Journal of the Nigerian Association of Mathematical Physics
Volume 29, (March, 2015), **pp 151 – 162**
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Mathematical Modelling of a Staged Progression HIV/AIDS Model with Control Measures

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Abstract

A staged-progression model for HIV/AIDS transmission dynamics is formulated and analyzed to study the impact of condom usage, HIV-related public health program and treatment. The local and global stability for the disease free equilibrium (DFE) was proved for Rc <1 and Kransnoselki sublinearity trick was used to show that the endemic equilibrium (EE) is locally asymptotically stable for a special case whenever R_c1 >1. Numerical simulation was also carried out for both EE and EE at special case to illustrate the idea of the results.

1.0 Introduction

Human Immunodeficiency virus (HIV) is an etiological agent of Acquired Immunodeficiency Syndrome (AIDS), an epidemic that constitutes one of the health and developmental challenges in the world today. Despite tremendous effort by researchers and scientists, HIV remains incurable, with no perfect prophylactic vaccine thereby causing the population of HIV/AIDS infectives to persist. Thus, there is need to develop an effective strategy in the prevention and control of HIV/AIDS infections, which is paramount in curbing its menance.

Mathematical models have been of great interest to Applied Mathematicians and Biologist to gain more insight on factors favoring the transmission of HIV/AIDS. Authors over the years have qualitatively studied the effect of HIV-related public health educational program, condom usage and treatment[1-5]. However aforementioned authors do not incorporate the staged-progression nature of HIV, which is an essential part of its transmission dynamics. Models of HIV/AIDS with staged-progression were studied in [6-7] with no control measure. And in [8,9] a staged progression HIV model with imperfect vaccine as the only control measures was studied.

Generally, many research have been carried out to analyze mathematically the role(s) of one or two of the above mentioned control measures on the spread of HIV/AIDS, we therefore presents deterministic model to complements and extend the work of a aforementioned authors by incorporating the staged progression nature of HIV/AIDS in the workdone in [4] and the rate of educating uncounseled AIDS individuals as shown in [3].

2.0 Model Formulation and Description

The total population of Nigeria at time t, denoted by $N(t)$ is subdivided into eight (8) mutually exclusive compartments of Susceptible individuals $S(t)$, non-counseled $E_u(t)$ and counseled $E_c(t)$ asymptomatic individuals, non-counseled $I_u(t)$ and counseled $I_c(t)$ symptomatic individuals, non-counseled $A_u(t)$ and counseled $A_c(t)$ individuals with AIDS symptoms and AIDS infected individuals receiving treatment $I_T(t)$, so that

$$N(t) = S(t) + E_u(t) + E_c(t) + I_u(t) + I_c(t) + A_u(t) + A_c(t) + I_T(t) \quad (1)$$

The recruitment rate of individuals (assumed susceptible) into the sexually active population is denoted ®, individuals acquire HIV infection, following effective contact with infected individuals in the $E_u, E_c, I_u, I_c, A_u, A_c$, and I_T classes at a rate Γ.

Where

$$\Gamma = \frac{(1-\in\sim)[E_u + ¤_u I_u + ¦_u A_u + ¯_c(E_c + ¤_c I_c + ¦_c A_c) + ¯_T I_T]}{N} \quad (2)$$

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is the force of infection.
The model takes the form of the following deterministic system of non-linear equations

$$
\begin{aligned}
\frac{dS}{dt} &= ® - (\Gamma + «)S \\
\frac{dE_u}{dt} &= \Gamma S - (¥_1 + w + «)E_u \\
\frac{dE_c}{dt} &= ¥_1 E_u - (w + «)E_c \\
\frac{dI_u}{dt} &= wE_u - (¥_1 + ± + «)I_u \\
\frac{dI_c}{dt} &= wE_c + ¥_1 I_u - (± + «)I_c \\
\frac{dA_u}{dt} &= ±I_u - (¥_2 + ²_u + ¢ + «)A_u \\
\frac{dA_c}{dt} &= ±I_c + ¥_2 A_u - (²_c + ¢ + «)A_c \\
\frac{dI_T}{dt} &= ²_c A_c + ²_u A_u - (« + · ¢)I_T
\end{aligned}
\qquad (3)
$$

The flowchart of the above differential equations is given in Figure1:

