyset(k,W)}.e^{-2krw}} = e^0.\frac{1-e^{-4i\emptyset(k,W)}}{1-e^{-4i\emptyset(k,W)}.e^0} = 1$

$$\therefore \tau(k, W, d) \to 1 \quad (61)$$

iii. $\therefore \rho_l(k) \to \rho(k, U)$ (62)

F $d \to \infty$

i- $\rho(k, W, d) = e^{-2i\emptyset(k,W)}.\frac{1-e^{-2krw}}{1-e^{-4i\emptyset(k,W)}.e^{-2krw}}$

$$\therefore \rho(k, W, d) \to \rho(k, w) \quad (63)$$

ii- $\tau(k, W, d) = e^{-2krw} . \frac{1-e^{-4i\emptyset(k,W)}}{1-e^{-4i\emptyset(k,W)}.e^{-2krw}}$ a e^{-2krw}

$= 0$

$$\therefore \tau(k, W, d) \to 0 \quad (64)$$

F $\rho(k, W, d) \propto -i$, we have:

$$\rho(k, W, d) = e^{-2i\emptyset(k,W)}.\frac{1-e^{2krw}}{1-e^{-4i\emptyset(k,W)}.e^{-2krw}} = -c$$

Where

$c = |\rho(k, w, d)| e^{i\emptyset(k,W,d)}$

$$\Longrightarrow \rho(k, W, d) = -i|\rho(k, W, d)| e^{i\emptyset(k,W,d)} \quad (65)$$

F $\tau(k, W, d) \propto ie^{-2krw}$; w ha

$$\tau(k, W, d) = e^{-2krw} \cdot \frac{1-e^{-4i\emptyset(k,W)}}{1-e^{-4i\emptyset(k,W)} \cdot e^{-2krw}} a\ e^{-krw}$$

$$\Longrightarrow \tau(k, w, d) = |\tau(k, W, d)| e^{i\emptyset(k,W,d)} \quad (66)$$

whe

$$|\tau(k, W, d)|^2 + |\rho(k, W, d)|^2 = 1 \quad (67)$$

and

$\emptyset(k,W,d) = \arctan\{\cot[2\emptyset(k,W)]\tanh(k'_W d)\}$ (68)

This phase has the following asymptotic behavior:

$$\emptyset(k,w,d) \to \begin{cases} 0 \ f \quad d \to 0 \\ \frac{\pi}{2} - 2\emptyset(k,w) \ f \quad d \to \infty \end{cases} \quad (69)$$

Substitution gives us

$$\rho_1 \ (k) = \frac{\rho(k,W,d) + \rho(k,U)[\iota^2(k,W,d) - \rho^2(k,W,d)]}{1 - \rho(k,U)\rho(k,W,d)}$$

Expansion gives us

$$\rho_1 \ (k) = -i|\rho(k,W,d)|e^{\iota\emptyset(k,W,d)} + e^{-2\iota\emptyset(k,U)} \cdot e^{2\iota\emptyset(k,W,d)} \quad (70)$$

But, $f \quad d \to 0, \emptyset(k,W,d) \to 0 \ a \quad \rho(k,W,d) \to 0$ thus equation (70) becomes

$$\rho_l(k) = e^{-2\iota\emptyset(k,U)} \cdot e^0 = e^{-2\iota\emptyset(k,U)} \quad (71)$$

Since we are considering the left wall, we assumed u to be zero equation (71) becomes

$$\rho_l(k) = e^{-2\iota\emptyset_l(k)}, \quad (72)$$

Comparison and some mathematical arrangement gives us

$$\emptyset_l(k) = \emptyset(k,U) - \emptyset(k,W,d) + \arctan\left\{\frac{|\rho(k,W,d)| \ c \quad [2\emptyset(k,U) - \emptyset(k,W,d)]}{1 + |\rho(k,W,d)| \ s \quad [2\emptyset(k,U) - \emptyset(k,W,d)]}\right\} \quad (73)$$

