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A Note on Quantitative Properties of Solution to a System Of Differential Equations Arising From Magnetohydrodynamic Fluid Flow in a Saturated Porous Medium

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Abstract

This study examines the qualitative properties of solutions to a system of ordinary differential equations arising from MHD forced convective flow in a saturated porous medium in the presence of thermal radiation. It is assumed that the forced convection is driven by buoyancy force similar to that of inviscidnon-conducting fluid flow over a non-isothermal circular cylinder. The viscosity and thermal conductivity of the fluid are assumed to vary as a linear function of temperature, whereas the contribution of thermal radiative heat loss is based on Rossel and approximation. The basic equations governing the flow are in the form of partial differential equations and consequently reduce to a set of non-linear coupled ordinary differential equations by applying suitable similarity transformations. Theorems are stated and proofs are provided on the qualitative properties of the new system of equation governing the physical model. The approximate analytical solution to this physical flow model is intended to be examined in our next article..

Keywords: MHD flow, variable viscosity, saturated porous medium, stability,existence and uniqueness, eigen values.

1.0 Introduction

The problems of heat and mass transfer in the boundary layer flow have attracted considerable attention during the last few decades due to their numerous applications in several industrial processes. The experimental and theoretical studies of Magnetohydrodynamic (MHD) viscous flows are important from a technological point of view, because they have many applications, such as high-temperature plasma, cooling of nuclear reactors, liquid metal fluids, MHD accelerators, Magnetohydrodynamic electrical power generation, geophysics, recovery of petroleum resources, cooling of underground electric cables, environmental impact of buried heat generating waste, hot-wire etc. As a result, a significant amount of interest has been carried out to study the effects of electrically conducting fluids in the presence of a magnetic field on the flow and heat transfer aspects in various geometries as seen in Wooding [1]. Likewise, the boundary layer analogies for convection in a porous medium had been proposed and used in Wooding [1].Study of Magnetohydrodynamic (MHD) heat transfer field can be divided into two classes: in first class, the electromagnetic fields are used to control the heat transfer as in the convection flows and hydrodynamics heating; while in the second class, the heating is produced by electromagnetic fields as MHD generator, pumps etc. This study, therefore focuses on the first class. Magneto convection plays an important role in various industrial applications. From boundary layer along materials handling conveyors, extrusion of plastic sheets, the cooling of an infinite plastic plate in cooling bath and glass blowing to the continuous casting and spinning of fibers, flows due to stretched surfaces find enormous applications. El-Amin [2] reported that the fundamental concept behind MHD is that magnetic fields can induce currents in a moving conductive fluid, which in turn creates forces on the fluid particles and also change the magnetic field itself. That is, MHD phenomenon is characterized by a mutual interaction between the fluid velocity field (hydrodynamic boundary layer) and the electromagnetic field. Satisfying the faraday laws, the fluid motion affects the magnet field and the magnet field affects the fluid motion. Hence he concluded that MHD is a combination of the Navier-Stoke equations of fluid dynamics and Maxwell's equations of Electromagnetism. However, the single role of MHD in most of these practical applications is to induce flow-that is to bring about forced convection.

Forced convective heat transfer through fluid flow in saturated porous media has been discussed in detail and review in Nield et al. [3]. A closed form solution of the Brinkman-Forchheimer equation for the forced convection in a fluid saturated porous medium with isothermal and isoflux boundaries was obtained in Nield et al. [3] , valid for all values of the Darcy number.

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They found that when the Darcy number is large and simultaneously the Forchheimer number is small, the velocity profile is approximately parabolic, and the effect of an increase in the viscosity ratio $\left(\frac{\mu_e}{\mu}\right)$ is due to in decrease the Nusselt number. Here the viscosity ratio depends on the structure of the porous matrix. Many problems of MHD convection flow in porous media were treated in [4-6]. Aldoss et al. [7] studied MHD mixed convection from horizontal circular cylinder.Makinde and Onyejekw [8] analyzed the effects of variable viscosity and electrical conductivity on MHD generalized Couette flow and heat transfer using numerical technique. Pantokratoras [9] made a theoretical study to investigate the effect of variable viscosity on flow and heat transfer on a continuous moving plate.Adegbie [10] obtained exact analytical solutions for the flow of temperature-dependent viscous fluid between parallel heated walls in the presence of viscous dissipation. Ahmad et al. [11] obtained non-similar solutions mixed convection boundary layer flow past an isothermal horizontal circular cylinder with temperature-dependent viscosity. They found that the flow and heat transfer characteristics are significantly influenced by temperature dependent viscosity. Fand et al. [12] carried out an experimental and analytical investigation on the free convective heat transfer from a horizontal cylinder embedded in porous medium filled with randomly packed glass sphere and the medium is made saturated with water and or silicon oil. They observed that when the saturating fluid is silicon oil, the heat transfer rate is significant due to viscous dissipating effect.

The study of magnetohydrodynamic has important applications, and may be used to deal with problems such as cooling of nuclear reactors by liquid sodium and induction flow meter, which depends on the potential difference in the fluid in the direction perpendicular to the motion of fluid and to the magnetic field in Ganesan and Palani [13]. Thermal radiation is one of the fundamental mechanisms of heat transfer and its effect plays a significant role in controlling heat transfer process in polymer processing industry. The quality of final product depends to a certain extent on heat controlling factors. Thermal radiation occurring within these systems is usually the result of emission by the hot walls and working fluid. These effects become more important when the difference between the surface and the ambient temperature is large. At high operating temperature, radiation effect can be quite significant and the knowledge of radiation heat transfer is very important for the design on pertinent equipment in Seddeek [14].Thus, thermal radiation is one of the vital factors controlling the heat and mass transfer in Dulal and Hiranmoy [15].

Observing various researches into the dynamics of MHD flow, it is seen that there are varying focuses on terms like viscosity and thermal diffusivity, heat generation or absorption coefficient, magnetic induction, electrical conductivity, Eckert number, heat dissipative term and a host of others. In a study carried out in Okedoyeet al. [16], velocity and magnetic fields of MHD flows with buoyancy in the presence of heat generation were examined with a new perspective. In their paper, the magnetic parameter is not constant. However, viscosity and thermal diffusivity are constant. The combination of variable viscosity and thermal conductivity of the fluid with thermal radiation arerare in most studies [14-17].

This study extends previous work of El-Amin [2], who investigated combined effects of viscous dissipation and joule heating on MHD forced convection flow over a non- isothermal horizontal cylinder in a fluid saturated porous medium, to account for the effects of variable viscosity and thermal conductivity of the fluid in the presence of thermal radiation. The focus of this study is to examine qualitative properties of solutions of coupled non-linear differential equations arising from the new MHD flow model. In most of the studies, of this type of problems, the viscosity and thermal conductivity of the fluid were assumed to be constant. However, it is known that these physical properties can change significantly with temperature. When the effects of variable viscosity and thermal conductivity are taken in to account, the flow characteristics are significantly changed compared to constant property case. To accurately predict the flow behavior, it is necessary to take into account the variation of viscosity and thermal conductivity with temperature. Motivated by the above investigation, the present study is proposed to investigate the problem of MHD forced convective flow over a non- isothermal horizontal circular cylinder in a fluid saturated porous medium having variable viscosity $\mu(T)$ and thermal conductivity $\lambda(T)$, which are taking to be linear function of temperature in the presence of radiation effect. The surface temperature T_w of the circular cylinder is higher than that of the ambient fluid temperature T_∞ . In formulating the equations governing the flow the viscosity and thermal conductivity of the fluid were assumed to be proportional to a linear function of temperature, a semi-empirical formula for the viscosity $\mu(T)$as seen in Bachelor [18]. The governing partial differential equations are reduced to local non-similar partial differential forms by adopting appropriate transformation variables. The transformed boundary layer equations are reduced into systems of first order differential equation. Qualitative properties such as existence and uniqueness of solution to the problem, and the nature of stability are examined.