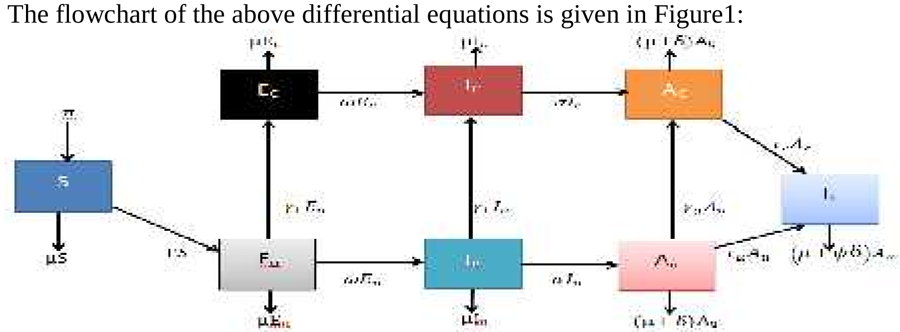

FIG 1: SCHEMATIC PRESENTATION OF THE MODEL

Table 1: Description of the Model Parameters

Parameters	Interpretation
®	Recruitment rate of humans
«	Natural death rate
	Effective contact rate
w	Progressive rate from E_u to I_u and from E_c to I_c classes
±	Progressive rate from I_u to A_u and from I_c to A_c classes
¢	Disease induced death rate of AIDs individuals
$²_u$	Treatment rate for individuals in A_u class
$²_c$	Treatment rate for individuals in A_c class
$¥_1$	Rate of counselling individuals in E_u, I_u class
$¥_2$	Rate of counselling individuals in A_u class
$¦_u, ¦_c$	Modification parameter associated with infection by AIDs individuals
$¯_T$	Modification parameter associated with infection by treated individuals
$¤_u, ¤_c$	Modification parameters associated with infection by individuals in I_u and I_c class
$¯_c$	Modification parameter associated with infection by counselled individuals
·	Modification parameter associated with reduced mortality of treated individuals.
$\in$	Condom efficacy
~	Condom compliance

Table 2: Hypothetical Valules of Model Parameters

Parameter	Hypothetical Value	References
®	$3000000yr^{-1}$	[3,4]
«	$0.02yr^{-1}$	[2]
	$0.3yr^{-1}$	[3,9]
¢	$0.333yr^{-1}$	[1]
$¯_c, ¯_T$	(0,1)	[9,10]
$¯_T$	(0,1)	[9,10]
∈	0.8	[4]
±	$0.6yr^{-1}$	[4]
·	0.75	[4]
$¦_u, ¦_c$	1.5	[4,11]
$¦_c$	1.5	[4]
~	(0,1)	[4]
$¥_1$	0.5	[3]
$¥_2$	0.5	[3]
$²_c, ²_u$	(0,1)	Estimated
$²_u$	(0,1)	Estimated
µ	0.6	Estimated
$¤_u, ¤_c$	1.2	[11]
$¤_c$	1.2	[11]

3.0 Equilibria State and Stability Analysis of the Model

4.0 Existence of Disease Free Equilibrium (DFE) and Effective Reproduction Number (R_c)

Let $£_{\circ}$ represent the equilibrium point at DFE. In the absence of infection, $E_u = E_c = I_u = I_c = A_u = A_c = I_T = 0$ and $S = \frac{®}{«}$ from (5). The model (3) has its DFE given by

$$£_{\circ} = (s^*, E_u^*, E_c^*, I_u^*, I_c^*, A_u^*, A_c^*, I_T^*) = \left(\frac{®}{«}, 0,0,0,0,0,0,0\right)$$

The stability of $£_{\circ}$ can be analyzed by the method of Effective Reproductive Number (R_c) which is determined by using the next generation method, on model (3) in the form of matrices F(non-negative) and V(non-singular). Where F denote the new infection terms and V the transition term at $£_{\circ}$. Therefore

$$F = \begin{bmatrix} (1-\in\sim) & ¯_c(1-\in\sim) & ¤_u(1-\in\sim) & C_1 & C_2 & C_3 & C_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where

$C_1 = {}^{-}{}_c {}^{\ddot{}}{}_c(1-\in\sim), C_2 = (1-\in\sim)¦_u, C_3 = {}^{-}{}_c ¦_c(1-\in\sim)$ and $C_4 = {}^{-}{}_T(1-\in\sim)$

$$V = \begin{bmatrix} k_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -¥_1 & k_2 & 0 & 0 & 0 & 0 & 0 \\ -\mu & 0 & k_3 & 0 & 0 & 0 & 0 \\ 0 & -\mu & -¥_1 & k_4 & 0 & 0 & 0 \\ 0 & 0 & -\pm & 0 & k_5 & 0 & 0 \\ 0 & 0 & 0 & -\pm & -¥_2 & k_6 & 0 \\ 0 & 0 & 0 & 0 & -²_u & -²_c & k_7 \end{bmatrix}$$

where

$$k_1 = ¥_1 + \mu + «,\quad k_2 = \mu + «,\quad k_3 = ¥_1 + \pm + «,\quad k_4 = « + \pm,$$
$$k_5 = ¥_2 + ²_u + ¢ + «, k_6 = ²_c + ¢ + «,\quad k_7 = « + \cdot ¢$$

$$R_c = \frac{(1-\in\sim)(A+B)}{k_1k_2k_3k_4k_5k_6k_7} \tag{6}$$

where

$$A = k_3k_4k_5k_6k_7(k_2 + ¥_1{}^{-}{}_c) + ¥_1{}^{\ddot{}}{}_c{}^{-}{}_c\mu k_5k_6k_7(k_2+k_3) + k_2k_4k_6k_7\mu({}^{\ddot{}}{}_u k_5 + ¦_u\pm)$$
$$+{}^{-}{}_c¦_c\mu\pm k_7[¥_1k_5(k_2+k_3) + ¥_2k_2k_4]$$
$$B = {}^{-}{}_T\mu\pm[²_u k_2k_4k_6 + ²_c[¥_1k_5(k_3+k_2) + ¥_2k_2k_4]]$$

and ° is the spectral radius (dominant eigen-value in magnitude) of the generation matrix, FV^{-1}. Hence the following result is established.

