Journal of the Nigerian Association of Mathematical Physics
***Volume 29,* (*March, 2015*), pp 167 – 172**

A Mathematical Model for the Control of Transmission of Typhoid Fever

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Abstract

Typhoid fever has become a major public health concern in developing countries like Nigeria, Ghana, Mali and Kenya.Most cases in developed countries are imported from endemic countries. Individuals in the greater parts of Africa, Asia and Central America are at a great risk of contracting the disease due to poor sanitation, inadequate portable water supply and health care. As a result of poor diagnosis of typhoid in these areas due to similar signs and symptoms with malaria, resulting in improper treatment and care for typhoid, this constitutes the major obstacle in managing the typhoid disease by the health workers. In this paper, SIR-Bmathematical model that addresses the control of the transmission and spread of typhoid is developed and analyzed. The human population is divided into three classes; the susceptible, the infected and recovered (and immune) classes for humans, and for the bacteria we only have infectious class.All the new born for humans are susceptible to the typhoid infection and there is no vertical transmission (i.e. mother to child transmission of typhoid Salmonella). The next generation approach is used to determine the epidemiological threshold known as the basic reproductive number R_0 where R_0 is the non-zero eigenvalue. We establish the existence of the disease free equilibrium and endemic equilibrium pointsin terms of the reproductive numbers R_0 typhoid. It isalso established that the disease free equilibrium is asymptotical stable using Jacobian method. Using Lynapunov function it is proved that the disease free equilibrium point E^0 is globally asymptotically stable when $R_0 < 1$ and the disease will always die out. For $R_0 > 1$, the disease free equilibriumE^0 becomes unstable and the endemic equilibriumE^* is locally and globally asymptotically stable. Therefore, typhoid fever persists in the population. Numerical experiment using data obtained from Central Intelligence Agency and Journal materials show that typhoid fever can be controlled at 70% vaccination rate.

1.0 Introduction

Typhoid fever is a major public health problem in the world. This disease continues to affect the poor countries of the world including Nigeria. Typhoid is an infectious disease caused by the *bacillus Salmonella typhi*. It can be transmitted by taking solid food contaminated by faeces of typhoid infected persons or of carrier persons, healthy individuals who carries germs without showing the symptoms of the disease. The World Health Organization (WHO) claims that about 16 million cases were reported annually causing about 600,000 deaths[1]. Typhoid is an infectious disease characterized by an acute illness, the first typical manifestations of which are fever, headache, abdominal pain, relative bradycardia, splenomegaly and leucopenia [2]. There are many mathematical models inform of ordinary differential equations (odes), partial differential equations (pdes) and integral equations describing transmission of typhoid, vaccination of the populace against typhoid and other diseases, treatment of typhoid disease. Most of them considered susceptible-Infectious-Recovered (*SIR*) relationship, but, our model considered susceptible-Infection-Recovered-Bacteria SIR-B where a compartment B is added to the *SIR*, so as to study the transmission mode of salmonella typhi from bacteria to the susceptible, S. the susceptible individuals become infected individuals with typhoid, I after interacting with *Salmonella* typhi bacteria, B.

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2.0 The Derivation of the Model

The human population*N*(t) is divided into the following classes: susceptible individuals *S*(t), infected individuals with typhoid, *I*(t) and recovered individuals while the total population of *Salmonella* bacteria typhi is represented by *B*(t). The total population of human is *N=S+I+R*. In this model, α denotes transmission rate of *Salmonella* bacteria typhi from bacteria salmonella, *B* to the susceptible, *S*; γ denotes recovery rate of typhoid infected individuals, *I*(t). The birth rate and death rate are represented byβ andμ respectively. We displayed the relationship between human and bacteria salmonella typhi in figure 1:

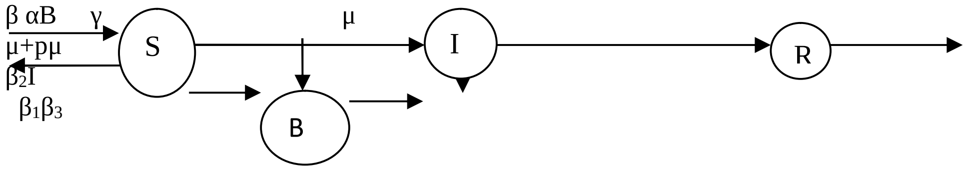

Fig 1: A diagram for a typhoid epidemic.