2.0 Mathematical Formulation

In this study, we consider a steady two-dimensional hydromagnetic flow of a viscous incompressible electrically conducting forced convection fluid flow over a horizontal impermeable circular cylinder in a saturated porous medium. A uniform magnetic field is acting normal to the cylinder surface. The magnetic Reynolds number is taken to be small enough such that the induced magnetic field along the flow is negligible. The x-axis is taken to be along the parallel surface of the plane measured from the lower stagnation point and the y-axis is normal to the surface. Under the above assumptions and invoking the Boussinesq and boundary layer approximations the boundary layer equations governing the flow and heat transfer of a viscous incompressible fluid can be written as follows;

$$\frac{\partial}{\partial}+\frac{\partial}{\partial}=0, \tag{2.01}$$

$$u\frac{\partial}{\partial}+v\frac{\partial}{\partial}=-\frac{1}{\mu_\infty}\frac{\partial}{\partial}\pm g\;\;(T-T_\infty)\sin\frac{x}{r}+\frac{1}{\mu_\infty}\frac{\partial}{\partial}\left(\mu(T)\frac{\partial}{\partial}\right)-\frac{\upsilon\beta_0^2 u}{\mu_\infty}-\frac{\kappa}{\mathrm{K}}-\frac{F\varepsilon^2 u^2}{K^{\frac{1}{2}}}, \tag{2.02}$$

$$u\frac{\partial}{\partial}+v\frac{\partial}{\partial}=\frac{1}{\mu_\infty}\frac{\partial}{\partial}\left(\lambda(T)\frac{\partial}{\partial}\right)+\frac{\upsilon\beta_0^2 u^2}{\mu_\infty C_p}+\frac{\mu(T)}{\mu_\infty C_p}\left(\frac{\partial}{\partial}\right)^2-\frac{1}{\mu_\infty C_p}\frac{\partial q_r}{\partial}\;. \tag{2.03}$$

Subject to the following boundary conditions:

$$\left.\begin{aligned} &y=0:\quad u=v,\quad T=T_w\\ &y\to\infty:\quad \frac{\partial}{\partial}=0.\quad T\to T_\omega \end{aligned}\right\} \tag{2.04}$$

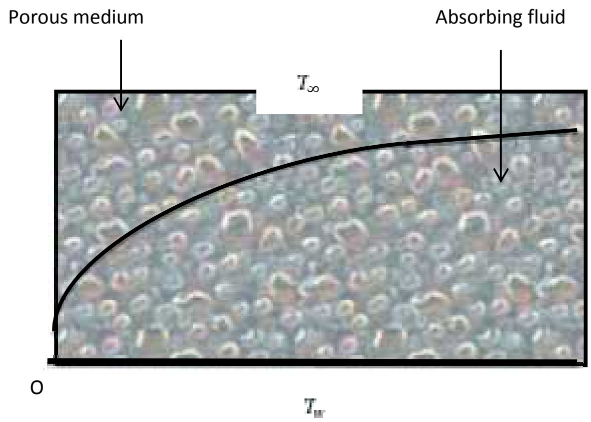

Figure 1: Physical configuration of the flow in a porous medium.

Where α, ρ_∞,λ and C_p are the thermal diffusivity, density, thermal conductivity and specific heat capacity at constant pressure of the fluid, respectively, k is the effective thermal conductivity of the saturated porous medium, w is the angular velocity, p is the pressure, T is the temperature, T_w is the surface temperature, T_∞ is the free stream temperature, B_0 is the radial magnetic field, $\beta$is the thermal expansion coefficient, υ is the kinematic viscosity of the fluid and g is the acceleration due to gravity. ε, F, q_r and K are the porosity, an empirical constant, radiative heat flux and permeability respectively. The last two terms in equation (2.02) represent the first- and second- order solid matrix resistance. The surface temperature is assumed to vary with the stream-wise direction, i.e.

$$T_w-T_\omega=b(1-\cos\omega)^m, \tag{2.05}$$

where b and m are constants, note that m = 0 represents the isothermal wall temperature case. To simplify the transformation of the given equations (2.01)-(2.04), the imposed pressure is assumed to be similar to that of the inviscid non conducting fluid flow about a circular cylinder, i.e.

$$-\frac{1}{\rho}\frac{\partial}{\partial}=\frac{4u_\infty^2}{r}\sin\omega\cos\omega. \tag{2.06}$$

Most of the effort in understanding fluid radiation is devoted to the derivation of reasonable simplification Aboeldahab [19]. One of this simplifications was made in Cogley[20] and assumed that the fluid is in the optically thin limit and, accordingly, the fluid does not absorb its own radiation but it only absorbs radiation emitted by the boundaries. For optically thick gas,the self-absorption rises and the situation become difficult. However, the problem can be simplified by using the Rosseland approximation (Rosseland 1936). Rosseland approximation requires that the media is optically dense media and radiation travels only a short distance before being scattered or absorbed. We are interested to study the radiation of heat within optically thick fluid before the heat is scattered, radiative heat transfer is taken into account, and Rosseland equation is used to account for the radiative thermal conductivity of the fluid. Rosseland equation is a simplified model of Radiative Transfer Equation (RTE). When material has a great extinction coefficient, it can be treated as optically thick. $q_r$is the radiative heat flux and is defined using the Rosseland approximation in Raptis[21], Sparrow and Cess [22] as

$$q_r=-\frac{4\sigma^*}{3K^*}\frac{\partial T^4}{\partial y}, \tag{2.07}$$

where $\sigma^*$is the Stefan-Boltzmann constant and $K^*$is known as the absorption coefficient. We assumed that the temperature differences within the flow are sufficiently small such that T^4 may be expressed as a linear function of the free stream temperature T_ω. This is obtained by expanding T^4 in a Taylor series about $T_\omega$and neglecting higher order terms. Considering the Taylor's series expansion of a function *f*(*x*) about x_0

$$f(x)=\;f(x0)+(x-x0)f'(x_0)+\frac{(x-x_0)^2}{2}f''(x_0)+\cdots\frac{(x-x_0)^n}{n!}f^n(x_0), \tag{2.08}$$

Likewise, expansion of T^4 about T_∞. Setting $f(x)=T^4$ and $f(x_0) = f(T_\infty) = T^4{}_\infty$ in above equation. Neglecting higher order, we obtained

$$T^4 = 4T_\infty^3 T - 3T_\infty^4. \qquad (2.09)$$

$$\frac{1}{\mu_\infty C_p}\frac{\partial q_r}{\partial} = -\frac{1}{3\mu_\infty C_p K^*}\frac{\sigma^* T_\infty^3}{}\frac{\partial^2 T}{\partial y^2} \qquad (2.10)$$

In this present problem, we consider a case where the viscosity and thermal conductivity of the fluid are linear functions of temperature. This assumption is valid since it is established that the physical properties of the fluid may change significantly with temperature. For lubricating fluids, heat generated by the internal friction and the corresponding rise in temperature affects the viscosity of the fluid and so the fluid viscosity can no longer be assumed constant. The increase of temperature leads to a local increase in the transport phenomena by reducing the viscosity across the momentum boundary layer and so the heat transfer rate at the wall is also affected greatly. In industrial systems, fluids can be subjected to extreme conditions such as high temperature, pressure, high shear rates and external heating (Ambient Temperature) and each of this factors can lead to high temperature being generated within the fluid. Hence, substituting the equations (2.05)-(2.10) into the governing equations (2.01)-(2.03), we obtain modified governing equations as follows:

$$\frac{\partial}{\partial} + \frac{\partial}{\partial} = 0, \qquad (2.11)$$

$$u\frac{\partial}{\partial} + v\frac{\partial}{\partial} = \frac{4u_\infty^2}{r}\sin\omega\cos\omega + \frac{1}{\mu_\infty}\mu(T)\frac{\partial^2 u}{\partial y^2} + \frac{1}{\mu_\infty}\frac{\partial\ (T)}{\partial}\frac{\partial}{\partial} \pm g\ \ (T - T_\infty)\sin w - \frac{\sigma\beta_0^2 u}{\mu_\infty} - \frac{\kappa}{\mathrm{K}} - \frac{F\varepsilon^2 u^2}{K^{\frac{1}{2}}}, \qquad (2.12)$$

$$u\frac{\partial}{\partial} + v\frac{\partial}{\partial} = \frac{1}{\mu_\infty}\lambda(T)\frac{\partial^2 T}{\partial y^2} + \frac{1}{\mu_\infty}\frac{\partial\ (T)}{\partial}\frac{\partial}{\partial} + \frac{\sigma\beta_0^2 u^2}{\mu_\infty C_p} + \frac{\mu(T)}{\mu_\infty C_p}\left(\frac{\partial}{\partial}\right)^2 + \frac{1}{3\mu_\infty C_p K^*}\frac{\sigma^* T_\infty^3}{}\frac{\partial^2 T}{\partial y^2} \qquad (2.13)$$

Subject to the following boundary conditions:

$$\left.\begin{array}{l} y = 0: \quad u = v, \quad T = T_w \\ y \to \infty: \quad \frac{\partial}{\partial} = 0, \quad T \to T_\infty \end{array}\right\} \qquad (2.14)$$