5.0 Local Stability of DFE

Theorem:1

The DFE of the model (3) is locally asymptotically stable (LAS) if $R_c < 1$ and unstable if $R_c > 1$

Proof

It is easy to prove that the above theorem holds by using

$S = N - E_u - E_c - I_u - I_c - A_u - A_c - I_T$ to reduce model (3) into 7 dimensional system.

$$\frac{dE_u}{dt} = \frac{(1-\in\sim)[E_u + {}^{\ddot{}}{}_u I_u + ¦_u A_u + {}^{-}{}_c(E_c + {}^{\ddot{}}{}_c I_c + ¦_c A_c) + {}^{-}{}_T I_T][N - E_u - E_c - I_u - I_c - A_u - A_c - I_T]}{N}$$
$$-k_1E_u$$

$$\begin{aligned} \frac{dE_c}{dt} &= ¥_1E_u - k_2E_c \\ \frac{dI_u}{dt} &= \mu E_u - k_3I_u = 0 \\ \frac{dI_c}{dt} &= \mu E_c + ¥_1I_u - k_4I_c \\ \frac{dA_u}{dt} &= \pm I_u - k_5A_u \\ \frac{dA_c}{dt} &= \pm I_c + ¥_2A_u - k_6A_c \\ \frac{dI_T}{dt} &= ²_uA_u + ²_cA_c - k_7I_T \end{aligned} \tag{7}$$

It follows that the Jacobian matrix of the system above, evaluated at $£_\circ$ is obtain as ;

$$J(£_{\circ}) = \begin{bmatrix} (1-\in\sim)-k_1 & ¯_c(1-\in\sim) & C_0 & C_1 & C_2 & C_3 & C_4 \\ ¥_1 & -k_2 & 0 & 0 & 0 & 0 & 0 \\ µ & 0 & -k_3 & 0 & 0 & 0 & 0 \\ 0 & µ & ¥_1 & -k_4 & 0 & 0 & 0 \\ 0 & 0 & ± & 0 & -k_5 & 0 & 0 \\ 0 & 0 & 0 & ± & ¥_2 & -k_6 & 0 \\ 0 & 0 & 0 & 0 & ²_u & ²_c & -k_7 \end{bmatrix}$$

where

$$C_0 = ¤_u(1-\in\sim)$$

Using the elementary row-transformation [5], we have

$$\therefore J(£_{\circ}) = \begin{bmatrix} (1-\in\sim)-k_1 & ¯_c(1-\in\sim) & C_0 & C_1 & C_2 & C_3 & C_4 \\ 0 & G_1 & G_2 & G_3 & G_4 & G_5 & G_6 \\ 0 & 0 & G_7 & G_8 & G_9 & G_{10} & G_{11} \\ 0 & 0 & 0 & G_{12} & G_{13} & G_{14} & G_{15} \\ 0 & 0 & 0 & 0 & G_{16} & G_{17} & G_{18} \\ 0 & 0 & 0 & 0 & 0 & G_{19} & G_{20} \\ 0 & 0 & 0 & 0 & 0 & 0 & G_{21} \end{bmatrix}$$

$$[\ (1-\in\sim)-k_1-ª][G_1-ª][G_7-ª][G_{12}-ª][G_{16}-ª][G_{19}-ª][G_{21}-ª]=0.$$

gives the characteristics equation which is obtained by using $|J-ªI|=0$. Hence, the characteristics roots (eigen-values) are

$$ª_1 = J_1, ª_2 = G_1, ª_3 = G_7, ª_4 = G_{12}, ª_5 = G_{16}, ª_6 = G_{19}, ª_7 = G_{21}$$

Where

$$G_1 = -\frac{J_2}{J_1}, G_{12} = -\frac{J_4}{J_3}, G_{16} = -\frac{J_5}{J_4}, G_7 = -\frac{J_3}{J_2}, G_{19} = -\frac{J_6}{J_5}, G_{21} = -\frac{J_7}{J_6}$$

$$J_1 = (1-£\sim)-K_1, J_2 = (1-£\sim)(k_2+¥_1¯_c)-K_1k_2, J_3 = k_3J_2 + (1-£\sim)µ¤_uK_2$$

$$J_4 = K_4J_3 + (1-£\sim)µ¥_1¯_c¤_c(K_2+K_3), J_5 = K_5J_4 + (1-£\sim)K_2k_4µ±¦_u,$$

$$J_6 = k_6J_5 + (1-£\sim)¯_c¦_cµ±[¥_1k_5(k_2+k_3)+¥_2k_2k_4]$$

$$J_7 = k_7J_6 + (1-£\sim)¯_Tµ±[²_uk_2k_4k_6 + ²_c[¥_1k_5(k_3+k_2)+¥_2k_2k_4]]$$

It is import to note that after many tedious algebraic substitution $J_7 = (1-\in\sim)(A+B)-k_1k_2k_3k_4k_5k_6k_7$

When $ª_i < 0$ for all $i = 1,2,\cdots,7,$ the system is said to be locally asymptotically stable (LAS) at $DFE(£_{\circ}),$ hence from G_{21} we obtained

$$\frac{(1-\in\sim)(A+B)}{k_1k_2k_3k_4k_5k_6k_7} < 1 \Rightarrow R_c < 1$$

This completes the proof.