After all these assumptions, the model has the following forms:

$$\frac{d}{d} = \beta - \alpha BS - (p + \mu)S$$
$$\frac{d}{d} = \alpha BS - (\gamma + \delta + \mu)I \qquad (2.1)$$
$$\frac{d}{d} = \gamma \quad - (\mu + \rho)R$$
$$\frac{d}{d} = \beta_1 B + \beta_2 I - \beta_3 B$$

Where *p* represents vaccination against typhoid and β_3 represents the death rate of *Salmonella typhi* bacteria.Note that both of them represent control measures against typhoid fever.

3.0 Model Analysis

4.0 Boundedness

The system of equations (2.1) is epidemiologically and mathematically well-posed in the domain

D=$\{(S, I, R, B)\epsilon R4: S \geq 0, I \geq 0, R \geq 0, B \geq 0\}$and the equilibrium points are defined.

3.2 Basic Reproduction Number, R_0

This number gives the number of secondary infective (new infection) cases of the typhoid disease produced by typhoid infected individuals during the effective period when introduced in a population of susceptibles[3--6].

We use the next generation method to find the basic reproduction of number of our typhoid model by finding the next generation matrix (operator), FV^{-1} of our typhoid infected individuals, *I* compartment and bacteria typhi, *B*compartment.

$$F = \begin{bmatrix} 0 & \alpha\ (0) \\ 0 & 0 \end{bmatrix} \qquad (2.2)$$

$$V = \begin{bmatrix} \gamma + \delta + \mu & 0 \\ -\beta_2 & \beta_3 - \beta_1 \end{bmatrix} \qquad (2.3)$$

Finding the eigenvalues of the characteristic equation of FV^{-1}, we have:

$$\lambda_1 = 0 \qquad (2.4)$$

and

$$\lambda_2 = \frac{\alpha\beta_2 S(0)}{(\gamma+\delta+\mu)(\beta_3-\beta_1)} \qquad (2.5)$$

The largest positive eigenvalue(λ_2) is thebasic reproduction number, therefore

$$R_0 = \frac{\alpha\beta_2 S(0)}{(\gamma + \delta + \mu)(\beta_3 - \beta_1)} \qquad (2.6)$$

where $\beta_3 - \beta_1 > 0$

5.0 Equilibrium Points

6.0 Disease Free Equilibrium and Endemic Equilibrium Points

By setting the system of equation (2.1) to zero, we have

$$\beta - [\alpha \quad + (p + \mu)]S = 0 \qquad (2.7)$$
$$\alpha \quad - (\gamma + \delta + \mu)I = 0 \qquad (2.8)$$
$$\gamma \quad - (\mu + \rho)R = 0 \qquad (2.9)$$

$$-(\beta_3-\beta_1)B+\beta_2 I=0 \quad (2.10)$$

Substituting (2.8) into (2.7) gives

$$B=\frac{(p+\mu)(\gamma+\delta+\mu)I}{\alpha[\beta-(\gamma+\delta+\mu)I]} \quad (2.11)$$

Substituting (2.11) into (2.10) gives

$$\{-(\beta_3-\beta_1)(p+\mu)(\gamma+\delta+\mu)+[\beta_2\alpha\ \ -(\gamma+\delta+\mu)\beta_2\alpha\ ]\}I=0 \quad (2.12)$$

Solving (2.12) gives

$I=0$,represents the situation where human population is free of typhoid fever infection or

$$I^*=\frac{(p+\mu)(\beta_3-\beta_1)}{\beta_2\alpha}\left(1-\frac{\beta R_L}{S(0)}\right) \quad (2.13)$$

WhereI^*biologically represent the number of infected human with typhoid disease.Substitu-ting (2.13) into (2.6) gives

$$B^*=\frac{(\mu+\mu)S(0)+\beta R_L}{\alpha\ (0)} \quad (2.14)$$

Equation (2.14) represents the number of bacteria *Salmonella typhi* that cause typhoid in human population. To find the number of susceptible individuals S^* in the population, we substitute (2.14) into (2.7) and this gives

$$S^*=\frac{(\beta_3-\beta_1)(\gamma+\delta+\mu)}{\alpha\beta_2 S(0)} \quad (2.15)$$

We therefore have two equilibrium points: diseases free equilibrium $E^0=(S,0\ 0\ 0)$ and endemic equilibrium $E^*=(S^*,I^*,R^*,B^*)$

