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Continuous Time Portfolio Optimization with Log Utility

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Abstract

A stochastic asset allocation problem on a finite horizon with log utility programming is considered. Our aim is to choose optimal investment and consumption strategies that would optimize the finite horizon expected discounted log utility. The method of dynamic programming principle is used to obtain the dynamic programming equation, which involve two ordinary differential equations. An explicit solution to the dynamic programming equation is obtained on a special case.

Keywords: Asset allocation, log utility, ordinary differential equation, power series.

1.0 Introduction

Due to the emergence of financial markets, there are numerous opportunities to transfer wealth between different points in time and different places of the universe. Investors must decide on how much to invest in the financial markets and also how to allocate the available amount of wealth to several financial securities. Traditionally, an investor dynamically allocates his wealth between two investment opportunities; risky and riskless assets, and chooses a rate of consumption that would maximize his expected utility of consumption.

Over the years, following the maiden paper of Merton [1], asset allocation has received tremendous attention from researchers and practitioners alike. Merton's portfolio optimization problem involves allocation of wealth to a risk asset with constant interest rate and a risky asset whose price fluctuates from time to time. Therefore, in this paper, we have allowed the interest rate for the riskless asset to fluctuate with time. In particular, we assume that the interest rate follows a Markov process. Examples of such interest rate process are the Vasicek [2], and Cox et al [3] models of interest rate, in which the interest rate is considered to be mean-reverting.

So many research works that are related to Merton's portfolio optimization problem have been carried out by many authors. Many, for example, Bielecki and Pliska [4], and Fleming and Sheu [5] have considered a case in which the investor does not withdraw consumption good from the investment and their aim was to maximize the utility based on the terminal wealth. Some authors such as Cambell and Viceira [6] obtained an approximate analytical solution for optimization on an infinite time horizon. Zion [7], considered a model in which an investor can determine a maximum probability of failing to reach a specific portfolio threshold. He argued that even if an investor has no idea about their attitude to risks, he can specify a stop-loss level and penetrate the attitude towards risks through it. Recently, Kasper and Gordon [8], considered utility maximization problem in a general semi martingale financial model subject to the number of shares held in each risky asset. Their result establishes among other things, the existence of optimal trading strategy under no smoothness requirements on the utility function.

The utility function used in this paper is similar to that of Pang [9], who considered optimization on an infinite time horizon but our method is different and time horizon used here is finite. We consider the case where the investor has log utility function of the form

$$U = \ln(c \quad) \tag{1.0}$$

Where c and w represent the investor's rate of consumption and total wealth respectively. We obtained a linear dynamic programming equations made up of a combination of two ordinary differential equations; a first order linear equation and an inhomogeneous second order linear equation with variable coefficients. This paper is organized as follows. In section 2, some useful preliminaries and assumptions are given and a stochastic portfolio optimization problem is formulated. In section 3, we consider the specific case where the evolution of the interest rate follows the CIR (1985) process. Section 4 concludes the work.

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2.0 Basic Assumptions and Problem Formulation

We consider the situation where the investor allocates his wealth to a risky asset, a stock for example, and a risk-free asset e.g. bond. The price of risky asset is assumed to follow the process

$$\frac{d}{p} = \mu d \; + \sigma_1 d_{\;1,t} \qquad (2.1)$$

where μ and σ_1 are constants representing asset's expected rate of return and volatility respectively. The standard one-dimensional Brownian motion $z_{1,t}$ captures the fluctuations in the stock price.

Also, the interest rate is not constant but is allowed to fluctuate according to the process.

$$d_{\;t} = f(r_t) + \sigma_2\sqrt{r_t}\, d_{\;1,t} \qquad r_0 = r \qquad (2.2)$$

We also assume that the market is complete so that the variations in the risk-free rate of return r_t, expected rate of returnμ, and variances and covariances defined by $\sigma_{1,}$ between the different rates of return are caused by the same one – dimensional Brownian motionz_t, that affects the risky asset price. Therefore, we take the market price of risk to be of the form.

$$\lambda = \frac{\mu - r_t}{\sigma_1} \qquad (2.3)$$

Let $w_t > 0$ be the investor's total wealth at time t. Moreover, let π_t be the fraction of the investor's total wealth invested on the risky asset and c_t be the rate at which the investor withdraws consumption goods from the investment. From the above assumptions, one can show that the wealth process w_t satisfies the stochastic differential equation.

$$d_{\;t} = w[r_t + (\mu - r_t)\pi] - c \;\; + \sigma_1\pi_t w_t d_{\;1,t}\;, \qquad w(0) = w_0 \qquad (2.4)$$

wherew_0 is the investor's initial wealth. We also require that the controls$\pi_t, c_t \in R^2$ and they aref_t – progressively measurable for some$z_t$adapted increasing σ - algebras. Finally, we require that the consumption process be an L^1 – process. That is