6.0 Global Stability of DFE

Theorem:2

The DFE of the model (3), is global asymptotically stable (GAS) in D if $R_c \le 1$

Proof

Consider the Lyapunov function

$$Q_1E_u + Q_2E_c + Q_3I_u + Q_4I_c + Q_5A_u + Q_6A_c + I_T$$

Where

$$Q_1 = \frac{k_7R_c}{(1-\in\sim)¯_T}$$

$$Q_2 = \frac{¯_c k_6 k_7 (k_4 + ¤_c µ) + ±µ(¯_c ¦_c k_7 + ¯_T ²_c)}{¯_T k_2 k_4 k_6}$$

$$Q_3 = \frac{k_4 k_6 k_7 (k_5 ¤_u + ± ¦_u) + ± k_4 [¯_c ¦_c k_7 ¥_2 + ¯_T (²_c ¥_2 + ²_u k_6)] + ¥_1 k_5 [k_7 ¯_c (k_6 ¤_c + ¦_c ±) + ¯_T ²_c ±]}{¯_T k_3 k_4 k_5 k_6}$$

$$Q_4 = \frac{¯_c k_7 (k_6 ¤_c + ¦_c ±) + ¯_T ²_c ±}{¯_T k_4 k_6}$$

$$Q_5 = \frac{k_6 (k_7 ¦_u + ¯_T ²_u) + (¯_c ¦_c k_7 + ¯_T ²_c) ¥_2}{¯_T k_5 k_6}$$

$$Q_6 = \frac{¯_c ¦_c k_7 + ¯_T ²_c}{¯_T k_6}$$

The time derivative of the Lyapunov function is given by (where a dot represents differentiation with respect to time t)

$$\dot{L} = Q_1 \dot{E}_u + Q_2 \dot{E}_c + Q_3 \dot{I}_u + Q_4 \dot{I}_c + Q_5 \dot{A}_u + Q_6 \dot{A}_c + \dot{I}_T$$

Using the coefficients, and with further simplifications to obtain

$$\dot{L} = \frac{k_7}{¯_T}\left(\frac{R_c S}{N} - 1\right) E_u + \frac{¯ c k_7}{¯_T}\left(\frac{R_c S}{N} - 1\right) E_c + \frac{¨_u k_7}{¯_T}\left(\frac{R_c S}{N} - 1\right) I_u + \frac{¯_c ¨_c k_7}{¯_T}\left(\frac{R_c S}{N} - 1\right) I_c + \frac{¦_u k_7}{¯_T}\left(\frac{R_c S}{N} - 1\right) A_u$$

$$+ \frac{¯_c ¦_c k_7}{¯_T}\left(\frac{R_c S}{N} - 1\right) A_c + k_7 \left(\frac{R_c S}{N} - 1\right) I_T$$

$$\dot{L} = \frac{k_7}{¯_T}\left(E_u + ¯_c E_c + ¤_u I_u + ¯_c ¤_c I_c + ¦_u A_u + ¯_c ¦_c A_c + ¯_T I_T\right)\left(\frac{R_c S}{N} - 1\right)$$

Since $S \le N$ in D, we now have

$$\dot{L} \le \frac{k_7}{¯_T}[E_u + ¯_c E_c + ¤_u I_u + ¯_c ¤_c I_c + ¦_u A_c + ¯_c ¦_c A_c + ¯_T I_T](R_c - 1)$$

Note that the quantity in square bracket is always positive.

Clearly, $\dot{L} \le 0$ when $R_c \le 1$ and $\dot{L} = 0$ if and only if $E_u = E_c = A_u = A_c = I_u = I_c = I_T = 0$,hence $\Gamma = 0$.

It follows from invariance principle , that every solution to the system (3) with initial conditions in D approaches $£_\circ$ as $t \to \infty$. Thus, since the region D is positively-invariant, the DFE is GAS in D if $R_c \le 1$.

7.0 Existence of Endemic Equilibrium Point (EEP)

In order to obtain the endemic equilibrium point of the model (3) (i.e, in the presence of infection, where at least one of the infected component of the model is non-zero). Let $£_1 = (S^{**}, E_u^{**}, E_c^{**}, I_u^{**}, I_c^{**}, A_u^{**}, A_c^{**} I_T^{**},)$ represents any arbitrary endemic equilibrium of the model (3). Solving equations in model (3) to yield the following

$$S^{**} = \frac{®}{\Gamma^{**} + «} \quad E_u = \frac{\Gamma^{**} ®}{k_1[\Gamma^{**} + «]} \quad E_c^{**} = \frac{¥_1 \Gamma^{**} ®}{k_1 k_2 [\Gamma^{**} + «]}$$