The following relations are introduce for *u* and *v* as $u = \frac{\partial}{\partial}$ and $v = -\frac{\partial}{\partial}$ respectively. Here $\psi(x, y)$ is the stream function. Introducing similarity variables

$$\xi(x) = 4[\sin(\tfrac{\omega}{2})]^2,\ \omega = \tfrac{x}{r},\ \eta(x,y) = y(\tfrac{2u_\infty}{\xi})^{1/2}\sin(\omega),\ \psi(x,y) = (2\vartheta\ \ u_\infty\xi)^{\frac{1}{2}}f(\xi,\eta), \qquad (2.15)$$

for the forced-convection-dominated regime. Dimensionless temperature, temperature dependent viscosity model in Bachelor [18] and temperature dependent thermal conductivity model in Salema and Fathy [23] are defined respectively as

$$\left.\begin{array}{l} \theta(x,y) = \frac{T - T_\infty}{T_w - T_\infty}, \\ \mu(T) = \mu_\infty\big[1 + \gamma[T_w - T]\big], \\ \lambda(T) = \lambda_\infty\big[1 + \alpha[T - T_\infty]\big]. \end{array}\right\} \qquad (2.16)$$

Where γ, α are constants and their values depend on the reference state and thermal properties of the fluid i.e. μ_∞ and λ_∞. In general, $\gamma > 0$ for liquids and $\gamma < 0$ for gases.The non-dimensional form of viscosity and thermal conductivity parameters L_γ and L_α can be written as,

$$\left.\begin{array}{l} L_\gamma = \gamma(T_w - T), \\ L_\alpha = \alpha(T_w - T) \end{array}\right\} \qquad (2.17)$$

Then by substituting the expressions in (2.15) and (2.16) into equations (2.11)-(2.13), the transformed momentum and energy equations take the form

$$\big(1 + L_\gamma[1-\theta]\big)f''' - L_\gamma f''\theta' + f''f + \left(\left(\frac{2\xi-4}{4-\xi}\right) - \left(\frac{2\xi}{\sqrt{(4\xi-\xi^2)}}k_2\right)\right)(f')^2 \pm \frac{N}{4-\xi}\theta + \frac{4-2\xi}{4-\xi} + \frac{2}{4-\xi}(M\ \ + k_1)f' =$$

$$2\xi\left(f'\frac{\partial f'}{\partial} - f''\frac{\partial}{\partial}\right), \qquad (2.18)$$

$$\left(\frac{1}{P}(1 + L_\alpha\theta) + \frac{4}{3}R\ \ ^*\right)\theta'' + \theta' f - 2nf'\theta + \frac{L_\alpha}{P}(\theta')^2 + 2\xi \qquad (f')^2 + E\quad (4-\xi)\big(1 + L_\gamma[1-\theta]\big)(f'')^2 =$$

$$2\xi\left(f'\frac{\partial}{\partial} - \theta'\frac{\partial}{\partial}\right). \qquad (2.19)$$

Subject to the following transformed boundary conditions;

$$f'(\xi,0) = 0, f(\xi,0) + 2\xi\frac{\partial\ \ (\xi,0)}{\partial} = 0, \theta(\xi,\infty) = 0, f''(\xi,\infty) = 0, \qquad \theta(\xi,0) = 1. \qquad (2.20)$$

In the above equations the primes denote the differentiation with respect to η.

Where $\vartheta = \frac{\mu_\infty}{\rho_\infty}$ is the reference kinematic viscosity, $G_r = \frac{g\beta(T_w - T_\infty)r^3\rho^2}{k^2}$ is the Grashof mass number, Re $= \frac{u_\infty r\rho}{\varepsilon}$ is the Reynold number, Ha $= \beta_u r\sqrt{\frac{\sigma}{k}}$ is the Hartmann number, Mn $= \frac{Ha^2}{R}$ is the Magnetic parameter, $k_1 = \frac{\varepsilon r\varepsilon}{Ku_\infty}$ is the first order solid matrix resistance, N $= \frac{G_r}{R^{\ 2}}$ is the Buoyancy parameter, $k_2 = \frac{\varepsilon(\rho_\infty)}{\mu_\infty}\left(\frac{F\ \varepsilon^2}{K^{\frac{1}{2}}}\right)$ is the second order solid matrix resistance parameter,

$\mathrm{Ec} = \frac{u_\infty^2}{B_1 c_p}$ is the Eckert number, $\mathrm{Ra} = \frac{4\sigma^* T_\infty^3}{\epsilon K^*}$ is the radiation parameter, $\mathrm{Pr} = \frac{\rho_\infty E}{\lambda_\infty}$ is the Prandtl number, $\mathrm{Ra}^* = \frac{\mathbb{R}}{c_p \rho_\infty}$ is the modified radiation parameter, $D_\alpha = \alpha B_1$ is the thermal conductivity parameter, $D_\gamma = \gamma B_1$ is the viscous parameter and $B_1 = (T_w - T_\omega)$ is the temperature difference between the wall and the flow stream temperature. The buoyancy parameter N measures the effects of buoyancy forces; for $N > 0$ implies that the surface temperature is larger than the free-stream temperature assisting flow. For $N < 0$ the opposite is true-opposing flow.
For practical applications, the major physical quantities of interest are the local skin friction coefficient and Nusselt number. The wall skin function and local Nusselt number defined are defined as follows:

$$C_f = \frac{4\sqrt{2s \quad 2 w f'' (\xi,0)}}{\sqrt{R}}, N \quad = \frac{q_w}{T_w - T_\infty}\left(\frac{r}{k}\right) = -\frac{(\sqrt{2}) s \quad (w)\, \theta'(\xi,0)}{\sqrt{(\xi/R \;)}}.$$

Where $q_w = -k \frac{d}{d} |_{y=0} = -k \left(B_1 \left(\frac{2u_\infty}{k}\right)^{\frac{1}{2}} \sin(\omega) \theta'(\xi,\eta) \right) = -k \quad {}_1 \left(\frac{2u_\infty}{k}\right)^{\frac{1}{2}} \sin(\omega) \theta'(\xi,0)$,

and $R \quad = \frac{u_\infty r}{k}$ is the local Reynolds number.
It is imperative to note that when $D_\alpha = D_\gamma = \mathrm{Ra}^* = 0$, in equations (2.18) and (2.19), we obtain the governing equations solved in El-Amin [2].

2.1 Transformation of the model equations to system of first order ODE

The transformed governing equations (2.18)-(2.19) are coupled non-linear differential equations, which are not amenable to exact analytical method of solutions. Therefore, a second level local non-similarity method is adopted to convert the non-similar equation (2.18)-(2.19) into system of ordinary differential equations. Consequently,equations (2.18)-(2.19) can now be written as:

$$(\alpha_1 + \alpha_2[\alpha_1 - \theta])f''' - \alpha_2 f'' \theta' + \alpha_1 f'' f + \alpha_3 (f')^2 + \alpha_4 + \alpha_5 \theta + \alpha_6 f' = \alpha_7 \left(f' \frac{\partial f'}{\partial} - f'' \frac{\partial}{\partial} \right), \qquad (2.21)$$

$$(\alpha_8[\alpha_1 + \alpha_9 \theta] + \alpha_1 \;)\theta'' + \alpha_1 \; (\theta')^2 + \alpha_1 \; (f')^2 + \alpha_1 \; \left[\alpha_1 + \alpha_2[1 - \theta]\right](f'')^2 + \alpha_1 \; f'\theta + \alpha_1 \theta' f = \alpha_7 \left(f' \frac{\partial}{\partial} - \theta' \frac{\partial}{\partial} \right), \qquad (2.22)$$

subject to

$$f'(\xi,0) = 0, f(\xi,0) + 2\xi \frac{\partial \;(\xi,0)}{\partial} = 0, \theta(\xi,\infty) = 0, \quad f''(\xi,\infty) = 0, \theta(\xi,0) = 1. \qquad (2.23)$$

Where

$$\left.\begin{array}{l} \alpha_1 = 1, \alpha_2 = D_\gamma, \alpha_3 = \left(\left(\frac{2\xi - 4}{4-\xi}\right) - \left(\frac{2\xi}{\sqrt{\xi(4-\xi)}} k_2 \right) \right), \alpha_4 = \frac{4-2\xi}{4-\xi}, \alpha_5 = \pm \frac{N}{4-\xi}, \\ \alpha_6 = -\frac{2}{4-\xi}[M \quad + k_1], \alpha_7 = 2\xi, \alpha_8 = \frac{1}{P}, \alpha_9 = D_\alpha, \alpha_1 \;\; = \frac{4}{3} R \;\; ^*, \alpha_1 \;\; = \frac{D_\alpha}{P}, \\ \alpha_1 \;\; = 2\xi \qquad , \alpha_1 \;\; = E \quad (4-\xi), \alpha_1 \;\; = -2n, \; \alpha_1 \;\; = \left(\left(\frac{4}{[4-\xi]^2}\right) - \left(\frac{4\xi}{(\xi(4-\xi))^{\frac{3}{2}}} k_2 \right) \right), \\ \alpha_2 \;\; = -\frac{4}{(4-\xi)^2}, \alpha_2 \;\; = \pm \frac{N}{(4-\xi)^2}, \alpha_2 \;\; = -\frac{2}{(4-\xi)^2}[M \quad + k_1], \alpha_2 \;\; = (4-2\xi)E \;\; , \\ \alpha_2 \;\; = 2E \qquad . \end{array}\right\} \qquad (2.24)$$

We desire to obtain an approximate transformation to the equations (2.21) and (2.22) with the boundary condition (2.23) based on the local non similarity method.

