Journal of the Nigerian Association of Mathematical Physics
Volume 29, (*March, 2015*), **pp 181 – 190**
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Optimal Plan for DC and DB Pension Schemes with Stochastic Income Under the Poisson Exponential-Trawl Process Model

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Abstract

In this paper, we apply a seemingly utility function derived for a pensioner whose fund management is via the defined contribution (DC) or defined benefits (DB) or both. We apply this utility function to investigate the retirement plan of the pensioner with view to determining the optimal plan. It is shown that the optimal plan follows the Integer-valued Poisson exponential-trawl process to capture the entry and exit of a member from the scheme as a jump Levy diffusion process. Techniques of Backward Stochastic Differential Equations (BSDE) to other utility functions are also applied to show that the optimal plan follows the trawl process.

Keywords: Optimal plan, Trawl process, Defined contribution, Defined benefits, BSDE.MSC: 98B28, 93E20, 60H10

1.0 Introduction

Count data are characterized as being non- negative, integer-valued, and often over-dispersed, meaning that the variance is typically greater than the corresponding mean. Count data appear in various applications, including medical science, epidemiology, meteorology, network modeling, actuarial science, econometrics and finance [1]. Various aspects of count data have attracted active research interest [2-4].

It is the view of this paper that the assessment of the pension fund by a pensioner generates count data. There are basically two different methods to design a pension fund scheme – the defined-benefit plan (hereinafter called DB) and the defined-contribution plan(hereinafter called DC).

The structure in DB , the benefits are fixed in advance by the sponsor and the contributions are adjusted continually to ensure that a required fund balance is maintained, where the associated financial risks are assumed by the sponsor agent. In DC, the contributions are fixed and benefits depend on the returns on the assets of the fund, where all the associated financial risks are borne by beneficiary.

Recently, owing to the demographic evolution and the development of equity markets, DC now plays a crucial role in the social pension fund management.

Our main objective in this work is to show that the optimal investment strategy of a pensioner whether by DB or DC or both follows the Poisson trawl process. We find the optimal processes in the case of DC, using a particular utility function and in the case of DB, using the mean-variance method, in solving the respective associated HJB equations. One is to maximize the accumulation of funds at the time of retirement and the other is to balance the return and the risk, that is, maximizing the fund size and minimizing the volatility of the accumulation.

The former goal involves three types of utility functions; (i) constant relative risk aversion (CRRA) (ii) constant absolute risk aversion(CARA), and (iii) quadratic loss functions. Concerning the CRRA utility function, some authors chose the power or logarithmic utility function as the objective function [5-6]. Concerning CARA utility function, some authors chose the exponential utility function as the objective function [7-8].

The latter goal includes the mean-variance utility and value-at-risk (V) utility. The optimal portfolio selection for a pension for a pension fund involves a long period, usually from 20 to 40 years, it is crucial to take into account the salary risk. The novel feature of our research is the derivation of a smilingly utility function giving the DC and DB. We apply the utility function to investigate the retirement plan of a DC, DB and both with view to determine the optimal plan. It is shown that the optimal plan follows the Integer-valued trawl process to capture the entry and exit of a member from the scheme as a jump Levy diffusion process.

We deal with a single cohort of workers, who enter the plan and retire at the same age and assume that the cohort is stable across an accumulation phase, that is, any member who withdraws is replaced by another.

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At retirement time, an accumulated fund is used to purchase life insurance policy, whose amounts are related to the final salaries of participants.

It is well known that Levy processes can easily reproduce heavy tails, skewness and other distributional properties of asset returns and can generate discontinuities in the price dynamics. As levy processes generate more realistic sample paths of stock prices, replacement of the celebrated Black-Scholes model, a Gaussian noise by Levy noise is worth the effort as investment returns is a crucial factor when constructing strategies for pension plan.

The techniques of Backward Stochastic Differential Equations (BSDE) are used in the presence of random market coefficient [9-11].

From the above , the rest of the work is to examine the optimal policy for a pension fund scheme using the Integer-valued Trawl process and show how the Levy jump process is solved by backward stochastic differential equation. In what follows, we make the following assumptions:

Assumptions1:

A1: the state size of the defined benefit is known

A2: the state size of the defined contribution is known.

A3: the portfolio process consists of risky (from defined contribution) and risk free (from defined benefit) assets

A4: the fund management policy to consume is not as admissible as the policy to save (or invest more).

2.0 The Trawl Process [1]

The trawl process is used to govern the contribution of funds from the participants.

