Journal of the Nigerian Association of Mathematical Physics
***Volume 29,* (*March, 2015*), pp 147 – 150**
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A Deterministic Model on the Existence and Uniqueness of Epidemic Model

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Abstract

A deterministic model for the transmission dynamics of endemic malaria which permits transmission by contact with infected host or infected vector with vaccination is developed and carefully analysed. We investigate the model for existence and uniqueness of its solution. The famous Leibnitz condition is employed to establish this existence and uniqueness.

1.0 Introduction

Malaria is a global deadly vector –borne disease [1], whose vectors are the mosquitoes. However, malaria is preventable and curable when treatment and prevention measures are sought early. Despite unrelented constant efforts of researchers though, there is no breakthrough yet for a malaria vaccine. Like other infectious diseases, malaria may be introduced by infected immigrants into the population. For instance, travelers from a malaria endemic region may act as a source of malaria to malaria free regions [2].
There are strong social and economic ways to the burden of the disease, which in so many ways affects fertility, population growth, saving and investment, worker productivity, absenteeism, premature mortality, and medical costs [3]. In areas where malaria is highly endemic, young children bear a larger burden in terms of the disease morbidity and mortality, and it affects fetal development during the early stages of pregnancy in woman due to the loss of immunity. Currently, strategies of controlling the disease include the use of chemo-therapy, intermittent preventive treatment for children and pregnant women (preventive doses of sulfadoxinepyrimethamine (IPT/ST)), the use of insecticides treated bednets, and insecticides against the vector [4].
Epidemiological mathematical models are useful in proposing and testing theories [5], and in comparing, planning, implementing and evaluating various detection, prevention, therapy and control programs. In this paper, we develop an SEIV epidemic model, to ascertain the impact of vaccination on infected human- host.

2.0 Model Equations

The model assumes a homogenous mixing of the human and vector (mosquito) populations, so that each mosquito bites has equal chance of transmitting the infection to susceptible human in the population. The total population is divided into four mutually- exclusive sub-populations of susceptible humans S, exposed E, infected I, and vaccination V.

The class S of susceptible is increased by birth or immigration at a rate Λ. It is decreased by infection following contact with infected individuals at a rate ® , and diminished by natural death at a rate « . The class E of exposed individuals is generated through contact with infected individuals at rate ® .
Applying definitions of variables and parameters with description of terms above, the differential equations describing the dynamics of malaria in the human and mosquito populations are formulated below:

$$\frac{dS}{dt} = \Lambda + \text{¥}V - \text{®}S - \text{«}S \qquad (1)$$

$$\frac{dE}{dt} = \text{®}S - \quad E - \text{«}E \qquad (2)$$

$$\frac{dI}{dt} = \quad E - \text{\textasciitilde}I - \text{«}I \qquad (3)$$

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$$\frac{dV}{dt} = \sim I - ¥V - «V \qquad (4)$$

3.0 Basic Properties of the Model

Since in the human population, all the associated parameters and state variables are non-negative i.e $t > 0$. It is easy to show that the state variables of the model remain non –negative for all non – negative initial conditions.
Consider the biological feasible region

$$\Psi = \left\{ (S, E, I, V : S + E + I + V + R \leq 1) \in \Re^4{}_+ \rightarrow \frac{\Lambda}{«} \right\}$$

Where $\frac{dN}{dt} = \Lambda - «N$

Lemma 1: The closed · is positively invariant and attracting.

Proof:

$\frac{dN}{dt} = \Lambda - «N$. The total human population (N) is bounded by $\frac{\Lambda}{«}$.

$$\frac{dN}{dt} + «N = \Lambda$$

$$Ne^{«t} = \Lambda \int e^{«t} + k$$

$$N(t) = \frac{\Lambda}{«} + ke^{-«t}$$

Hence, at $t = 0$,

$N(t) = \frac{\Lambda}{«} + \left(N_o - \frac{\Lambda}{«} \right) e^{-«t}$, where

$N(0) = \frac{\Lambda}{«}$ if $\qquad N(t) = \frac{\Lambda}{«}$.

Hence, the region · is positively invariant and attracts all solutions in $\Re^4$.