$\int_0^T c_t d \; < \infty$with probability one

and

$E\int_0^T \pi_t^2 d \; < \infty.$

Using (1.0), our aim is to maximize the objective function

$$V = E\int_0^T e^{-\beta}\, l\!\!\;\; (c \;\;)d \qquad (2.5)$$

Applying the dynamic programming principle, the HJB equation becomes

$$\beta \;\; (w,r,t) = s \quad \begin{Bmatrix} \ln(c \;\;) + \frac{d}{d} + (w[r_t + (\mu - r)\pi] - c)\frac{d}{d} + \frac{w^2\pi^2\sigma_1^2}{2}\frac{d^2V}{d^{\;\,2}} \\ +f(r)\frac{d}{d} + \frac{1}{2}\sigma_2^2 r\frac{\partial^2 V}{\partial r^2} + w \;\; \sigma_1\sigma_2\sqrt{r}\frac{\partial^2 V}{d} \end{Bmatrix} \qquad (2.6)$$

Let the value function $V(w,r,t)$ be given by

$$V(w,r,t) = A \quad w + g(r) + h(t)\;, \qquad A > 0 \qquad (2.7)$$

Substituting relevant derivatives of $V(w,r,t)$into (2.6), we obtain

$$\beta\big(A \qquad + g(r) + h(t)\big) = \ln c + \ln w + h'(t) + A\left(r + (\mu - r)\pi - \frac{c}{w}\right) - \frac{A}{2}\pi^2\sigma_1^2$$

$$+f(r)g'(r) + \frac{1}{2}\delta_2^2 r g''(r) \qquad (2.8)$$

Comparing both sides of (2.8), we see that

$A = \frac{1}{\beta}.$

The first order conditions with respect to c and π show that the optimal consumption and investment strategies are given by

$$c^{m} \quad = \beta \qquad (2.9)$$

$$\pi^{m} \quad = \frac{\mu - r}{\sigma_1^2} \qquad (2.10)$$

Rearranging (2.8), we get

$$h'(t) - \beta h(t) + \ln\beta \quad - w + \frac{1}{2}\sigma_2^2 r g''(r) + f(r)g'(r) - \beta \;\; (r) + \frac{1}{\beta}\left(r + \frac{(\mu - r)^2}{\sigma_1^2}\right) = 0 \qquad (2.11)$$

Note that (2.11) is a sum of two ordinary differential equations in different variables, which can be written in the form

$$F(t) + G(r) = 0 \qquad (2.12)$$

Where$F(t)$ and $G(r)$ are the ordinary differential equations in t and r respectively.

From (2.12), we obtain

$F(t) = -G(r) = a\, c \qquad (s \quad k).$

Hence, we have

$$h'(t) - \beta h(t) + \frac{\mu^2}{2\sigma_1^2\beta} = k, \quad h(0) = 0 \qquad (2.13)$$

$$\frac{1}{2}\sigma_2^2 r g''(r) + f(r)g'(r) - \beta \;\; (r) + \frac{r}{\beta}\left(1 - \frac{\mu}{\sigma_1^2}\right) + \frac{r^2}{2\sigma_1^2\beta} + l \qquad - w = k \qquad (2.14)$$

The solution to (2.13) is

$$h(t) = \frac{1}{\beta}\left(\frac{\mu^2}{2\sigma_1^2\beta} - k\right)\left(1 - e^{\beta}\ \right) \tag{2.15}$$

3.0 One Factor CIR Dynamics

In this section, we consider as an example, the special case in which the interest rate follows the CIR (1985) interest rate model

$$d\ \ = \alpha(\gamma - r)d\ \ + \sigma\sqrt{r}dz_t \tag{3.1}$$

This is a mean-reverting process in which the interest rate r_t is made to reverts to a long term average $\gamma > 0$ by a force equal to $\alpha > 0$.That is in (2.2), we let $f(r) = \alpha(\gamma - r)$. It has been shown that the bond price in the CIR model satisfies the partial differential equation.

$$\frac{1}{2}\sigma^2 r\frac{\partial^2 P}{\partial\ ^2} + \frac{\partial}{\partial} + \left(\alpha\ \ - r(\alpha + ^{a})\right)\frac{\partial}{\partial} - r\ \ = 0(3.2)$$

Where $P(r, t, T) = A(t, T)e\ \ \ \left(B(t, T)(-r)\right)$

Here

$$A(t, T) = \left[\frac{2¦e^{(\alpha+¦)(T-t)/2}}{(\alpha + ¦)(e^{¦(T-t)} - 1) + 2¦}\right]$$

$$B(t, T) = \frac{2\left(e^{¦(T-t)} - 1\right)}{(\alpha + ¦)(e^{h(T-t)} - 1) + 2¦}$$

and $¦ = \sqrt{\alpha^2 + 2\sigma^2}$

Substituting $f(r) = \alpha(\gamma - r)$ in (2.14), we obtain.