The Integer-valued Trawl process (IVT) ensures that funds are built up to maximize the expected returns. The trawl movement may or may not be determined by the direction of the wave (this is a stochastic process). The net is allowed to hold a certain number of fishes and usually the small ones may or may not escape from the net

This work introduces new method for accounting for stochastic volatility in the context of Integer-Valued stochastic processes.

Trawl process have recently been introduced [12], here focus is on the important sub-class of Integer-valued trawl (IVT) process.

The DB pension scheme is viewed as an IVT process, which can be described using BSDE.

The PFM make the following assumptions:

Assumptions 2:

B1: A member will exit on retirement from the scheme

B2: A member can exit before retirement (by death or dismissal)

B3: A new member enters into the scheme ($(T \geq 0)$

The above process is governed by the IVT processes

Definition 1: A trawl is a Borel set $A \subset R \times (-\infty, 0]$ such that

$$leb(A) < \infty, \ \ then \ \ we \ \ set \ \ A_t = A + (0, t) \tag{2.1}$$

The trawl is ensured non-anticipative, where A is of the form

$A = \{(x, s) : s \leq 0, \ 0 \leq x \leq d(s)\} \quad where \ \ d : (-\infty, 0] \rightarrow R$ is a continuous function such that $leb(A) < \infty$.

Then, we define

$$d^* = leb(A) = \int_{-\infty}^{0} \int_{0}^{d(s)} dx ds = \int_{-\infty}^{0} d(s) ds \tag{2.2}$$

That is ,

$$A_t = A + (0, t) = \{(x, s) : s \leq t, \ 0 \leq x \leq d(s - t)\} \tag{2.3}$$

d is monotonically non-decreasing, then A is a monotonic trawl as in (2.3).

At time zero, it is considered the set $A = A_0 \subset R^2$ (with finite Lebseque measure), the pension member has entered into the scheme with minimal initial contribution to be invested. At time t, we consider the

The connection between Integer-Valued Trawl (IVT) process and Integer-Valued Levy (IVL) basis is defined below.

Definition 2.2: A stationary integer-valued trawl(IVT) process $(Y_t)_{t \in R}$ is defined as:

$$Y_t = L(A_t) = \int_{R\times R} I_A(x, s-t)L(dx, ds)$$

$$where \;\; L \;\; is \;\; defined \;\; as \;\; L(dx,dt) = \int_{-\infty}^{\infty} yN(dy,dx,dt) \qquad (2.4)$$

$$and \quad A_t = A + (0,t) \;\; defined \;\; in \;\; (2.1)$$

The key component for an IVT process is an integer-valued, homogeneous Levy basis, which we will define in terms of a Poisson random measure.

Definition 2.3: Let N be a homogeneous Poisson random measure on R^3 with compensator $E(N(dy,dx,dt)) = ¬(dy)dxdt, \quad where \;\; ¬$ is a Levy measure satisfying

$$\int_{-\infty}^{\infty} \min(1,|y|)¬(dy) \le \infty. \qquad (2.5)$$

In addition , suppose that the Poisson random measure is integer-valued, which implies that the Levy measure is concentrated on $Z \setminus \{0\}$. Then we define an integer-valued, homogeneous Levy basis on R^2 in terms of Poisson random measure as

$$L(dx,dt) = \int_{-\infty}^{\infty} yN(dy,dx,dt) \qquad (2.6)$$

L is infinitely divisible with characteristic function

$$E(\exp(i¯L(dx,dt)) = \exp(C(¯ \pm L(dx,dt)) \qquad (2.7)$$

where

$$C(¯ \pm L(dx,dt)) = \int_R (e^{i¯y} - 1)¬(dy)dxdt \qquad (2.8)$$

denotes the corresponding cumulant function.

We have a probability space (Ω, F, P) and a fixed time horizon $T \in (0,\infty)$ so all processes defined on this space are indexed by $[0,T]$, and all random times take their values in $[0,T] \cup \{\infty\}$

This space is endowed with a non-explosive multivariate point processes(also called marked point process) on $[0,T]\times E \;\;,\;\; where \;\; (E,£)$ is a LUSIN SPACE i.e. a sequence (S_n, X_n) of points , with distinct times of occurrence S_n and with marks X_n, so it can be viewed as a random measure of the form

$$«(dt,dx) = \sum_{n\ge 1, S_n \le T} £_{(S_n,X_n)}(dt,dx) \qquad (2.9)$$

Put in IVT process, S_n is the time a fish enters in the trawl and X_n is the size of the fish (representing the entry of a participant in the pension scheme andthe initial contribution or the entry point).