7.0 Stability Analysis

8.0 Local stability of Disease-Free Equilibrium

We test our model local stability by applying Jacobian method to our model (2.1) as follows

$$\begin{vmatrix}-(\rho+\mu+\lambda) & 0 & -\alpha\ (0)\\ 0 & -(\gamma+\delta+\mu+\lambda) & \alpha\ (0)\\ 0 & \beta_2 & (\beta_1-\beta_3)-\lambda\end{vmatrix}=0 \quad (2.16)$$

$$[(\rho+\mu)+\lambda][\lambda^2+(\gamma+\delta+\mu+[\beta_3-\beta_1])\lambda+(1-R_0)]=0 \quad (2.17)$$

Solving equation (2.17), we obtain three eigenvalues(λ ’) that are negative provided $R_0<1$,therefore, our typhoid model (2.1) is locally asymptotically stable.

4.2 Global stability of disease-free equilibrium

Let us consider the Lynapunov function[7]

$$V=\ S^0\left(\frac{S}{S^L}-1-\ln(\frac{S}{S^L})\right)+I^0\left(\frac{I}{I^L}-1-\ln(\frac{I}{I^L})\right) \quad (2.18)$$

It is easy to see that *V* is positive in the positive cone R^3 and attains zero at E^0. We therefore show that $\dot{V}$ is negative definite. Differentiating *V* at the point E^0, we obtain

$$\dot{V}=\ S'\left(1-\frac{S^L}{S}\right)+I'\left(1-\frac{I^L}{I}\right) \quad (2.19)$$

Substituting equations (2.7) and (2.8) and rearranging the related gives

$$\dot{V}=\ \alpha B^0S^0\left(2-\frac{S^L}{S}-\frac{B}{B^LS^L}\right)+(p+\mu)S^0\left(2-\frac{S^L}{S}-\frac{S}{S^L}\right)+I(\mu+\gamma+\delta)(R_0-1) \quad (2.20)$$

If $R_0<1$. By this and the relation of geometric and arithmetic means, we conclude $\dot{V}\le 0$, with equality holding at the equilibrium E^0. Therefore, E^0 is globally asymptotically stable if $R_0<1$.

9.0 Local stability of Endemic Equilibrium

We test our model local stability by applying Jacobian method to our model (2.1) as follows

$$\begin{vmatrix}-(\alpha B^*+\rho+\mu+\lambda) & 0 & -\alpha S^*\\ \alpha B^* & -(\gamma+\delta+\mu+\lambda) & \alpha S^*\\ 0 & \beta_2 & (\beta_1-\beta_3)-\lambda\end{vmatrix}=0 \quad (2.21)$$

Solving equation (2.18), we obtain the characteristic equation of degree 3:

$$\lambda^3+A_1\lambda^2+A_2\lambda+A_3=0 \quad (2.22)$$

where

$$A_1=\alpha B^*+\rho+2\mu+\gamma+\delta+\beta_1-\beta_3 \quad (2.23)$$

$$A_2=(\alpha B^*+\rho+2\mu+\delta+\gamma)(\beta_1-\beta_3)+(\alpha B^*+\rho+\mu)(\gamma+\delta+\mu)+\beta_2\alpha S^* \quad (2.24)$$

$$A_3 = (\alpha B^* + \rho + \mu)[\gamma + \mu + \delta][\beta_1 - \beta_3] + \beta_2 \alpha S^*(\rho + \mu) \quad (2.25)$$

It is obvious that A_1in (2.22) and A_2in (2.22) are positive, but, A_3in (2.22) is positive provided $R_0 > 1$. Solving equation (2.22), the three eigenvalues (λs') are all negative. Also, by applying Routh-Hurwith criterion[8,9]to (2.22), we have the following:

$A_1 > 0$,

$A_3 > 0$,

and

$A_1 A_2 - A_3 > 0$.

Provided $R_0 > 1$.Therefore, the model E^* is locally asymptotically stable.