$$I_u^{**} = \frac{µ \Gamma^{**} ®}{k_1 k_3 [\Gamma^{**} + «]}$$

$$I_c{**} = \frac{µ ¥_1 \Gamma^{**} ® (K_2 + k_3)}{k_1 k_2 k_3 k_4 [\Gamma^{**} + «]} \quad A_u^{**} = \frac{µ ± \Gamma^{**} ®}{k_1 k_3 k_5 [\Gamma^{**} + «]} \qquad (8)$$

$$A_c^{**} = \frac{± µ \Gamma^{**} ® [¥_1 k_5 (k_2 + k_3) + ¥_2 k_2 k_4]}{k_1 k_2 k_3 k_4 k_5 k_6 [\Gamma^{**} + «]}$$

$$I_T^{**} = \frac{± µ \Gamma^{**} ® [²_c (¥_1 k_5 (k_2 + k_3) + ¥_2 k_2 k_4) + ²_u k_2 k_4 k_6]}{k_1 k_2 k_3 k_4 k_5 k_6 k_7 [\Gamma^{**} + «]}$$

where Γ^{**} is expressed at equilibrium, as

$$\Gamma^{**} = \frac{(1-\in\sim)[E_u^{**} + ¤_u I_u^{**} + ¦_u A_u^{**} + ¯_c(E_c^{**} + ¤_c I_c^{**} + ¦_c A^{**}) + ¯_T I_T^{**}]}{N}$$

and (6) gives the associated effective reproductive number.

8.0 Existence and Local Stability of EEP: Special Case

The existence and local stability of endemic equilibrium is explored for a special case where there is no AIDS induced death ($¢ = 0$) or assumed to be negligible [2,4,12,13] In the continuing absence of HIV/AIDS cure, this assumption and corresponding analysis has no public health meaningful insights [10] but it allows us to investigate the worst scenario as the accumulation of AIDS individuals is at its maximum [14], Thus under this setting (with $¢ = 0$), model(3) has a unique endemic equilibrium point $£_1'$, of the form $£_1' = (S', E_u', E_c', I_u', I_c', A_u', A_c', I_T')$ where

$$\begin{aligned}
S' &= \frac{®}{\Gamma' + «} \quad E_u' = \frac{\Gamma' S'}{k_1} \quad E_c' = \frac{¥_1\Gamma' S'}{k_1 k_1} \quad I_u' = \frac{\mu\Gamma' S'}{k_1 k_3} \\
I_c' &= \frac{\mu ¥_1 \Gamma' S'(k_2 + k_3)}{k_1 k_2 k_3 k_4} \quad A_u' = \frac{\pm\mu\Gamma' S'}{k_1 k_3 k_5'} \\
A_c' &= \frac{\pm\mu\Gamma' S'[¥_1 k_5(k_2 + k_3) + ¥_2 k_2 k_4]}{k_1 k_2 k_3 k_4 k_5' k_6'} \\
I_T' &= \frac{\pm\mu\Gamma S'[²_c(¥_1 k_5(K_2 + k_3) + ¥_2 k_2 k_4) + ²_u k_2 k_4 k_6]}{« k_1 k_2 k_3 k_4 k_5' k_6'}
\end{aligned} \tag{9}$$

and the associated Reproductive Number

$$R_{c_1} = \frac{(1-\in\sim)(A_1 + B_1)}{« k_1 k_2 k_3 k_4 k_5' k_6'} \tag{10}$$

where

$$\begin{aligned}
A_1 &= « k_3 k_4 k_5' k_6'(k_2 + ¥_1 ¯_c) + ¥_1 ¤_c ¯_c \mu k_5' k_6' «(k_2 + k_3) + k_2 k_4 k_6' « \mu(¤_u k_5' + ¦_u \pm) \\
&+ ¯_c ¦_c \mu \pm «[¥_1 k_5'(k_2 + k_3) + ¥_2 + k_2 k_4] \\
B_1 &= ¯_T \mu \pm \left[²_u k_2 k_4 k_6' + ²_c[¥_1 k_5'(k_2 + k_3) + ¥_2 k_2 k_4]\right]
\end{aligned}$$

From(1) $\frac{dN}{dt} = ® - « N$ since $¢ = 0$, so that $N \to \frac{®}{«} = N'$ as $t \to \infty$. Therefore the force of infection Γ' at the special case of endemic equilibrium can be expressed as

$$\Gamma' = \frac{(1-\in\sim)[E_u' + ¤_c I_u' + ¦_u A_u' + ¯_c(E_c' + ¤_c I_c' + ¦_c A_c') + ¯_T I_T]}{N'} \tag{11}$$