Here $S_n \in (0,T] \cup \{\infty\}$ - valued and the $X_n{}'s$ are E - valued and

$$S_1 > 0 \quad and \quad S_n < S_{n+1} \;\; if \;\; S_n < T, \quad and \quad S_n \le S_{n+1} \quad and \quad \Omega = \bigcup \{S_n > T\}.$$

We denote by ¬ the predictable compensator of the measure «, relative to the filtration (F_t) .

The measure ¬ admits the disintegration:

$$¬(w,dt,dx) = dA_t(w)¤_{w,t}(dx) \qquad (2.10)$$

where ¤ is a transition probability from $(\Omega \times [0,T], P)$ into $(E,£), \;\; and \;\; A$ is an increasing Cadlag predictable process starting at $A_0 = 0$ which is also the predictable compensator of the univariate point process.

$$N_t = «([0,t]\times E) = \sum_{n\ge} 1_{\{S_n \le t\}} \qquad (2.11)$$

As a special case, the multivariate point process « reduces to the univariate $N \;\; when \;\; E$ is a singleton.

3.0 The Poisson Trawl Process

As a good starting point for exploring the wide class of IVT processes, we choose the Poisson trawl process, which we obtain by choosing a Poisson basis for the Levy basis L or, equivalently, by choosing the Levy seed $L' \sim Poi(v)$ for an intensity parameter $v > 0$. Then

$L(A_t) \sim Poi(vleb(A))$, since $leb(A_t) = leb(A)$ for all $t \in IR$.

The intuition behind such a basic model is the following one: If we consider a realization of L again, then we obtain a countable set R of points (y, x, t) in $\{1\} \times IR^2$, where-as before-the last two coordinates (t, x) are uniformly distributed over IR^2 while y is the value of the basis at that point, which in the case of a Poisson basis is always equal to 1. Hence as soon as time reaches t the value of the process Y increases by 1. As time progresses, the point eventually drop out of the trawl again, which will result in the value of the process dropping by one. This finding can be used to set up a simulation algorithm for such processes.

4.0 Backward Stochastic Differential Equations

To find a generalization of the Feynman-Kac formula, more precisely to be able to represent semilinear PDEs of the type

$$\partial_t u(t,x) + L \ (t,x) + f(t,x,u(t,x), D_x u(t,x) = 0, \qquad (2.12)$$

assume that u is the solution of this equation with no initial condition for now. If one can describe the dynamics of $Y_t = u(t, X_t)$, then we could for every (t,x) consider a version $X^{t,x}$ of X that starts at time t in x. This would imply $u(t,x) = u(t, X_t^{t,x}) - Y_t^{t,x}$. What would the dynamics of Y have to be? By Ito's formula

$$dY_t = \left(\partial_t u(t,X_t) + L \ (t,X_t)\right)d \ + D_x u(t,X_t)\sigma(X_t)dW_t = -f\left(t, X_t, Y_t, D_x u(t,X_t)\right)d \ + D_x u(t,X_t)\sigma(X_t)dW_t$$

This suggests to consider equations of a slightly less general type than above:

$$\partial_t u(t,x) + L \ (t,x) + f(t,x,u(t,x), D_x u(t,X_t)\sigma(X_t) = 0 \qquad (2.13)$$

If u solves this equation, one obtains for $Y_t = u(t,X_t)$ and $Z_t = D_x u(t,X_t)\sigma(X_t)$:

$$dY_t = -f(t, X_t, Y_t, Z_t)d \ + Z_t dW_t. \qquad (2.14)$$

A solution to this equation consists of a pair of processes (Y, Z). Note that this equation does not make any sense if we consider it as a forward equation: For $f = 0$ one obtains

$$dY_t = Z_t dW_t, \ t \leq T; \ y_0 = y \in R^d. \qquad (2.15)$$

Of course then one can choose Z independently of the initial condition, and therefore there would be infinitely many solutions. However, if one considers it as a backward equation, then there is hope: one considers again the case $f = 0$.If Y is adapted one gets for any adapted and square integrable Z

$$Y_t = \mathbb{E}(Y_t | F_t) = \mathbb{E}\left(\xi - \int_t^T Z_s dW_s | F_t\right) = \mathbb{E}(\xi | F_t). \qquad (2.16)$$