Substitution gives the following equation for eigenvalues $k_n O \quad k$:

$$k \quad - \emptyset_r(k) - \emptyset_l(k) = n$$

$$\Rightarrow k \quad - \emptyset(k,U) - \emptyset(k,U) + \emptyset(k,W,d)$$

$$+ \arctan\left\{\frac{|\rho(k,W,d)| \ c \quad [2\emptyset(k,U) - \emptyset(k,W,d)]}{1 + |\rho(k,W,d)| \ s \quad [2\emptyset(k,U) - \emptyset(k,W,d)]}\right\} = n \quad (74)$$

6.0 The Generalized Bohr – Sommerfeld quantization rule is [15]

$$\int_a^b k(x) d \quad - \pi/_2 = n \quad (75)$$

Let us see how well this rule is satisfied in the potential well (Fig.1)

In this case, $k(x) = k$ does not depend on x, we obtained

$$\int_{-L/_2}^{L/_2} k \quad = k[x]_{L/_2}^{L/_2} = k\left(\frac{L}{2} + \frac{L}{2}\right) = k$$

$$\therefore \int_{-L/_2}^{L/_2} k \quad = k$$

And the precise quantization rule becomes (41). This means that $\pi/_2$ must be replaced by 2arccos$\left(k/_{\sqrt{U}}\right)$. The last quantity becomes π as$u \to \infty$; thus the quantization rule for infinite u becomes.

$$K \quad - \pi = n$$

With integer,$n \geq 0$. The precise quantization rule (41) deviates from (75). This shows that the quantization rule (75) should be modified to

$$\int_a^b k(x) d \quad - \emptyset_l(x) - \emptyset_r(k) = n \quad (76)$$

Where $\emptyset_l(x) a \quad \emptyset_r(x)$ are the reflection phases at the turning points $x_l a \quad x_r$, where $k(x_l) = k(x_r) = 0$.

In view of that, we consider a simple case of a potential well shownin Fig.6.

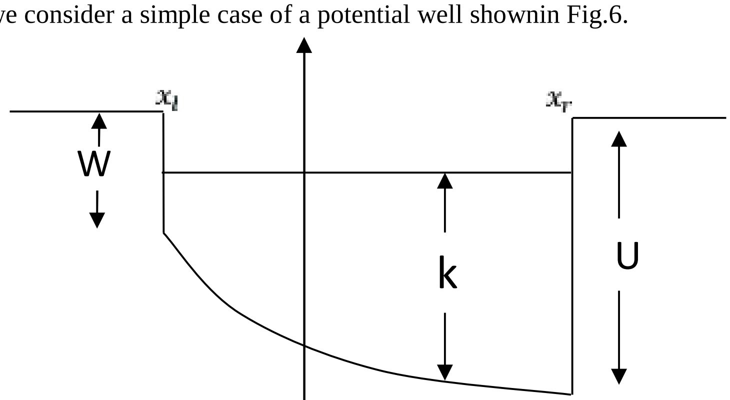

Fig. 6: Non-symmetrical potential well.

The reflection phase for such a potential are:

$$\emptyset_l = a \qquad \sqrt{\frac{k^2 - V(x_l)}{W}} \tag{77}$$

$$\emptyset_r = a \qquad \sqrt{\frac{k^2 - V(x_r)}{U}}$$

and the quantization rule (75) becomes

$$\int_a^b k(x)d \quad - a \qquad \sqrt{\frac{k^2 - V(x_l)}{W}} - a \qquad \sqrt{\frac{k^2 - V(x_r)}{U}} = n \tag{78}$$

7.0 Conclusion

In conclusion therefore, in this paper our contribution to the existing knowledge in the literature is that we were able to modify **Bohr – Sommerfeld quantization rule** by considering the reflection and transmission of particles within the potential well of depth V_0 and width L. Our result shows that for symmetrical potential, the one in Fig.1, the quantization rule will be given by equation (41) rather than (75). For non-symmetrical potential well shown in Fig.3, we see that the general rule for quantization will be given by (76) and not equation (75).

8.0 References

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