4.0 Existence and Uniqueness of Solution for the Model

For the mathematical model to predict the future of the system from its current state at time t_o, the initial value problem.

$$x^1 = f(t, x), \qquad x(t_o) = x_o \qquad (5)$$

Must have a solution that exist and also unique.
In this subsection, we shall establish condition for the existence and uniqueness of solution for the model of equations. Let

$$\begin{aligned} f_1(t, x) &= \Lambda + ¥v + ®s - «s \\ f_2(t, x) &= ®s - \quad e - «e \\ f_3(t, x) &= \quad e - \sim i - «i \\ f_4(t, x) &= \sim i - ¥v - «v \end{aligned} \qquad (6)$$

So that

$$x^1 = f(t, x), = f(x) \qquad (7)$$

Theorem 1 (see [6]). Let D^1 denotes the region

$|t - t_o| \leq a$, $\|x - x_o\| \leq b$, $x = (x_1, x_2, ..., x_n)$, $\quad x_0 = (x_{10}, x_{20}, ..., x_{n0})$ (8)

Suppose that $f(t, x)$ satisfies the Lipchitz condition

$$\| f(t, x_1) - f(t, x_2) \| \leq k \| x_1 - x_2 \|, \qquad (9)$$

Whenever, the pairs (t, x_1) and (t, x_2) belong to D^1, where k is a positive constant. Then, there exist a constant ¢ > 0, such that there exist a unique continuous vector solution $\bar{x}(t)$ of the system (5) in the interval $|t - t_0| \leq$ ¢.

Lemma 2. If $f(t, x\ )$ has continuous partial derivative $\frac{\partial f_i}{dx_j}$ on a bounded closed convex domain R, then it satisfies a Lipchitz condition in R.

We are interested in the region

$$1 \leq £ \leq R \tag{10}$$

We look for a bounded solution of the form

$$0 < R < \infty \tag{11}$$

We shall prove the following existence theorem:

Theorem 2: Let D^1 denote the region defined in (9) such that (10) and (11) hold. Then there exist a solution of model system (6) which is bounded in the region D^1.

Proof Let

$$f_1 = \Lambda + ¥v + ®s - «s$$

$$f_2 = ®s - \quad e - «e$$

$$f_3 = \quad e - \sim i - «i$$

$$f_4 = \sim i - ¥v - «v$$

It suffices to show that $\frac{\partial f_i}{dxj}$, $i, j = 1,2,3,4$ are continuous.

Consider the partial derivatives

$$\frac{\partial f_1}{ds} = -® - « , \left|\frac{\partial f_1}{\partial s}\right| = |-® - «| < \infty,$$

$$\frac{\partial f_1}{de} = 0 , \left|\frac{\partial f_1}{\partial e}\right| = |0| < \infty,$$

$$\frac{\partial f_1}{di} = 0 , \left|\frac{\partial f_1}{\partial i}\right| = |0| < \infty,$$

$$\frac{\partial f_1}{dv} = ¥ , \left|\frac{\partial f_1}{\partial v}\right| = |¥| < \infty,$$

Also,

$$\frac{\partial f_2}{ds} = \Lambda , \left|\frac{\partial f_2}{\partial s}\right| = |\Lambda| < \infty,$$

$$\frac{\partial f_2}{de} = - \quad - « , \left|\frac{\partial f_2}{\partial e}\right| = |- \quad - «| < \infty,$$

$$\frac{\partial f_2}{di} = 0 , \left|\frac{\partial f_2}{\partial i}\right| = |0| < \infty,$$

$$\frac{\partial f_2}{dv} = 0 , \left|\frac{\partial f_2}{\partial v}\right| = |0| < \infty,$$

$$\frac{\partial f_3}{ds} = 0 , \left|\frac{\partial f_3}{\partial s}\right| = |0| < \infty,$$

$$\frac{\partial f_3}{de} = \quad , \left|\frac{\partial f_3}{\partial e}\right| = \left| \quad \right| < \infty,$$

$$\frac{\partial f_3}{di} = -\sim - \text{«} , \left|\frac{\partial f_3}{\partial e}\right| = \left|-\sim - \text{«}\right| < \infty,$$

$$\frac{\partial f_3}{dv} = 0 , \left|\frac{\partial f_3}{\partial v}\right| = \left|0\right| < \infty,$$

Lastly,

$$\frac{\partial f_4}{ds} = 0 , \left|\frac{\partial f_4}{\partial s}\right| = \left|0\right| < \infty,$$

$$\frac{\partial f_4}{de} = 0 , \left|\frac{\partial f_3}{\partial e}\right| = \left|0\right| < \infty,$$

$$\frac{\partial f_4}{di} = \sim , \frac{\partial f_4}{di} = 0 ,$$

$$\frac{\partial f_4}{dv} = -\text{¥} - \text{«} , \frac{\partial f_4}{dv} = -\text{¥} - \text{«} .$$

Clearly, all these partial derivatives are continuous and bounded, hence, by theorem (2), there exist a unique solution of (6) in the D^1.

5.0 Conclusion

In this paper, we have considered the variable population SEIV epidemic model. We showed that there exists a positive invariant, where the system is biologically meaningful. Also, the existence and uniqueness of the model is established using Lipchitz's condition.

6.0 References

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