Therefore Y is a martingale. If the filtration F is now generated by the Brownian motion W, then by the martingale representation property there exists a unique predictable process Z such that $Y_t = Y_0 + \int_0^t Z_s dW_s$ which yields

$$Y_t = Y_T + \int_t^T Z_s dW_s = \xi - \int_t^T Z_s dW_s. \qquad (2.17)$$

In what follows, we let Y represent the DC and $Z$the DB. The we propose;

Proposition 1.1: Let us consider the Picard sequence (Y^k, Z^k) recursively by $(Y^0 = 0, Z^0 = 0)$, and

$$-dY_t^{k+1} = g(t, Y_t^k, Z_t^k)d \ - Z_t^{k+1}dW_t \, , Y_T^{k+1} = \xi_T \qquad (2.18)$$

This sequence (Y^k, Z^k) converges in $\mathcal{H}_m^2 \otimes \mathcal{H}_{m\times d}^2$, and $d \ \otimes d\mathbb{P} - a.e$ to the solution (Y, Z) of the BSDE (g, ξ_T). Moreover, the sequence (Y^k) converges uniformly almost surely[11]

5.0 Mathematical Formulation

In this section, we introduce the market structure and define the stochastic dynamics of asset's prices and the salary.

We consider a complete and frictionless financial market which is continuously open over the fixed time interval $[0, T]$, *where* $T > 0$ denotes the retirement time of a representative member.

We assume that , the pension fund manager(PFM) can invest into two financial assets. The PFM is at liberty to invest in either of the assets. The PFM can invest in risk-less asset (bond or bank account) at time t, which is governed by a dynamics

$$dB_t = r_0 B_t dt \quad , \quad B(0) = B_0, \quad r_0 > 0 \qquad (3.1)$$

where B_0 is the initial price value of the risk-less asset, r_0 is the discount rate.

We also assume that the price of the risky asset is a continuous time stochastic process governed by the dynamics

$$dS_t = r_1 S_t dt + \pm_1 S(t) dW(t) \ , \ S(0) = S_0 \qquad (3.2)$$

where S_0 is the initial stock price of the risky asset, r_1 *and* $\pm_1$ are constant parameters,

r_1 is the expected instantaneous rate of return of the risky asset, satisfying the general condition $r_1 > r_0$, $\pm_1 S(t)$ is the volatility, $\{W(t) : t \geq 0\}$ is a standard 1-dimensional Brownian motion defined on a complete probability space (Ω, F, P).

The filtration $F = \{F_t\}$ is a right continuous filtration of $\sum$ - algebra on the space.

The stochastic salary dynamics at time t follows the SDE which is influenced by the financial market.

$$dL(t) = r_2 L(t) dt + \pm_2(t) L(t) dW(t) \quad , \quad L(0) = L_0 \qquad (3.3)$$

where L_0 is the initial salary, r_2 is the expected growth rate of the salary, $\pm_2(t)$ is the volatility and hedgeable whose risk source belongs to the defined set of the financial market risk source.

The wealth process under consideration are the contribution made into the pension fund at a rate of ª$L(t)$.

Let the wealth of the pension fund at time $t \in [0, T]$ be ®$_t$ (stock) and $1 -$®$_t$ (bank account) respectively. Then , the optimal investment amount from the contributors is given as

$$dX(t) = (1 - ®_t) X(t) \frac{dB(t)}{B(t)} + ®_t X(t) \frac{dS(t)}{S(t)} + ªL(t), \quad X(0) = X_0 \qquad (3.4)$$

Using (3.1), (3.2) and (3.3) above, the evolution of the pension fund manager (PFM) wealth is given by:

$$dX(t) = [(1 - ®_t) X(t) r_0 + ®_t X(t) r_1 + ªL(t)] dt + ®_t X(t) \pm_1 S(t) dW(t) \qquad (3.5)$$

The contribution term ª$(t)dt$ in (3.4) is the same as already established in [13].Based on the sets of assumptions A1-A3 and B1-B3

Let

$$P^i_{t+1} = ª_t P^i_t + R_t , \qquad (3.6a)$$

Where the LHS of (3.6a) is the total income for $t - period$, ª$_t$ is the percentage of the contribution in $t - period$.

The investment part of the total income is given by

$$R_t = ¸ + \ \ P^i_{t+1} , \qquad (3.6b)$$

where ¸ may be policy of fund manager to invest or not to invest and   is the investment rate.

The investment policy becomes

$R_t = ¸ + \ \ P^i_{t+1}$ as in (A2).

Recall from (A1)

$P^i_{t+1} = ª_t P^i_t + R_t ,$

where ª$_t$ *and* R_t are assumed independent random variables, by assumption of A3, that (A2) follows.