2.2 Local Non-similarity method

The local non-similarity method was developed in Sparrow and Yu [24] and has been applied by many investigators, for example inMinkowycz and Sheikh [25], to solve various non-similar boundary layer problems. This method embodies two essential feathers. First the non-similar solution at any specific stream wise location is found (i.e. each solution is locally autonomous). Second, the local solutions are found from differential equations. These equations can be solved numerically by well-established techniques, such as forward integration (e.g. a Runge-Kutta scheme) in conjunction with a shooting procedure to determine the unknown boundary conditions at the wall. The method also allows some degree of self-checking for accuracy of the numerical results.Before proceeding to the local non-similarity method, it is useful to examine equations (2.21) and (2.22) from the standpoint of local similarity. The local similarity solutions give first order estimates of the heat and mass transfer rates in boundary layers. Therefore, the local similarity solution is also known as the first level truncation as seen inDulal and Harmony [15]. By this concept, the right-hand sides of these equations are assumed to be small and thus can be neglected. The equations on the left sides are treated as ordinary differential equations and solved. Thus,

$$(\alpha_1 + \alpha_2[\alpha_1 - \theta])f''' - \alpha_2 f'' \theta' + \alpha_1 f'' f + \alpha_3 (f')^2 + \alpha_4 + \alpha_5 \theta + \alpha_6 f' = 0, \qquad (2.25)$$

$$(\alpha_8[\alpha_1 + \alpha_9 \theta] + \alpha_1 \;)\theta'' + \alpha_1 \; (\theta')^2 + \alpha_1 \; (f')^2 + \alpha_1 \; \left[\alpha_1 + \alpha_2[1 - \theta]\right](f'')^2 + \alpha_1 \; f'\theta + \alpha_1 \theta' f = 0. \qquad (2.26)$$

Turning to the rationale of the local similarity concept, the reduction of equations (2.21) and (2.22) is clearly justifiable for values of ξthat are very close to zero. On the other hand, when $\xi$is not small, local similarity is based on the postulate that derivatives involving $\xi$are very small El-Amin [2]. Uncertainty on whether to neglect the right-hand side of equation or not is the weakness of this local similarity concept. To correct the drawbacks of the local similarity method, Sparrow and Yu [24] presented a local non-similarity method in obtaining the solutions for the non-similar boundary layer equations. Accordingly, the terms on the right-hand sides of equations (2.21) and (2.22) are all retained. Auxiliary differential equations are introduced to approximate them. These auxiliary equations are obtained simply by differentiating equations (2.29) and (2.30) with respect to ξ and defining the new dependent variables G and ϕ as

$$G = \frac{d}{d}, \phi = \frac{d}{d}. \qquad (2.27)$$

These represent two additional unknown functions; therefore it is necessary to deduce two further equations to determine G and ϕ. This is accomplished by creating subsidiary equations by differentiating the transformed conservation equations and boundary conditions with respect toξ. The subsidiary equations for G and ϕ contain terms ϕ, G and their derivatives. That is,to close the system of equations at this second level, the terms involving$\frac{d}{d}, \frac{d}{d}$ or higher order are neglected. When these terms are ignored theequations (2.21) and (2.22) are reduce to a system of ordinary differential equations that provides locally autonomous solutions in the stream wise direction. This form of the local non-similarity method is referred to as the second level of truncation, because approximations are made by dropping terms in the second level equation.

The procedure as described above in the formulation of the local non-similarity method can result in a large number of ordinary differential equations that may require simultaneous solution. For example, at the second level of truncation there will be four equations. It is expected that the accuracy of the local non-similarity method results will depend upon the truncation level. Below is the simplification of the method. Let introduce two auxiliary variables $G = \frac{d}{d}$ and $\phi = \frac{d}{d}$ into the equation (2.21) and (2.22) we obtain:

$$(\alpha_1 + \alpha_2[\alpha_1 - \theta])f''' - \alpha_2 f''\theta' + \alpha_1 f''f + \alpha_3(f')^2 + \alpha_4 + \alpha_5\theta + \alpha_6 f' = \alpha_7(f'G' - f''G), \qquad (2.28)$$

$$(\alpha_8[\alpha_1 + \alpha_9\theta] + \alpha_1)\theta'' + \alpha_1(\theta')^2 + \alpha_1(f')^2 + \alpha_1\left[\alpha_1 + \alpha_2[1 - \theta]\right](f'')^2 + \alpha_1 f'\theta + \alpha_1\theta' f = \alpha_7(f'\phi - \theta' G), \qquad (2.29)$$

$$(\alpha_1 + \alpha_2[\alpha_1 - \theta])G''' - \alpha_2\phi f''' - \alpha_2 f''\phi' - \alpha_2\theta' G'' + \alpha_1 f'' G + \alpha_1 f G'' + \alpha_1 f'^2 + 2\alpha_3 f'G' + \alpha_2 + \alpha_5\phi + \alpha_2\theta + \alpha_6 G' + \alpha_2 f' = 2(f'G' - f''G) + \alpha_7\left(G'^2 - G'G''\right), \qquad (2.30)$$

$$(\alpha_1 + \alpha_1\theta + \alpha_1)\phi'' + \alpha_1\phi\theta'' + 2\alpha_1\theta'\phi' + 2\alpha_1 f'G' + \alpha_2 f'^2 + 2\alpha_1 f''G'' + \alpha_2 s_1 f''^2 - \alpha_2\alpha_2\theta f''^2 + 2\alpha_1 f''G'' + \alpha_2\alpha_2 f''^2 - 2\alpha_1\theta f''G'' - \alpha_1 f''^2\phi + \alpha_1 f'\phi + \alpha_1\theta G' + \alpha_1 f\phi' + \alpha_1\theta' G = 2(f'\phi - \theta' G) + \alpha_7(\phi G' - G\phi'). \qquad (2.31)$$

Subject to

$$f(0) + \alpha_1 G(0) = 0, f'(0) = 0, \; f''(\infty) = 0, \; G(0) = 0, G'(0) = 0, \quad G''(\infty) = 0, \phi(0) = 0, \phi(\infty) = 0, \; \theta(0) = 1, \theta(\infty) = 0. \qquad (2.32)$$

To reduce the equations (2.28)-(2.31) subject to (2.32) to system of first order ordinary differential equations, we let $f_1 = f, f_2 = f', f_3 = f'', f_4 = G, f_5 = G', f_6 = G'', f_7 = \theta, f_8 = \theta', f_9 = \varphi, f_1 = \varphi'$

Consequently, we get

$$f_1' = f' = f_2, \qquad (2.33)$$

$$f_2' = f'' = f_3, \qquad (2.34)$$

$$f_3' = f''' = \frac{1}{[\alpha_1 + \alpha_2(\alpha_1 - f_7)]}\{\alpha_7[(f_2 f_5) - (f_3 f_4)] + \alpha_2 f_3 f_8 - \alpha_1 f_3 f_1 - \alpha_3 f_2^2 - \alpha_4 - \alpha_5 f_7 - \alpha_6 f_2\} \qquad (2.35)$$

$$f_4' = G' = f_5, \qquad (2.36)$$

$$f_5' = G'' = f_6, \qquad (2.37)$$

$$f_6' = G''' = \frac{1}{\alpha_1 + \alpha_2(\alpha_1 - f_7)}[2(f_5 f_2 - f_4 f_3) + \alpha_7(f_5^2 - f_5 f_6) + \alpha_2 f_9[\frac{1}{[\alpha_1 + \alpha_2(\alpha_1 - f_7)]}\{\alpha_7[(f_2 f_5) - (f_3 f_4)] + \alpha_2 f_3 f_8 - \alpha_1 f_3 f_1 - \alpha_3 f_2^2 - \alpha_4 - \alpha_5 f_7 - \alpha_6 f_2\}] + \alpha_2 f_3 f_1 + \alpha_2 f_8 f_6 - \alpha_1 f_3 f_4 - \alpha_1 f_1 f_6 - \alpha_1 f_2^2 - 2\alpha_3 f_2 f_5 - \alpha_2 - \alpha_5 f_9 - \alpha_2 f_7 - \alpha_6 f_5 - \alpha_2 f_2, \qquad (2.38)$$