Substituting the expressions in (9) into (11) with further simplification gives

$$N'\Gamma' = \frac{(1-\in\sim)\Gamma' S'}{« k_1 k_2 k_3 k_4 k_5 k_6}\left\{\begin{aligned}
&« k_3 k_4 k_5' k_6'(k_2 + ¥_1 ¯_c) + ¥_1 ¤_c ¯_c \mu k_5' k_6' «(k_2 + k_3) \\
&+ k_2 k_4 k_6' « \mu(¤_u k_5' + ¦_u \pm) + ¯_c ²_c \mu \pm «[¥_1 k_5'(k_2 + k_3) \\
&+ ¥_2 k_2 k_4] + ¯_T \mu \pm [²_u k_2 k_4 k_6' + ²_c[¥_1 k_5'(k_2 + k_3) + ¥_2 k_2 k_4]]
\end{aligned}\right\} \tag{10}$$

$$\begin{aligned}
N'\Gamma' &= \frac{(1-\in\sim)\Gamma' S'(A_1 + B_1)}{« k_1 k_2 k_3 k_4 k_5 k_6} \\
N' &= S' R_{c,} \\
&\Rightarrow \frac{S'}{N'} = \frac{1}{R_{c,}}
\end{aligned} \tag{12}$$

$$\begin{aligned}
\frac{R_{c,}®}{\Gamma' + «} &= \frac{®}{«} \\
«®(R_{c,} - 1) &= ®\Gamma'
\end{aligned}$$

$$\Gamma^{'} = «(R_{c} - 1) > 0 \quad whenever \;\; R_{c} > 1 \qquad (13)$$

The component of $£_1$ can be obtained by substituting the unique value of $\Gamma^{'}$, given by (13), into the expression (9). Thus, the following result is established.

Lemma:1

The model(3) with $¢ = 0$ has a unique endemic (positive) equilibrium, given by $£_1$, whenever $R_{c,} > 1$

Theorem:3

The associated unique endemic equilibrium $£_1^{'}$ of the model (3) with $¢ = 0$ is LAS if $R_{c,} > 1$

Proof

Using $S = N - E_u - E_c - I_u - I_c - A_u - A_c - I_T$, the model (3) with $¢ = 0$ can be re-written as:

$$\frac{dE_u}{dt} = \frac{(1- \in \sim)[E_u + ¨_u I_u + ¦_u A_u + ¯_c(E_c + ¨_c I_c + ¦_c A_c) + ¯_T I_T][N - E_u - E_c - I_u - I_c - A_u - A_c - I_T]}{N}$$
$$-k_1 E_u$$
$$\frac{dE_c}{dt} = ¥_1 E_u - k_2 E_c$$
$$\frac{dI_u}{dt} = \mu E_u - k_3 I_u = 0$$
$$\frac{dI_c}{dt} = \mu E_c + ¥_1 I_u - k_4 I_c \qquad (14)$$
$$\frac{dA_u}{dt} = \pm I_u - k_5^{'} A_u$$
$$\frac{dA_c}{dt} = \pm I_c + ¥_2 A_u - k_6^{'} A_c$$
$$\frac{dI_T}{dt} = {}^2{}_u A_u + {}^2{}_c A_c - « I_T$$

Linearizing the model (14) around the endemic equilibrium, $£_1^{'}$,gives

$$\frac{dE_u}{dt} = (p_1 - p_2 - k_1)E_u + (¯_c p_1 - p_2)E_c + (¨_u p_1 - p_2)I_u + (¯_c ¨_c p_1 - p_2)I_c + (¦_u p_1 - p_2)A_u$$
$$+(¯_c ¦_c p_1 - p_2)A_c + (¯_T p_1 - p_2)I_T$$
$$\frac{dE_c}{dt} = ¥_1 E_u - k_2 E_c$$
$$\frac{dI_u}{dt} = \mu E_u - k_3 I_u = 0$$
$$\frac{dI_c}{dt} = \mu E_c + ¥_1 I_u - k_4 I_c \qquad (15)$$
$$\frac{dA_u}{dt} = \pm I_u - k_5^{'} A_u$$
$$\frac{dA_c}{dt} = \pm I_c + ¥_2 A_u - k_6^{'} A_c$$
$$\frac{dI_T}{dt} = {}^2{}_u A_u + {}^2{}_c A_c - « I_T$$

where

$$p_1 = \frac{(1- \in \sim)S^{'}}{N^{'}}$$

$$p_2 = \frac{(1-\in\sim)[E_u^{'} + ¤_u I_u^{'} + ¦_u A_u^{'} + ¯_c(E_c^{'} + ¤_c I_c^{'} + ¦_c A_c^{'}) + ¯_T I_T^{'}]}{N^{'}}$$

The associated Jacobian matrix of the system (15), evaluated at $£_1^{'}$ is given by

$$J(£_1^{'}) = \begin{bmatrix} p_1 - p_2 - k_1 & ¯_c p_1 - p_2 & ¨_u p_1 - p_2 & ¯_c ¨_u p_1 - p_2 & ¦_u p_1 - p_2 & C_6 & C_5 \\ -¥_1 & -K_2 & 0 & 0 & 0 & 0 & 0 \\ \mu & 0 & -k_3 & 0 & 0 & 0 & 0 \\ 0 & \mu & ¥_1 & -k_4 & 0 & 0 & 0 \\ 0 & 0 & \pm & 0 & -K_5 & 0 & 0 \\ 0 & 0 & 0 & \pm & ¥_2 & -k_6 & 0 \\ 0 & 0 & 0 & 0 & ²_u & ²_c & -« \end{bmatrix} \qquad (16)$$