$$\frac{1}{2}\sigma_2^2 r g''(r) + \alpha(\gamma - r)g'(r) - \beta\ \ (r) + \frac{r}{\beta}\left(1 - \frac{\mu}{\sigma_1^2}\right) + \frac{r^2}{2\sigma_1^2\beta} + h\ \ \ \ \ \ - w = k \tag{3.3}$$

This equation has a regular singular point at $r = 0$.We assume that its homogenous part has a solution of the form

$$g(r) = \sum_{n=0}^{\infty} a_n r^{n+\delta} \tag{3.4}$$

Substituting the derivatives of $g(r)$ into the homogenous part of (3.3), we obtain the indicial equation

$$\frac{1}{2}\sigma_2^2\delta(\delta - 1) + \alpha\ \ \ \ = 0 \tag{3.5}$$

This gives

$$\delta = 0\ a\ \ \ \ \delta = 1 - \frac{2\alpha}{\sigma_2^2}$$

The recurrence relations corresponding to the indices $\delta = 0\ a\ \ \ \ \delta = 1 - \frac{2\alpha}{\sigma_2^2}$ are

$$a_{n+1} = \frac{2(n\ \ +\beta)}{(n+1)(n\sigma_2^2+2\alpha\ \ )}a_n, n \geq 0 \tag{3.6}$$

and

$$b_{n+1} = \frac{\sigma_2^2(n\ \ +2\alpha+\beta)-2\alpha^2\gamma}{\sigma_2^4(n^2+2n+2)-2\alpha\ \ \sigma_2^2(n+1)}b_n\ , \qquad n\geq 0 \tag{3.7}$$

respectively. Theinhomogeneous part of (3.3) is quadratic. Hence, we assumea solution of the form

$$g_p(r) = d_2 r^2 + d_1 r + d_0 \tag{3.8}$$

Substituting the derivatives of $g_p(r)$ into (3.3) and equating powers of r, we obtain

$$(2\alpha + \beta)d_2 = \frac{1}{2\beta\ \ ^2_1} \tag{3.9}$$

$$2(\alpha\ \ + \sigma_2^2)d_2 - (\alpha + \beta)d_1 = \frac{\mu}{\beta\ \ ^2_1} - \frac{1}{\beta} \tag{3.10}$$

$$\alpha\ \ d_1 - \beta d_0 = k + w - h \tag{3.11}$$

Hence, we get the coefficients as

$$d_2 = \frac{1}{2\sigma_1^2\beta(2\alpha+\beta)} \tag{3.12}$$

$$d_1 = \frac{1}{\beta(\alpha+\beta)}\left[\frac{\alpha\ \ +\sigma_2^2}{2\sigma_1^2(2\alpha+\beta)} - \frac{\mu}{\sigma_1^2} + 1\right] \tag{3.13}$$

$$d_0 = \frac{\alpha}{\beta^2(\alpha+\beta)}\left[\frac{\alpha\ \ +\sigma_2^2}{\sigma_1^2(2\alpha+\beta)} - \frac{\mu}{\sigma_1^2} + 1\right] + \frac{1}{\beta}(k + w - h\ \ \ \ \ \ ) \tag{3.14}$$

With these above, we can write

$$g(r) = c_1 g_1(r) + c_2 g_2(r) + g_p(r) \tag{3.15}$$

Where

$$g_1(r) = a_0 + \sum_{n=0}^{\infty} a_{n+1}$$

$$g_2(r) = t_0 + \sum_{n=0}^{\infty} t_{n+1}$$

Therefore the optimal value function is given as

$$V(w, r, t) = \frac{1}{\beta} ln \quad + h(t) + g(r)$$

4.0 Summary

The second order condition with respect to consumption rate c and investment rate π gives $\frac{-1}{c^2}$ and $-\sigma_1^2$ respectively. Since the second derivatives at these points are less than zero, we conclude that the strategies are indeed optimal and therefore give optimal wealth and value of the investment. More so, it is clear that the optimal consumption rate is proportional to the investor's current wealth with constant of proportionality β as the investor's subjective time reference rate. On the other hand, the optimal investment on the risky asset is proportional to the market price of risk in the following way;

$$\pi^* = \frac{\mu - r}{\sigma_1^2} = \frac{1}{\sigma_1}\left(\frac{\mu - r}{\sigma_1}\right) = k\frac{\mu - r}{\sigma_1}$$

Where $\frac{\mu - r}{\sigma_1}$ is the market price of risk. This shows that investments should be made more, on risky assets than on risk-free assets.

5.0 References

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