Then,

$$P^i_{t+1} = ª_t P^i_t + ¸ + \ \ P^i_{t+1} , \qquad (3.7)$$

or

$$P^i_{t+1} = \frac{ª_t P^i_t + ¸}{(1 - \ \ )} . \qquad (3.8)$$

The PFM assumed that a member will exit from the fund system on retirement at time T and is risk averse, the utility function $U(x)$ is increasing and concave $(U^{II}(x) < 0)$.

The PFM is interested in maximizing the utility of the member's investor's terminal wealth. For the given strategy ®$_t$, we define the value function by the PFM from the space state x *at time* t as

$$H_{®_t}(t, s, l, x) = E_{®_t} [U(X(T)) | S(t) = s, L(t) = l, X(t) = x] \qquad (3.9)$$

The objective is to find the optimal value function

$H(t,s,l,x) = Sup\, H_{®_t}(t,s,l,x)$ and the strategy $®_t^*$ such that

$$H_{®_t^*}(t,s,l,x) = H(t,s,l,x) \qquad (3.10)$$

is satisfied.

1. UTILITY FUNCTION

To investigate of the optimal plan, we consider a utility function which incorporating both the DE and the DB. Given (3.8), the special utility function is derived in [14] as

$$U(X) = \frac{X^{-(1+¢)}}{1-}. \qquad (4.1)$$

Put $¥ = 1 + ¢$ then

$$U(X) = \frac{X^{-¥}}{1-}. \qquad (4.2)$$

Notice that for $\beta = 0$ and $\gamma = 2$, we have (4.2) becoming $U(X) = \frac{1}{X^2}$ which is the reciprocal of the Melton's quadratic utility function.

The objective of the PFM is to choose an investment allocation of the contributions made in (3.5) so as to maximize the expected utility of the terminal wealth on exit from the scheme.

The HJB equation associated with the problem is defined as

$$0 = \max_{\{®_t\}}\left[\frac{\partial u}{\partial t} + (®(t)(r_1 - r_0) + r_0)X(t)\frac{\partial u}{\partial X} + \frac{1}{2}®^2 ±^2 X^2(t)\frac{\partial u}{\partial X^2}\right] \qquad (4.3)$$

Put $u(X) = \frac{b(t)X^{-¥}}{\sim}$.

$$\frac{\partial u}{\partial t} = \frac{b'(t)X^{-¥}}{\sim}, \frac{\partial u}{\partial X} = -\frac{¥b(t)}{\sim}X^{-(¥+1)}$$

and

$$\frac{\partial^2 u}{\partial z^2} = \frac{¥(¥+1)b(t)}{\sim}z^{-(¥+2)} \qquad (4.4)$$

$$\Rightarrow 0 = \frac{b'(t)X^{-¥}}{\sim} + \max_{\{®_i\}}\left\{-\left(r_0 + (r_1 - r_0)®\right)\frac{X¥b(t)}{\sim}X^{-(1+¥)} + \tfrac{1}{2}®^2 ±^2 X^2 \frac{¥(¥+1)}{\sim}b(t)X^{-(¥+2)}\right.$$

$$\Rightarrow 0 = b' + b¥ \max_{®_t}\left\{-r_0 - (r_1 - r_0)® + \frac{1}{2}®^2 ±^2 (¥+1)\right\}$$

$$= b' + b¥\left[-r_0 - \tfrac{1}{2}\frac{(r_1 - r_0)^2}{(1+¥)±^2}\right]. \qquad (4.5)$$

Where the optimal policy is

$$® = \frac{(r_1 - r_0)}{(1+¥)±^2}. \qquad (4.6)$$

That is

$$0 = b'(t) + b(t)¥\left[r_0 + \tfrac{1}{2}\frac{(r_1 - r_0)^2}{(1+¥)±^2}\right]$$

$$\int \frac{db}{b} = \int_t^® ¥\left[r_0 + \tfrac{1}{2}\frac{(r_1 - r_0)^2}{2(1+¥)±^2}\right]dt$$

$$\Rightarrow b(t) = b_0 \exp\left\{¥\left(r_0 + \frac{(r_1 - r_0)}{2±^2(1+¥)}\right)(T-t)\right\}.$$

Equation (4.6) implies that if the effect of the volatility is allowed to increase, it will affect the optimal policy generated from the risky assets.