where

$C_5 = ¯_T p_1 - p_2 \quad and \quad C_6 = ¯_c ¦_c p_1 - p_2$

The proof of the above theorem is based on using the Kranoselski's sub linearity trick, as given by Hethcote and Thieme (1985); Adewale et al (2009), Garba and Gummel (2010), Esteva and Vargas(2000), Sun et al (2012). This technique essentially entails showing that the linearized system (15), around the equilibrium $£_1^{'}$ has no solution of the form

$$\bar{Z}(t) = \bar{Z}_{\circ} e^{Xt} \qquad (17)$$

with $z_{\circ} \in X^8 \setminus \{0\}, X \in X, z_i \in X, z_{\circ} = (Z_1,\ Z_2,\ Z_3,\ Z_4,\ Z_5,\ Z_6,\ Z_7)$ and $R_e(x) \geq 0$, where X denotes the complex number. Substituting a solution of the form (17) into the linearized system of (15), at $£_1^{'}$, gives the following linear system.

$$\begin{aligned} XZ_1 &= (p_1 - p_2 - k_1)Z_1 + (¯_c p_1 - p_2)Z_2 + (¨_u p_1 - p_2)Z_3 + (¯_c ¨_c p_1 - p_2)Z_4 + \\ &\quad (¦_u p_1 - p_2)Z_5 + (¯_c ¦_c p_1 - p_2)Z_6 + (¯_T p_1 - p_2)Z_7 \\ XZ_2 &= ¥_1 Z_1 - k_2 Z_1 \\ XZ_3 &= \mu Z_1 - k_3 Z_3 \\ XZ_4 &= \mu Z_2 + ¥_1 Z_3 - k_4 Z_4 \\ XZ_5 &= \pm Z_3 - k_5^{'} Z_5 \\ XZ_6 &= \pm Z_4 + ¥_2 Z_5 - k_6^{'} Z_6 \\ XZ_7 &= ²_u Z_5 + ²_c Z_6 - « Z_7 \end{aligned} \qquad (18)$$

Solving the last five equations of (18) for $Z_3,\ Z_4,\ Z_5,\ Z_6\ and\ Z_7$, and substituting the results in the first two equations, after some algebraic manipulations we obtain the following system

$$\begin{aligned} &Z_1[1 + F_1(X)] + Z_2[1 + F_2(X)] = (M\bar{Z})_1 + (M\bar{Z})_2 \\ &Z_3[1 + F_3(X)] = (M\bar{Z})_3 \\ &Z_4[1 + F_4(X)] = (M\bar{Z})_4 \\ &Z_5[1 + F_5(X)] = (M\bar{Z})_5 \\ &Z_6[1 + F_6(X)] = (M\bar{Z})_6 \\ &Z_7[1 + F_7(X)] = (M\bar{Z})_7 \end{aligned} \qquad (19)$$

Where

$$F_1(X) = \frac{X + P_2}{k_1} + \frac{P_2}{k_1} T_1, \quad F_2(X) = \frac{X}{k_2} + \frac{P_2}{k_1} T_2, \quad F_3(X) = \frac{X}{k_3},,$$

$$F_4(X) = \frac{X}{k_4},\ F_5(X) = \frac{X}{k_5^{'}}, \quad F_6(X) = \frac{X}{k_6^{'}}, \quad F_7(X) = \frac{X}{«},$$

with

$$M=\begin{bmatrix} \frac{P_1}{k_1} & \frac{\bar{\ }_c P_1}{k_1} & \frac{\ddot{\ }_u P_1}{k_1} & \frac{\bar{\ }_c \ddot{\ }_c P_1}{k_1} & \frac{\dot{\ }_u P_1}{k_1} & \frac{\bar{\ }_c \dot{\ }_c P_1}{k_1} & \frac{\bar{\ }_T P_1}{k_1} \\ \frac{¥}{k_2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{µ}{k_3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{µ}{k_4} & \frac{¥_1}{k_4} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{±}{k_5} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{±}{k_6} & \frac{¥_2}{k_6} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{²_u}{«} & \frac{²_c}{«} & 0 \end{bmatrix}$$

$$T_1=\frac{µ}{X+k_3}+\frac{¥_1 µ}{(X+k_3)(X+k_4)}+$$

and $\frac{±µ}{(X+k_3)(X+k_5^{'})}+\frac{±µ[¥_1(X+k_5)+¥_2(X+k_4)]}{(X+k_3)(X+4)(X+k_5^{'})(X+k_6^{'})}+$

$$\frac{±²_u}{(X+«)(X+k_3)(X+k_5^{'})}+\frac{²_c±µ[¥_1(X+k_5)+¥_2(X+k_4)]}{(X+k_3)(X+4)(X+k_5^{'})(X+k_6^{'})}$$

$$T_2=1+\frac{µ}{X+k_4}+\frac{±µ}{(X+k_4)(X+k_6)}+\frac{±²_c µ}{(X+«)(X+k_4)(X+k_6^{'})}$$

Note that the matrix M has non-negative entries and $£_1^{'}=(E_u^{'},E_c^{'},I_u^{'},I_c^{'},A_u^{'},A_c^{'},I_T^{'})$ satisfies $£_1^{'}=M£_1^{'}$. Hence if Z is any solution of (19), then it is possible to find a minimal positive real number r, such that

$$\|Z\|\le r£_1^{'} \qquad (20)$$

where, $\|Z\|=(\|Z_1\|,\|Z_2\|,\|Z_3\|,\|Z_4\|,\|Z_5\|,\|Z_6\|,\|Z_7\|)$, with the lexicographic order, and $\|.\|$ is a norm in X. The primary objective is to show that if $R_e(X)<0$, then the linearized system (15) has a solution of the form (17). By contradiction, we show that $R_e(X)\ge 0$ is not satisfied which will then be sufficient to concluded that $R_e(X)<0$. Hence consider the two general cases for $X=0$ and $X\neq 0$.