$$f_7' = \theta' = f_8, \qquad (2.39)$$

$$f_8' = \theta'' = \frac{1}{[\alpha_8(\alpha_1 + \alpha_9(f_7 - 1)) + \alpha_1]}[\alpha_7(f_2 f_9 - f_8 f_4) - \alpha_1 f_8^2 - \alpha_1 f_2^2 - \alpha_1\left(\alpha_1 + \alpha_2(1 - f_7)\right)f_3^2 - \alpha_1 f_2 f_7 - \alpha_1 f_8 f_1], \qquad (2.40)$$

$$f_9' = \varphi' = f_1, \qquad (2.41)$$

$$f_1' = \varphi'' = \frac{1}{\alpha_1 + \alpha_1 + \alpha_1(f_7 - 1)}[2[f_2 f_9 - f_8 f_4] + \alpha_7\{f_9 f_5 - f_4 f_1\} - \alpha_1 f_9\left[\frac{1}{\{\alpha_8(\alpha_1 + \alpha_9(f_7 - 1)) + \alpha_1\}}[\alpha_7(f_2 f_9 - f_8 f_4) - \alpha_1 f_8^2 - \alpha_1 f_2^2 - \alpha_1\left(\alpha_1 + \alpha_2(\alpha_1 - f_7)\right)f_3^2 - \alpha_1 f_2 f_7 - \alpha_1 f_8 f_1]\right] - 2\alpha_1 f_8 f_1 - 2\alpha_1 f_2 f_5 - \alpha_2 f_2^2 - 2\alpha_1 f_3 f_6 - \alpha_1\alpha_2 f_3^2 + \alpha_2\alpha_2 f_7 f_3^2 - 2\alpha_1 f_3 f_6 - \alpha_2\alpha_2 f_3^2 + 2\alpha_1 f_7 f_3 f_6 + \alpha_1 f_3^2 f_9 - \alpha_1 f_2 f_9 - \alpha_1 f_7 f_5 - \alpha_1 f_1 f_1 - \alpha_1 f_8 f_4]. \qquad (2.42)$$

Subject to the initial conditions:
$f_1(0) = 0, f_2(0) = 0, f_3(0) = \beta_1, f_4(0) = 0\ , f_5(0) = 0, f_6(0) = \beta_2, f_7(0) = 1$
$f_8(0) = \beta_3,\ f_9(0) = 0,\ f_1\ \ (0) = \beta_4$ (2.43)

3.0 Properties of Solution

In this section, we formulate some basic theorems relating to properties of solutions such as existence and uniqueness ofproblem (2.28)-(2.31)subject to (2.32) and establish the proof respectively.

3.1 Theorem on existence and uniqueness of solution

Let $\beta_i(i = 1, \dots 4) \geq 0$, $\xi < 4$and $0 \leq X_1 < \infty$, $0 \leq X_2 < a_1$, $0 \leq X_3 < a_2$, $b_1 \leq X_4 < a_3$, $0 \leq X_5 < a_4$, $0 \leq X_6 < a_5$, $b_2 \leq X_7 < a_6$, $0 \leq X_8 < a_7$, $b_3 \leq X_9 < a_8$, $0 \leq X_1\ \ < a_9$,
$b_4 \leq X_1\ \ < a_1$ where $a_i(i = 1,2, \dots ,10)$ a $b_j(j = 1,..,4)$ are positive constants. Then, there exists a unique solution of problem (2.28)-(2.31)subject to (2.32).

Proof:

Let $X_1 = \eta$, $X_2 = f$, $X_3 = f'$, $X_4 = f''$, $X_5 = G$,$X_6 = G'$, $X_7 = G''$,$X_8 = \theta$, $X_9 = \theta'$
$X_1\ \ = \varphi, X_1\ \ = \varphi'$

Then, we obtain that

$$\begin{pmatrix} X_1' \\ X_2' \\ X_3' \\ X_4' \\ X_5' \\ X_6' \\ X_7' \\ X_8' \\ X_9' \\ X_{10}' \\ X_{11}' \end{pmatrix} = \begin{pmatrix} 1 \\ X_3 \\ X_4 \\ A_1 \\ X_6 \\ X_7 \\ A_2 \\ X_9 \\ A_3 \\ X_{11} \\ A_4 \end{pmatrix} \tag{3.01}$$

Subject to initial condition

$$\begin{pmatrix} X_1(0) \\ X_2(0) \\ X_3(0) \\ X_4(0) \\ X_5(0) \\ X_6(0) \\ X_7(0) \\ X_8(0) \\ X_9(0) \\ X_{10}(0) \\ X_{11}(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ {}_1 \\ 0 \\ 0 \\ {}_2 \\ 1 \\ {}_3 \\ 0 \\ {}_4 \end{pmatrix} \tag{3.02}$$

where

$A_1 = \frac{(A_1\ \ +A_1\ \ +A_1\ \ )}{A_4}$,$A_1\ \ = a_7(X_3X_6 - X_4X_5)$, $A_1\ \ = a_2X_4X_9 - a_1X_4X_2$,
$A_1\ \ = -a_3X_3^2 - a_4 - a_5X_8 - a_6X_3$,$A_1\ \ = 2(X_6X_3 - X_4X_5)$, $A_1\ \ = a_7(X_6^2 - X_6X_7)$,
$A_1\ \ = a_2X_1\ \ A_1 + a_2X_4X_1$,$A_1\ \ = a_2X_9X_7 - a_1X_4X_5 - a_1X_2X_7$,
$A_1\ \ = -a_1\ \ X_3^2 - 2a_3X_3X_6 - a_2\ \ - a_5X_1$, $A_1\ \ = -a_2\ \ X_8 - a_6X_6 - a_2\ \ X_3$,
$A_2\ \ = a_7(X_3X_1\ \ - X_9X_5)$, $A_2\ \ = -a_1\ \ X_9^2 - a_1\ \ X_3^2$, $A_2\ \ = -a_1\ \ A_3X_4^2$,
$A_2\ \ = -\alpha_1\ \ X_3X_8 - \alpha_1X_9X_2$,$A_2\ \ = 2(X_3X_1\ \ - X_9X_5)$, $A_2\ \ = \alpha_7(X_1\ \ X_6 - X_5X_1\ \ )$,
$A_2\ \ = -a_1\ \ X_1\ \ A_3 - 2a_1\ \ X_9X_1$,$A_2\ \ = -2a_1\ \ X_3X_6 - a_2\ \ X_3^2 - 2a_1\ \ X_4X_7$,
$A_2\ \ = -\alpha_1\alpha_2\ \ X_4^2 + \alpha_2\alpha_2\ \ X_8X_4^2 - 2\alpha_1\ \ X_4X_7$,$A_2\ \ = -\alpha_2\alpha_2\ \ X_4^2 + 2\alpha_1\ \ X_4X_8X_7 + \alpha_1\ \ X_4^2X_1$,
$A_3\ \ = -\alpha_1\ \ X_3X_1\ \ - \alpha_1\ \ X_8X_6 - \alpha_1X_2X_1\ \ - \alpha_1X_9X_5$, $A_2 = \frac{(B_1+B_2)}{A_6}$, $A_3 = \frac{(A_2\ \ +A_2\ \ +A_2\ \ +A_2\ \ )}{A_5}$,
$A_4 = \frac{B_3+B_4}{A_5}$, $A_5 = \frac{1}{\alpha_1\ \ +\alpha_1\ \ +\alpha_1\ \ (X_2-1)}$, $A_6 = \frac{1}{\alpha_1+\alpha_2(\alpha_1-X_2)}$,

$B_1 = A_1 + A_1 + A_1$, $B_2 = A_1 + A_1 + A_1$,
$B_3 = A_2 + A_2 + A_2 + A_2$,$B_4 = A_2 + A_2 + A_3$.
Where
$\alpha_k (k = 1,2, \ldots ,23)$ are real constants defined earlier and $\beta_j (j = 1,2,3,4)$ are guess values that must satisfy the boundary conditions.
Let

$$\begin{pmatrix} X_1' \\ X_2' \\ X_3' \\ X_4' \\ X_5' \\ X_6' \\ X_7' \\ X_8' \\ X_9' \\ X_{10}' \\ X_{11}' \end{pmatrix} = \begin{pmatrix} f_1(X_1, X_2, X_3, ..., X_{11}) \\ f_2(X_1, X_2, X_3, ..., X_{11}) \\ f_3(X_1, X_2, X_3, ..., X_{11}) \\ f_4(X_1, X_2, X_3, ..., X_{11}) \\ f_5(X_1, X_2, X_3, ..., X_{11}) \\ f_6(X_1, X_2, X_3, ..., X_{11}) \\ f_7(X_1, X_2, X_3, ..., X_{11}) \\ f_8(X_1, X_2, X_3, ..., X_{11}) \\ f_9(X_1, X_2, X_3, ..., X_{11}) \\ f_{10}(X_1, X_2, X_3, ..., X_{11}) \\ f_{11}(X_1, X_2, X_3, ..., X_{11}) \end{pmatrix} \quad (3.03)$$