2. STOCHASTIC CONTROL PROBLEM

We consider the stochastic control problem on finite horizon. Let X be a controlled diffusion on R^n governed by

$$dX_s = b(X_s, \sim_s)ds + \pm(X_s, \sim_s)dW_s \qquad (5.1)$$

where W is a d -dimensional standard Brownian motion, and, the control process, is a progressively measurable valued in A.

Let S be as in section 2where we have a controller who intervenes on a system S whose evolution when non-controlled is described by a R^d −valued stochastic income process $(x_t)_{t\le T}$ solution of the SDE(2.15). Assume now that the reward functional is given by the following expression:

$$\forall u \in U, J(u) := \mathbb{E}^u\left[\exp\theta\left\{\int_0^T h(s, x., u_s)d \ + \Psi(x_T)\right\}\right]$$

where θ is a real parameter. The problem we are interested in is to find an optimal strategy for the controller, i.e. an admissible control u^* which maximizes the reward $J(u)$. Since we have a utility function in the reward, which is of exponential type, we call this problem of Risk-Sensitive type. The parameter θ stands for the sensitiveness of the controller with respect to risk. He/she is risk averse (resp. seeking) if $\theta < 0$ (resp. $\theta > 0$). There are several works related to those types of control problems and especially their applications [15-18]. Moreover we can write risk-sensitive control problem in term of dynamic entropic risk measure which is studied in [19]. Indeed, let$\left(e_{u,r,t}\right)$ (resp.$\left(e_{r,t}\right)$) be a $\mathbb{P}$-dynamics entropic risk measure (resp. $\mathbb{P}^u$-dynamic entropic risk measure) given for any ξ_T bounded by:

$e_{u,r,t}(\xi_T) = r \ln \mathbb{E}^u\left[e \ \left(-\frac{1}{r}\xi_T\right)|F_t\right].\left(r \quad .e_{r,t}(\xi_T) = r \ln \mathbb{E}\left[e \ \left(-\frac{1}{r}\xi_T\right)|F_t\right]\right).$

Taking $r = 1/\theta$ and $\xi_T = -\int_0^T h(s, x., u_s)d \ - \Psi(x_T)$, we get that $J(u) = e \ (\frac{1}{r}e_{u,r,0}(\xi_T)$.

Actually it has been proved [19] that the dynamic entropic measure $\left(e_{r,t}(\xi_T)\right)$ is $\mathbb{P}$-solution of the following BSDE with the quadratic coefficient $g(t, z) = \frac{1}{2r}\|z\|^2$ and terminal bounded condition ξ_T.

$-de_{r,t}(\xi_T) = \frac{1}{2r}\|z\|^2 d \ - Z_t dW_t, \mathbb{P} - a.s, \ \ e_{r,T}(\xi_T) = -\xi_T.$

We are going now to get similar result that is the link between $J(u)$ and $\mathbb{P}$-solution of some BSDEs. Actually we have the following result.

The gain functional to maximize the terminal wealth is

$$J_{(\sim)} = E\left[\int_0^T f(t, X_t, \sim_t)dt + g(X_T)\right] \qquad (5.2)$$

where $f : [0,T] \times R^n \times A \to R$ is continuous in (t, x) $\forall a \in A,$ $g : R^n \to R$ is a concave C' function, and f, g satisfy a quadratic growth condition in x.

The generalized Hamiltonian H is defined as: $H : [0,T] \times R^n \times A \times R^n \times R^{n\times d} \to R$ by

$$H(t, x, a, y, z) = b(x, a)y + tr(\pm'(x, a)z + f(t, x, a) \qquad (5.3)$$

H is differentiable in $x(D_x H)$.

For each $\sim \in A$, the BSDE, called the adjoint equation

$$-dY_t = D_x H(t, x_t, \sim_t, Y_t, Z_t)dt - Z_t dW_t, \ \ Y_T = D_x g(X_T) \qquad (5.4)$$

THEOREM 5.1;

Let $\hat{\sim} \in A$ and $\hat{X}$ be associated controlled diffusion. Suppose that there exists a solution $(\hat{Y}, \hat{Z})$

T0 the associated BSDE (5.4) such that

$$H(t, \hat{X}_t, \hat{\sim}_t, \hat{Y}_t, \hat{Z}_t) = \max_{a\in A} H(t, \hat{X}_t, a, \hat{Y}_t, \hat{Z}_t), \ 0 \le t \le T, a..s \qquad (5.5)$$

and

$$(x, a) \to H(t, x, a, \hat{Y}_t, \hat{Z}_t) \qquad (5.6)$$

is a concave function $\forall t \in [0, T]$.