Case 1: X=0

In this case, equation (18) becomes a homogenous linear system of the form

$$\overline{O}=GZ_i \; i=1,2,3,4,5,6,7.$$

The determinant of the matrix (16) corresponds to that of the Jacobian of the system (18) givenby

$$\Delta=-p_2[«µ±k_2k_4(k_6^{'}+¥_2)+¥_1µ±k_5^{'}(k_2+k_3)(«+²_c)+«µk_2k_5^{'}k_6^{'}(k_4+¥_1)+$$

$$«k_3k_4k_5^{'}k_6^{'}(k_2+¥_1)+µ±k_2k_4(²_uk_6+²_c¥_2)+«µk_3k_5^{'}k_6^{'}¥_1]+«k_1k_2k_3k_4k_5^{'}k_6^{'}\left(\frac{S^{'}R_{c,}}{N^{'}}-1\right)$$

Using equation (12) to show that $\Delta<0$. Since $\Delta<0,$ it follows that system (18) has a unique solution, given by $Z=\overline{0}$ (which corresponds to the DFE)

Case 2: $X\neq 0$

By assumption, we have $R_e(X)>0$, then $|1+F_i(X)|>1$ for all $i=1,2,\cdots,7$. We define $F(X)=min|1+F_i(X)|$ (for all $i=1,2,\cdots,7$), then $F(X)>1$ and hence $\frac{r}{F(X)}<r$. The minimality of r implies $\|\overline{Z}\|>\frac{r}{F(X)}£_1^{'}$. Taking norm of both sides of the equation of (18), and using the fact that the matrix M is non-negative, gives;

$$F(X)\|Z_3\| \le M(\|Z\|)_3 \le r(M\|£_1^{'}\|)_3 \le rI_c^{'} \qquad (21)$$

Then if follows from the (2) that $\|Z_3\| \le \frac{r}{F(X)} I_c^{'}$, which contradicts $R_e(F_i(X)) \ge 0$. Hence $R_e(X) < 0$, so that the endemic equilibrium $£_1^{'}$ is LAS if $R_{c,} > 1$. This completes the proof.

9.0 Numerical Simulation and Discussion of Results

10.0 Numerical Simulation

The role played by some important epidemiological parameters, are investigated with the aid of Maple software for the numerical simulation by comparing the model Effective Reproductive Number, the parameters used, their estimated values and appropriate source are given in table (2.1).

Table3: Numerical Simulation of the Model at Endemic Equilibruim State.

—$_c$	—$_T$	~	²$_c$	²$_u$	R_c	$R_{c,}$	Remark
0.1	0.9	0.1	0.1	0.9	3.0385	55.7491	Stable
0.2	0.8	0.2	0.2	0.8	2.6178	117.332	Stable
0.3	0.7	0.3	0.3	0.7	2.3462	84.9945	Stable
0.4	0.6	0.4	0.4	0.6	2.1656	69.4375	Stable
0.5	0.5	0.5	0.5	0.5	2.0464	60.9960	Stable
0.6	0.4	0.6	0.6	0.4	1.9729	56.3550	Stable
0.7	0.3	0.7	0.7	0.3	1.9365	54.1951	Stable
0.8	0.2	0.8	0.8	0.2	1.9332	54.0069	Stable
0.9	0.1	0.9	0.9	0.1	1.963	55.7491	Stable

11.0 Discussion of Results

From the table above, it is observed that $R_{C_1} > R_C > 1$ having the same remark, implying that the disease will be more persistent in the absence of AIDS-induced death when compared to the other. Thus the most effective control strategy from the table above is achievable when —$_c = 0.8$, —$_T = 0.2$, $\sim = 0.8$, ²$_c = 0.8$, ²$_u = 0.2$ to give the least persistent threshold for both R_{C_1} and $R_C > 1$.

12.0 Conclusion

In this study,a staged-progression HIV/AIDS model coupled with condom usage, public health education program and treatment is designed. Some of the main findings of this study are:

(i) The model has a global asymptotically stability at DFE whenever $R_C < 1$;

(ii) The EE is stable for a special case whenever $R_{C_1} > 1$;

(iii) The numerical simulation result clearly shows that setting ¢ to zero does not affect the stability of endemic equilibrium since $R_{C_1} > R_C > 1$ gives the same remark although,the disease will be more persistent in the absence of AIDS-induced death.

Thus, this study shows that HIV/AIDS will persist and be eliminated from the population whenever $R_{C_1} > 1$ and $R_C < 1$ respectively.

13.0 References

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