10.0 Global Stability of Endemic Equilibrium

Let us consider the Lynapunov function

$$V = S^*\left(\frac{S}{S^*} - 1 - \ln\left(\frac{S}{S^*}\right)\right) + I^*\left(\frac{I}{I^*} - 1 - \ln\left(\frac{I}{I^*}\right)\right) \quad (2.26)$$

It is easy to see that V is positive in the positive cone R^3 and attains zero at E^*. We therefore show that $\dot{V}$ is negative definite. Differentiating V at the point E^*, we obtain

$$\dot{V} = S'\left(1 - \frac{S^*}{S}\right) + I'\left(1 - \frac{I^*}{I}\right) \quad (2.27)$$

Substituting equations (2.15) and (2.13) and rearranging the related gives

$$\dot{V} = \alpha B^* S^*\left(2 - \frac{S^*}{S} - \frac{B}{B^* S^*}\right) + (p + \mu)S^*\left(2 - \frac{S^*}{S} - \frac{S}{S^*}\right) + I(\mu + \gamma + \delta)\left(\frac{S^* R_0}{S(0)} - 1\right) \quad (2.28)$$

Equation (2.15) combined together with the relation of geometric and arithmetic means, implies that $\dot{V} \leq 0$, with equality holding at the equilibrium E^*. By applying Lyapunov-Lasalle theorem[7], we therefore conclude that E^0 is globally asymptotically stable inR^3.

11.0 Numerical Result

Here, we consider the parameter values gotten from Central Intelligent Agency (CIA) and Journal materials for our model. Table 1 displays the parameter values and theirreferences.

Table 1: Parameter values

S/n	Parameter Symbol	Parameter Description	Value	reference
1	β	Recruitment rate for human beings	0.0001063 day^{-1}	[10]
2	α	Transmission rate of typhoid from Bacteria to susceptible human and infected host	0.000197 day^{-1}	[11] [12]
3	μ	Natural mortality rate for human	0.000052 day^{-1}	[10]
4	p	Vaccination rate	0.7	
5	ρ	Removal rate of human from typhoid infected state to susceptible state	(0.6 - 0.8)	[12] [13]
6	δ	Typhoid induced death	(0.5 - 0.8)	[11]
7	γ	Recovery rate for typhoid	0.000548 day^{-1}	[11] [12]
8	β_1	Growth rate of Salmonella bacteria	0.000904 day^{-1}	[14]
9	σ	Rate of discharge of bacteria by Infected human	0.0000022 day^{-1}	[14]
10	d	Natural/Drug Induced death of Bacteria	0.00247 day^{-1}	[14]
11	N	Total humans	155215600	[10]

To support analytical results in this study, we carried out numerical experiment on the typhoid model using the MATLAB ODE solver, ode45 and parameter values in Table 1.

Figure1 shows that the number of susceptible humans is decreasing with time for R_0>1, due to control measure introduced (i.e. vaccination) to control the spread of typhoid *Salmonella* typhi in the population.Figure2 shows that the number of typhoid infected is on the decrease due to the fact that there is less number of susceptible individuals to be attacked by the bacteria. Figure3 shows that the number of recovered humans is increasing with time for, due to treatment of infected individuals. Figure4 shows that the number of bacteria *Salmonella typhi* is decreasing exponentially withtime; this is as a result vaccination and preventive measure like washing of hands.

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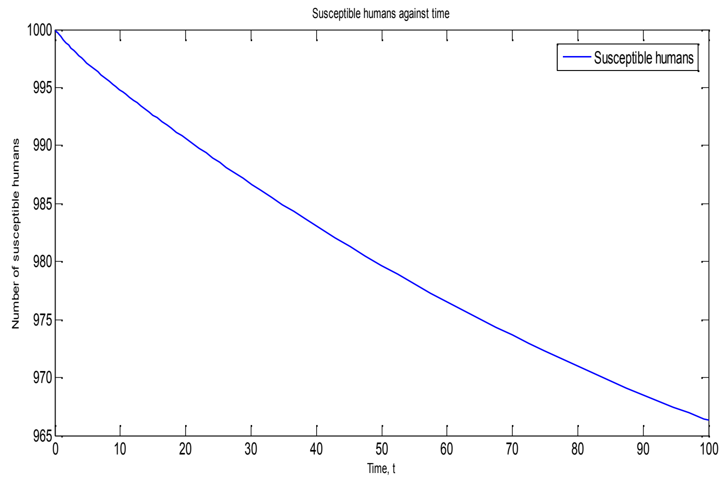