It is obvious that $f_i(X_1, X_2, X_3, \ldots, X_{11})$ for each $i = 1,2,3 \ldots ,11$ is Lipchitzcontinuous. Thus, we show that $\frac{\partial f_i}{\partial x_j} \forall\, i, j = 1,2,3 \ldots ,11$ are bounded and there exist K such that $\left|\frac{\partial f_i}{\partial x_j}\right| \leq K$ where $0 \leq K < \infty$. Hence, differentiating $f_i(X_1, X_2, X_3, \ldots, X_{11})$ for each $i = 1,2,3,\ldots.11$ with respect to $X_1, X_2, X_3, \ldots, X_{11}$, we have

$$\left|\frac{\partial f_1}{\partial X_1}\right| = \left|\frac{\partial f_1}{\partial X_2}\right| = \left|\frac{\partial f_1}{\partial X_3}\right| = \left|\frac{\partial f_1}{\partial X_4}\right| = \left|\frac{\partial f_1}{\partial X_5}\right| = \left|\frac{\partial f_1}{\partial X_6}\right| = \left|\frac{\partial f_1}{\partial X_7}\right| = \left|\frac{\partial f_1}{\partial X_8}\right| = \left|\frac{\partial f_1}{\partial X_9}\right| = 0 ,$$
$$\left|\frac{\partial f_1}{\partial X_{10}}\right| = \left|\frac{\partial f_1}{\partial X_{11}}\right| = 0 . \quad (3.04)$$

$$\left|\frac{\partial f_2}{\partial X_1}\right| = \left|\frac{\partial f_2}{\partial X_2}\right| = \left|\frac{\partial f_2}{\partial X_4}\right| = \left|\frac{\partial f_2}{\partial X_5}\right| = \left|\frac{\partial f_2}{\partial X_6}\right| = \left|\frac{\partial f_2}{\partial X_7}\right| = \left|\frac{\partial f_2}{\partial X_8}\right| = \left|\frac{\partial f_2}{\partial X_9}\right| = \left|\frac{\partial f_2}{\partial X_{10}}\right| = 0 ,$$
$$\left|\frac{\partial f_2}{\partial X_{11}}\right| = 0, \left|\frac{\partial f_2}{\partial X_3}\right| = 1. \quad (3.05)$$

$$\left|\frac{\partial f_3}{\partial X_1}\right| = \left|\frac{\partial f_3}{\partial X_2}\right| = \left|\frac{\partial f_3}{\partial X_3}\right| = \left|\frac{\partial f_3}{\partial X_5}\right| = \left|\frac{\partial f_3}{\partial X_6}\right| = \left|\frac{\partial f_3}{\partial X_7}\right| = \left|\frac{\partial f_3}{\partial X_8}\right| = \left|\frac{\partial f_3}{\partial X_9}\right| = \left|\frac{\partial f_3}{\partial X_{10}}\right| = 0 ,$$
$$\left|\frac{\partial f_3}{\partial X_{11}}\right| = 0, \left|\frac{\partial f_3}{\partial X_4}\right| = 1. \quad (3.06)$$

$$\left|\frac{\partial f_4}{\partial X_1}\right| = 0, \quad \left|\frac{\partial f_4}{\partial X_2}\right| = \left|\frac{-\alpha_1 X_4}{\alpha_1 + \alpha_2(\alpha_1 - X_8)}\right| \leq T_1, \left|\frac{\partial f_4}{\partial X_3}\right| = \left|\frac{\alpha_7 X_6 - 2\alpha_3 X_3 - \alpha_6}{\alpha_1 + \alpha_2(\alpha_1 - X_8)}\right| \leq T_2,$$
$$\left|\frac{\partial f_4}{\partial X_4}\right| = \left|\frac{-\alpha_7 X_5 + \alpha_2 X_9 - \alpha_1 X_2}{\alpha_1 + \alpha_2(\alpha_1 - X_8)}\right| \leq T_3, \left|\frac{\partial f_4}{\partial X_5}\right| = \left|\frac{-\alpha_7 X_4}{\alpha_1 + \alpha_2(\alpha_1 - X_8)}\right| \leq T_4, \left|\frac{\partial f_4}{\partial X_6}\right| = \left|\frac{-\alpha_7 X_3}{\alpha_1 + \alpha_2(\alpha_1 - X_8)}\right| \leq T_5 ,$$
$$\left|\frac{\partial f_4}{\partial X_7}\right| = 0, \left|\frac{\partial f_4}{\partial X_8}\right| = \left|\frac{-\alpha_1 + \alpha_2(\alpha_1 - X_8) + \alpha_2(\alpha_7(X_6 X_3 - X_4 X_5) + \alpha_2 X_4 X_9 - \alpha_1 X_4 X_2 - \alpha_3 X_3^2 - \alpha_4 - \alpha_5 X_8 - \alpha_6 X_3)}{\alpha_1 + \alpha_2(\alpha_1 - X_8)}\right|$$
$$\leq T_6 ,$$
$$\left|\frac{\partial f_4}{\partial X_9}\right| = \left|\frac{\alpha_2 X_4}{\alpha_1 + \alpha_2(\alpha_1 - X_8)}\right| \leq T_6', \left|\frac{\partial f_4}{\partial X_{10}}\right| = \left|\frac{\partial f_4}{\partial X_{11}}\right| = 0. \quad (3.07)$$

$$\left|\frac{\partial f_5}{\partial X_1}\right| = \left|\frac{\partial f_5}{\partial X_2}\right| = \left|\frac{\partial f_5}{\partial X_3}\right| = \left|\frac{\partial f_5}{\partial X_4}\right| = \left|\frac{\partial f_5}{\partial X_5}\right| = \left|\frac{\partial f_5}{\partial X_7}\right| = \left|\frac{\partial f_5}{\partial X_8}\right| = \left|\frac{\partial f_5}{\partial X_9}\right| = \left|\frac{\partial f_5}{\partial X_{10}}\right| = 0,$$
$$\left|\frac{\partial f_5}{\partial X_{11}}\right| = 0, \left|\frac{\partial f_5}{\partial X_6}\right| = 1. \quad (3.08)$$

$$\left|\frac{\partial f_6}{\partial X_1}\right| = \left|\frac{\partial f_6}{\partial X_2}\right| = \left|\frac{\partial f_6}{\partial X_3}\right| = \left|\frac{\partial f_6}{\partial X_4}\right| = \left|\frac{\partial f_6}{\partial X_5}\right| = \left|\frac{\partial f_6}{\partial X_6}\right| = \left|\frac{\partial f_6}{\partial X_8}\right| = \left|\frac{\partial f_6}{\partial X_9}\right| = \left|\frac{\partial f_6}{\partial X_{10}}\right| = 0,$$
$$\left|\frac{\partial f_6}{\partial X_{11}}\right| = 0, \left|\frac{\partial f_6}{\partial X_7}\right| = 1. \quad (3.09)$$

$$\left|\frac{\partial f_7}{\partial X_1}\right| = 0 , \left|\frac{\partial f_7}{\partial X_2}\right| = \left|\frac{-\alpha_2 \alpha_1 X_{11} X_4}{\left(\alpha_1 + \alpha_2(\alpha_1 - X_8)\right)^2} - \frac{\alpha_1 X_7}{\alpha_1 + \alpha_2(\alpha_1 - X_8)}\right| \leq T_7,$$

$$\left|\frac{\partial f_7}{\partial X_3}\right| = \left|\frac{2X_6}{\alpha_1 + \alpha_2(\alpha_1 - X_8)} + \frac{\alpha_2 \alpha_7 X_{11} X_6}{\left(\alpha_1 + \alpha_2(\alpha_1 - X_8)\right)^2} - \frac{2\alpha_3 \alpha_2 X_{11} X_3}{\left(\alpha_1 + \alpha_2(\alpha_1 - X_8)\right)^2} - \frac{\alpha_6 \alpha_2 X_{11}}{\left(\alpha_1 + \alpha_2(\alpha_1 - X_8)\right)^2} - \frac{2\alpha_{11} X_3}{\alpha_1 + \alpha_2(\alpha_1 - X_8)} \right.$$
$$\left. - \frac{2\alpha_3 X_6}{\alpha_1 + \alpha_2(\alpha_1 - X_8)} - \frac{\alpha_2}{\alpha_1 + \alpha_2(\alpha_1 - X_8)}\right| \leq T_8,$$