Then $\hat{\sim}$ is an optimal control:

$$J(\hat{\sim}) = \sup_{\sim \in A} J(\sim).$$

6.0 Criterion for Portfolio Selection

We consider the usual Black-Scholes financial model. There are basically two types of assets under consideration. The risk less asset is of price process

$$dS_t^0 = rS_t^0 dt \qquad (5.7)$$

and the risky asset governed by the price dynamics

$$dS_t = bS_t dt + \pm S_t dW_t \qquad (5.8)$$

with constants $b > r \ and \ \pm > 0$.

Consider an agent who invest at any time t an amount $\sim_t$ in the risky asset, the wealth process is governed by

$$dX_t = \sim_t \frac{dS_t}{S_t} + (X_t - \sim_t)\frac{dS_t^0}{S_t^0}$$
$$= \left[rX_t + \sim_t (b-r)\right]dt + \pm \sim_t dW_t, \quad X_0 = x, \qquad (5.9)$$

where A is the set of progressively measurable processes $\sim$ valued in R, such that

$$E\left[\int_0^T \left|\sim_t\right|^2 dt\right] < \infty. \qquad (5.10)$$

By mean-variance approach which is suitable in the optimization of portfolio selection in DB pension scheme, which consists in minimizing the variance of the wealth under the constraint that its expectation is equal to given constant. Mean-variance objective is equally applied in solving portfolio selection problem for self-financing wealth processes. The techniques of BSDE in the presence of random market coefficients;

$$V(m) = \inf_{\sim \in A}\{Var(X_T) : E(X_T) = m\}, m \in R \qquad (5.11)$$

Problem (5.11) is reduced to the resolution of an auxiliary control problem

$$\tilde{V}(ª) = \inf_{\sim \in A} E\left[X_T - ª\right]^2, \quad ª \in R. \qquad (5.12)$$

By stochastic maximum principle, (5.3) takes the form

$$H(x,a,y,z) = \left[rx + a(b-r)\right]y + \pm az. \qquad (5.13)$$

The adjoint BSDE (5.4) is written as for any $\sim \in A$

$$-dY_t = rY_t dt - Z_t dW_t, \quad Y_t = 2(X_T - ª). \qquad (5.14)$$

Put $\hat{\sim} \in A$ as a candidate for the optimal control, and $\hat{X}, (\hat{Y}, \hat{Z})$ the corresponding processes.
Then,

$$H(x,a,\hat{Y}_t,\hat{Z}_t) = rx\hat{Y}_t + a\left[(b-r)\hat{Y}_t + \pm\hat{Z}_t\right]. \qquad (5.15)$$

By theorem(5.1) , (5.15) is linear in a , then,

$$(b-r)\hat{Y}_t + \pm\hat{Z}_t = 0, \quad 0 \le t \le T \quad , \quad a.s. \qquad (5.16)$$

We seek solution for $(\hat{Y}, \hat{Z})$ to (5.14) in the form of

$$\hat{Y}_t = ¨(t)\hat{X}_t + \cdot(t) \qquad (5.17)$$

for some deterministic C' function $¨ \ and \ \cdot$. Using (5.14) and (5.10), we have $¨, \cdot \ and \ \hat{\sim}$ satisfying

$$¨'(t)\hat{X}_t + ¨(t)(r\hat{X}_t + \hat{\sim}_t(b-r) + ¨'(t) = -r(¨(t)\hat{X}_t + \cdot(t), \qquad (5.18)$$

$$¨(t)\pm\hat{\sim}_t + \hat{Z}_t, \qquad (5.19)$$

together with the terminal conditions

$$\cdot(T) = 2 \ , \ \cdot(T) = -2ª. \qquad (5.20)$$

Using (3.10),(5.17) and (5.19), we have expression for the candidate $\hat{\sim}$ as :

$$\hat{\sim}_t = \frac{(r-b)\hat{Y}_t}{\pm^2 ¨(t)} = \frac{(¨'(t) + 2r¨(t)\hat{X}_t + \cdot(t))}{\pm^2 ¨(t)}. \qquad (5.21)$$

Also from (5.18), we obtained

$$\hat{\sim}_t = \frac{(¨'(t) + 2r¨(t)\hat{X}_t + \cdot'(t) + r\cdot(t)}{(r-b)¨(t)}. \qquad (5.22)$$