Figure 1: The population of susceptible humans with respect to time

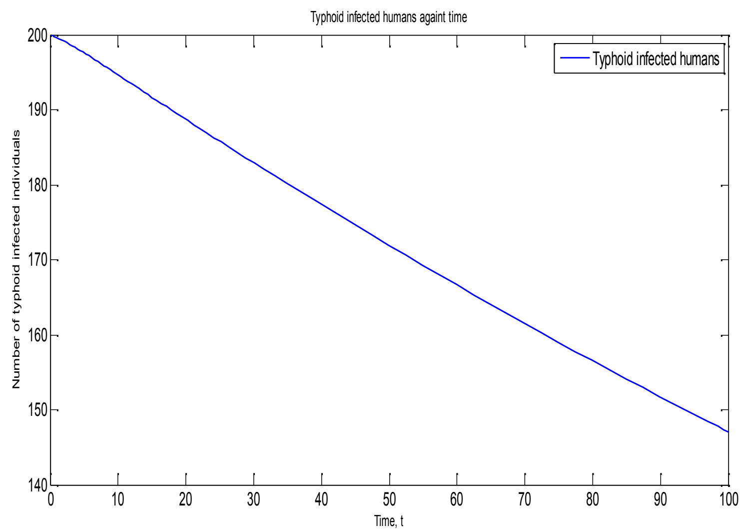

Figure 2: The population of typhoid infected humans with respect to time

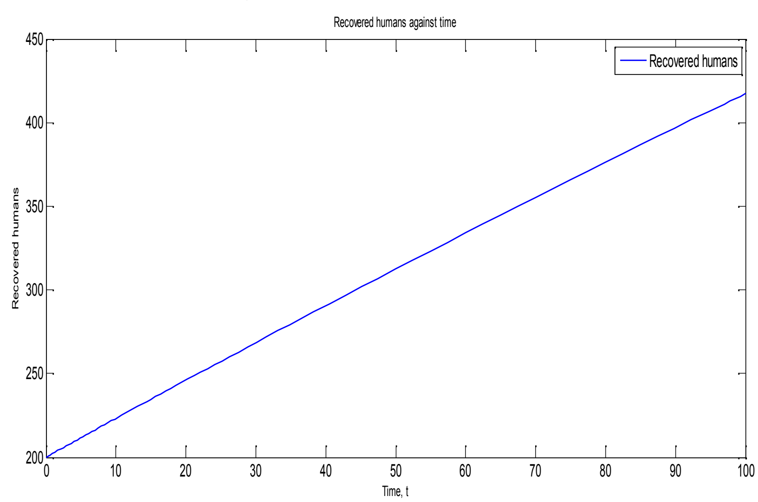

Figure 3: The population of recovered humanswith respect to time

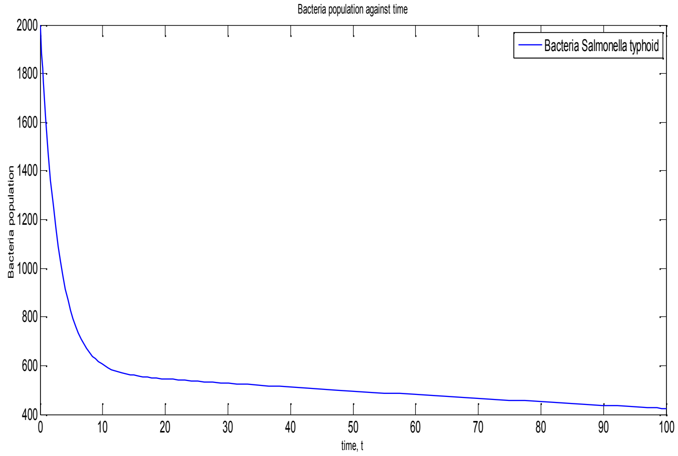

Figure 4: The population of *Salmonella* typhi bacteria with respect to time

12.0 Discussion

In this paper, we propose a system of four ordinary differential equations to model the control of typhoid fever with vaccination for the susceptible individuals in the population. We investigated the following:i. existence of disease-free equilibrium ii. existence of endemic equilibrium and iii. their stabilities. It was discovered in this paper that global stabilities of disease-free and endemic equilibriums exist unlike other papers where only local stability exist for disease-free equilibrium. We also discovered that vaccination against typhoid and thorough washing of hands and fruits can reduced bacteria *Salmonella typhi* in the community there by reducing the spread and transmission of the bacteria to the susceptible subpopulation.

13.0 References

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