$$\left|\frac{\partial f_7}{\partial X_4}\right| = \left|\frac{-2X_5}{\alpha_1+\alpha_2(\alpha_1-X_8)} - \frac{\alpha_2\alpha_7X_1\ X_5}{\left(\alpha_1+\alpha_2(\alpha_1-X_8)\right)^2} + \frac{\alpha_2^2X_9X_1}{\left(\alpha_1+\alpha_2(\alpha_1-X_8)\right)^2} - \frac{\alpha_2\alpha_1X_1\ X_2}{\left(\alpha_1+\alpha_2(\alpha_1-X_8)\right)^2} + \frac{\alpha_2X_1}{\alpha_1+\alpha_2(\alpha_1-X_8)} - \frac{\alpha_1X_5}{\alpha_1+\alpha_2(\alpha_1-X_8)}\right|$$

$$\le \left|\frac{-2X_5}{\alpha_1+\alpha_2(\alpha_1-X_8)}\right| + \left|\frac{\alpha_2\alpha_7X_1\ X_5}{\left(\alpha_1+\alpha_2(\alpha_1-X_8)\right)^2}\right| + \left|\frac{\alpha_2^2X_9X_1}{\left(\alpha_1+\alpha_2(\alpha_1-X_8)\right)^2}\right| + \left|\frac{-\alpha_2\alpha_1X_1\ X_2}{\left(\alpha_1+\alpha_2(\alpha_1-X_8)\right)^2}\right| + \left|\frac{\alpha_2X_1}{\alpha_1+\alpha_2(\alpha_1-X_8)}\right| + \left|\frac{\alpha_1X_5}{\alpha_1+\alpha_2(\alpha_1-X_8)}\right| \le T_5,$$

$$\left|\frac{\partial f_7}{\partial X_5}\right| = \left|\frac{-2X_4}{\alpha_1+\alpha_2(\alpha_1-X_8)} - \frac{\alpha_2X_1\ \alpha_7X_4}{\left(\alpha_1+\alpha_2(\alpha_1-X_8)\right)^2} - \frac{\alpha_1X_4}{\alpha_1+\alpha_2(\alpha_1-X_8)}\right| \le T_1\ ,$$

$$\left|\frac{\partial f_7}{\partial X_6}\right| = \left|\frac{2X_3}{\alpha_1+\alpha_2(\alpha_1-X_8)} + \frac{2X_6\alpha_7}{\alpha_1+\alpha_2(\alpha_1-X_8)} - \frac{\alpha_7X_7}{\alpha_1+\alpha_2(\alpha_1-X_8)} + \frac{\alpha_2\alpha_7X_3X_1}{\left(\alpha_1+\alpha_2(\alpha_1-X_8)\right)^2} - \frac{2\alpha_3X_3}{\alpha_1+\alpha_2(\alpha_1-X_8)} - \frac{\alpha_6}{\alpha_1+\alpha_2(\alpha_1-X_8)}\right| \le T_1\ ,$$

$$\left|\frac{\partial f_7}{\partial X_7}\right| = \left|\frac{-\alpha_7X_6}{\alpha_1+\alpha_2(\alpha_1-X_8)} + \frac{\alpha_2X_9}{\alpha_1+\alpha_2(\alpha_1-X_8)} - \frac{\alpha_1X_2}{\alpha_1+\alpha_2(\alpha_1-X_8)}\right| \le T_1\ ,$$

$$\left|\frac{\partial f_7}{\partial X_8}\right| = \left|\frac{\alpha_2X_1\ X_4'}{\alpha_1+\alpha_2(\alpha_1-X_8)} - \frac{\alpha_2}{\alpha_1+\alpha_2(\alpha_1-X_8)} + \frac{\alpha_2X_2}{\left(\alpha_1+\alpha_2(\alpha_1-X_8)\right)^2}\right| \le T_1\ ,$$

$$\left|\frac{\partial f_7}{\partial X_9}\right| = \left|\frac{\alpha_2^2X_4X_1}{\left(\alpha_1+\alpha_2(\alpha_1-X_8)\right)^2} + \alpha_2X_7\right| \le T_1\ ,$$

$$\left|\frac{\partial f_7}{\partial X_1}\right| = \left|\frac{\alpha_2\alpha_7(X_3X_6-X_4X_5)+\alpha_2X_4X_9-\alpha_1X_4X_2-\alpha_3X_3^2-\alpha_4-\alpha_5X_8-\alpha_6X_3}{\left(\alpha_1+\alpha_2(\alpha_1-X_8)\right)^2} - \frac{\alpha_5}{\alpha_1+\alpha_2(\alpha_1-X_8)}\right|$$

$$\le \left|\frac{\alpha_2\alpha_7(X_3X_6-X_4X_5)+\alpha_2X_4X_9-\alpha_1X_4X_2-\alpha_3X_3^2-\alpha_4-\alpha_5X_8-\alpha_6X_3}{\left(\alpha_1+\alpha_2(\alpha_1-X_8)\right)^2}\right| + \left|\frac{-\alpha_5}{\alpha_1+\alpha_2(\alpha_1-X_8)}\right|$$

$\le T_1\ ,$

$$\left|\frac{\partial f_7}{\partial X_1}\right| = \left|\frac{\alpha_2X_4}{\alpha_1+\alpha_2(\alpha_1-X_8)}\right| \le T_1 \qquad (3.10)$$

$$\left|\frac{\partial f_8}{\partial X_1}\right| = \left|\frac{\partial f_8}{\partial X_2}\right| = \left|\frac{\partial f_8}{\partial X_3}\right| = \left|\frac{\partial f_8}{\partial X_4}\right| = \left|\frac{\partial f_8}{\partial X_5}\right| = \left|\frac{\partial f_8}{\partial X_6}\right| = \left|\frac{\partial f_8}{\partial X_7}\right| = \left|\frac{\partial f_8}{\partial X_8}\right| = \left|\frac{\partial f_8}{\partial X_1}\right| = \left|\frac{\partial f_8}{\partial X_1}\right| = \left|\frac{\partial f_8}{\partial X_1}\right| = 0\ ,$$
$$\left|\frac{\partial f_8}{\partial X_9}\right| = 1. \qquad (3.11)$$

$$\left|\frac{\partial f_9}{\partial X_1}\right| = 0\ , \left|\frac{\partial f_9}{\partial X_2}\right| = \left|\frac{-\alpha_1X_9}{\alpha_1+\alpha_1+\alpha_1\ (X_8-1)}\right| \le T_1$$

$$\left|\frac{\partial f_9}{\partial X_3}\right| = \left|\frac{\alpha_7X_1}{\alpha_1+\alpha_1+\alpha_1\ (X_8-1)} - \frac{2\alpha_1\ X_3}{\alpha_1+\alpha_1+\alpha_1\ (X_8-1)} - \frac{\alpha_1\ X_8}{\alpha_1+\alpha_1+\alpha_1\ (X_8-1)}\right|$$

$$\le \left|\frac{\alpha_7X_1}{\alpha_1+\alpha_1+\alpha_1\ (X_8-1)}\right| + \left|\frac{-2\alpha_1\ X_3}{\alpha_1+\alpha_1+\alpha_1\ (X_8-1)}\right| + \left|\frac{-\alpha_1\ X_8}{\alpha_1+\alpha_1+\alpha_1\ (X_8-1)}\right|$$

$\le T_1$

$$\left|\frac{\partial f_9}{\partial X_4}\right| = \left|\frac{-2\ \alpha_1\ X_4(\alpha_1+\alpha_2(\alpha_1-X_8))}{\alpha_1+\alpha_1+\alpha_1\ (X_8-1)}\right| \le \frac{|-2\ \alpha_1\ X_4(\alpha_1+\alpha_2(\alpha_1-X_8))|}{|\alpha_1+\alpha_1+\alpha_1\ (X_8-1)|} \le T_1$$

$$\left|\frac{\partial f_9}{\partial X_5}\right| = \left|\frac{-\alpha_7X_5}{\alpha_1+\alpha_1+\alpha_1\ (X_8-1)}\right| \le T_2\ , \left|\frac{\partial f_9}{\partial X_6}\right| = \left|\frac{\partial f_9}{\partial X_7}\right| = 0\ ,$$

$$\left|\frac{\partial f_5}{\partial X_6}\right| = \left|\frac{\alpha_1\ \alpha_2 X_4^2}{\alpha_1\ +\alpha_1\ +\alpha_1\ (X_6-1)} - \frac{\alpha_1\ X_3}{\alpha_1\ +\alpha_1\ +\alpha_1\ (X_6-1)} - \frac{\alpha_1\ \alpha_7 X_3 X_1}{(\alpha_1\ +\alpha_1\ +\alpha_1\ (X_6-1))^2} + \frac{\alpha_1\ \alpha_1\ X_3^2}{(\alpha_1\ +\alpha_1\ +\alpha_1\ (X_6-1))^2}\right.$$
$$+\frac{\alpha_1\ \alpha_1\ X_3^2}{(\alpha_1\ +\alpha_1\ +\alpha_1\ (X_6-1))^2} + \frac{\alpha_1\ \alpha_1\ \left(\alpha_1+\alpha_2(1-X_6)\right)X_4^2}{(\alpha_1\ +\alpha_1\ +\alpha_1\ (X_6-1))^2} + \frac{\alpha_1\ \alpha_1\ X_3 X_6}{(\alpha_1\ +\alpha_1\ +\alpha_1\ (X_6-1))^2}$$
$$\left.+\frac{\alpha_1\ \alpha_7 X_3 X_5}{(\alpha_1\ +\alpha_1\ +\alpha_1\ (X_6-1))^2} + \frac{\alpha_1 X_3 X_2}{(\alpha_1\ +\alpha_1\ +\alpha_1\ (X_6-1))^2}\right| \le T_2\ ,$$