By (5.21) we obtained the ODE satisfied by $¨$ *and* $\cdot$:

i.e. $$¨'(t) + (2r - \frac{(b-r)^2}{\pm^2})¨(t) = 0 \ , \ ¨(T) = 2 \qquad (5.23)$$

$$\cdot'(t) + (r - \frac{(b-r)^2}{\pm^2})\cdot(t) = 0 \ , \ ¨(T) = -2ª. \qquad (5.24)$$

Solutions to (5.23) and (5.24) are: $(\cdot - \cdot_ª)$, *depending on* ª

$$¨(t) = 2\exp\left[(2r - \frac{(b-r)^2}{\pm^2})(T-t)\right] \qquad (5.25)$$

$$\cdot_ª(t) = ª\cdot_1(t) = -2ª\exp\left[(r - \frac{(b-r)^2}{\pm^2})(T-t)\right]. \qquad (5.26)$$

With appropriate choice of $¨, \cdot_ª$, *the process* $(\hat{Y}, \hat{Z})$ is the solution of the BSDE(5.14), satisfying the maximum principle in Theorem (5.1).

Then, the optimal control is given by (5.12), in Markovian form is

$$\hat{\sim}_ª(t,x) = \frac{(r-b)(¨(t)x + \cdot_ª(t)}{\pm^2 ¨(t)}. \qquad (5.27)$$

7.0 Computation of The Value Function $\tilde{V}(ª)$:

For any $\sim \in A$, applying Ito's formula to $\frac{1}{2}¨(t)X_t^2 + \cdot_ª(t)X_t$ and using (5.11) , the ODE (5.23) to (5.24) is satisfied by $¨$ *and* $\cdot_ª$.

By expectation:

$$E[X_T - ª]^2 = \frac{1}{2}¨(0)x^2 + \cdot_ª(0)x + ª^2 + E\left[\int_0^T \frac{¨(t)\pm^2}{2}(\sim_t - \frac{(r-b)(¨(t)X_t + \cdot_ª(t))}{\pm^2 ¨(t)})^2 dt\right] - \frac{1}{2}\int_0^T (\frac{b-r}{\pm})^2 \frac{\cdot_ª(t)^2}{¨(t)} dt \qquad (5.28)$$

The above result implies that the optimal control in (5.11) and the value function is

$$\tilde{V}(ª) = \frac{1}{2}¨(0)x^2 + \cdot_ª(0)x + ª^2 - \frac{1}{2}\int_0^T (\frac{b-r}{\pm})^2 \frac{\cdot_ª(t)^2}{¨(t)} dt. \qquad (5.29)$$

We conjecture that

$$\tilde{V}(ª) = e^{-(\frac{b-r}{\pm^2})^2}(ª - e^{rT}x)^2 \ , \ ª \in R. \qquad (5.30)$$

8.0 Interpretation of Result and Conclusion

We have viewed the DC and DB pension scheme as integer-valued trawl (IVT) processes and also solve using the BSDE to obtain the optimal plan. A Poisson exponential-trawl process can be constructed by trawling a Poisson basis using (see section 2);

$$A_t = \{(x,s): s \le t, 0 \le x \le e^{-\lambda(t-s)}\}. \qquad (6.1)$$

In our case , we construct a Poisson exponential- trawl using equations (4.5, 5.25 and 5.26). For example, for

$\lambda = \gamma\left(r_0 + \frac{(r_1 - r_L)}{2\sigma^2(1+\gamma)}\right)$,

equation (4.5) becomes;

$$L(t) = \{(x,T): T \le t, 0 \le x \le L_0 e^{-\lambda(t-s)}\}. \quad (6.2)$$

Similarly, equations (5.25 and 5.26) are respectively

$$\varphi(t) = \{(x,T): T \le t, 0 \le x \le 2e^{-\theta(t-s)}\}, \quad (6.3)$$

and

$$\psi_\lambda(t) = \{(x,T): T \le t, 0 \le x \le 2\lambda^{-\aleph(t-s)}\}. \quad (6.4)$$

Where $\lambda = 2r - \frac{(b-r)^2}{\sigma^2}$ and $\aleph = r - \frac{(b-r)^2}{\sigma^2}$.

Consider the problem of a PFM with self-financed wealth process X_t who wants to minimize $E\left[(\tilde{X}_T)^2\right]$ in the complete market. His optimal strategy is the usual Merton portfolio allocation for a quadratic function $U(x) = -x^2$ defined by

$$\bar{\pi}_t = -\frac{b-r}{\sigma^2}\bar{X}_t\ , \ 0 \le t \le T. \quad (6.5)$$

The optimal strategy for the problem of (5.12) is the same as (5.24) the wealth process.

9.0 References

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