$$\left|\frac{\partial f_9}{\partial X_9}\right| = \left|\frac{-\alpha_7 X_5}{(\alpha_1\ +\alpha_1\ +\alpha_1\ (X_8-1)} - \frac{2\alpha_1\ X_9}{(\alpha_1\ +\alpha_1\ +\alpha_1\ (X_8-1)} - \frac{\alpha_1 X_2}{(\alpha_1\ +\alpha_1\ +\alpha_1\ (X_8-1)}\right|$$
$$\le T_2\ ,$$
$$\left|\frac{\partial f_9}{\partial X_1}\right| = \left|\frac{s_7 X_3}{(\alpha_1\ +\alpha_1\ +\alpha_1\ (X_8-1)}\right| \le \frac{|s_7 X_3|}{|(\alpha_1\ +\alpha_1\ +\alpha_1\ (X_8-1)|} \le T_2\ ,$$
$$\left|\frac{\partial f_5}{\partial x_1}\right| = 0. \qquad (3.12)$$
$$\left|\frac{\partial f_1}{\partial X_1}\right| = \left|\frac{\partial f_1}{\partial X_2}\right| = \left|\frac{\partial f_1}{\partial X_3}\right| = \left|\frac{\partial f_1}{\partial X_4}\right| = \left|\frac{\partial f_1}{\partial X_5}\right| = \left|\frac{\partial f_1}{\partial X_6}\right| = \left|\frac{\partial f_1}{\partial X_7}\right| = \left|\frac{\partial f_1}{\partial X_8}\right| = \left|\frac{\partial f_1}{\partial X_9}\right| = 0$$
$$\left|\frac{\partial f_1}{\partial x_1}\right| = 0\ ,\ \left|\frac{\partial f_1}{\partial x_1}\right| = 1\ . \qquad (3.13)$$
$$\left|\frac{\partial f_1}{\partial x_1}\right| = 0,$$
$$\left|\frac{\partial f_1}{\partial X_2}\right| = \left|\frac{\alpha_1\ \alpha_1 X_1\ X_9}{(\alpha_1\ +\alpha_1\ +\alpha_1\ (X_8-1))^2} - \frac{\alpha_1 X_1}{(\alpha_1\ +\alpha_1\ +\alpha_1\ (X_8-1)}\right|$$
$$\le \left|\frac{\alpha_1\ \alpha_1 X_1\ X_9}{(\alpha_1\ +\alpha_1\ +\alpha_1\ (X_8-1))^2}\right| + \left|\frac{-\alpha_1 X_1}{(\alpha_1\ +\alpha_1\ +\alpha_1\ (X_8-1)}\right| \le T_2\ ,$$

$$\left|\frac{\partial f_1}{\partial X_3}\right| = \left|\frac{2X_1}{(\alpha_1\ +\alpha_1\ +\alpha_1\ (X_8-1)} - \frac{\alpha_1\ \alpha_7 X_1^2\ +2\alpha_1\ \alpha_1\ X_3 X_1\ +\alpha_1\ \alpha_1\ X_1\ X_8}{(\alpha_1\ +\alpha_1\ +\alpha_1\ (X_8-1))^2} - \frac{2X_6\alpha_1}{(\alpha_1\ +\alpha_1\ +\alpha_1\ (X_8-1)}\right.$$
$$\left.-\frac{2\alpha_2\ X_3}{(\alpha_1\ +\alpha_1\ +\alpha_1\ (X_8-1)} - \frac{\alpha_1\ X_1}{(\alpha_1\ +\alpha_1\ +\alpha_1\ (X_8-1)}\right|$$
$$\le \left|\frac{2X_1}{(\alpha_1\ +\alpha_1\ +\alpha_1\ (X_8-1)}\right| + \left|\frac{\alpha_1\ \alpha_7 X_1^2}{(\alpha_1\ +\alpha_1\ +\alpha_1\ (X_8-1))^2}\right| + \left|\frac{2\alpha_1\ \alpha_1\ X_3 X_1}{(\alpha_1\ +\alpha_1\ +\alpha_1\ (X_8-1))^2}\right|$$
$$+\left|\frac{\alpha_1\ \alpha_1\ X_1\ X_8}{(\alpha_1\ +\alpha_1\ +\alpha_1\ (X_8-1))^2}\right| + \left|\frac{-2X_6\alpha_1}{(\alpha_1\ +\alpha_1\ +\alpha_1\ (X_8-1)}\right| + \left|\frac{-2\alpha_2\ X_3}{(\alpha_1\ +\alpha_1\ +\alpha_1\ (X_8-1)}\right|$$
$$+\left|\frac{-\alpha_1\ X_1}{(\alpha_1\ +\alpha_1\ +\alpha_1\ (X_8-1)}\right| \le T_2\ ,$$

$$\left|\frac{\partial f_1}{\partial X_4}\right| = \left|\frac{2\alpha_1\ \alpha_1\ X_4 X_1\ (\alpha_1+\alpha_2(1-X_8))}{(\alpha_1\ +\alpha_1\ +\alpha_1\ (X_8-1))^2} - \frac{2\alpha_1\ X_7}{(\alpha_1\ +\alpha_1\ +\alpha_1\ (X_8-1)} - \frac{2\alpha_1\alpha_2\ X_4}{(\alpha_1\ +\alpha_1\ +\alpha_1\ (X_8-1)}\right.$$
$$+\frac{2\alpha_2\alpha_2\ X_8 X_4}{(\alpha_1\ +\alpha_1\ +\alpha_1\ (X_8-1)} - \frac{2\alpha_1\ X_7}{(\alpha_1\ +\alpha_1\ +\alpha_1\ (X_8-1)} - \frac{2\alpha_2\alpha_2\ X_4}{(\alpha_1\ +\alpha_1\ +\alpha_1\ (X_8-1)}$$
$$\left.+\frac{2\alpha_1\ X_7 X_8}{(\alpha_1\ +\alpha_1\ +\alpha_1\ (X_8-1)} + \frac{2\alpha_1\ X_4 X_1}{(\alpha_1\ +\alpha_1\ +\alpha_1\ (X_8-1)}\right|$$
$$\le \left|\frac{2\alpha_1\ \alpha_1\ X_4 X_1\ \left(\alpha_1+\alpha_2(1-X_8)\right)}{(\alpha_1\ +\alpha_1\ +\alpha_1\ (X_8-1))^2}\right| + \left|\frac{-2\alpha_1\ X_7}{(\alpha_1\ +\alpha_1\ +\alpha_1\ (X_8-1)}\right| + \left|\frac{-2\alpha_1\alpha_2\ X_4}{(\alpha_1\ +\alpha_1\ +\alpha_1\ (X_8-1)}\right|$$
$$+\left|\frac{-2\alpha_2\alpha_2\ X_8 X_4}{(\alpha_1\ +\alpha_1\ +\alpha_1\ (X_8-1)}\right| + \left|\frac{-2\alpha_1\ X_7}{(\alpha_1\ +\alpha_1\ +\alpha_1\ (X_8-1)}\right| + \left|\frac{-2\alpha_2\alpha_2\ X_4}{(\alpha_1\ +\alpha_1\ +\alpha_1\ (X_8-1)}\right|$$
$$+\left|\frac{2\alpha_1\ X_7 X_8}{(\alpha_1\ +\alpha_1\ +\alpha_1\ (X_8-1)}\right| + \left|\frac{2\alpha_1\ X_4 X_1}{(\alpha_1\ +\alpha_1\ +\alpha_1\ (X_8-1)}\right| \le T_2\ ,$$

$$\left|\frac{\partial f_1}{\partial X_5}\right| = \left|\frac{-2X_9}{(\alpha_1\ +\alpha_1\ +\alpha_1\ (X_8-1)} - \frac{\alpha_7 X_1}{(\alpha_1\ +\alpha_1\ +\alpha_1\ (X_8-1)} + \frac{\alpha_1\ X_9 X_1\ \alpha_7}{(\alpha_1\ +\alpha_1\ +\alpha_1\ (X_8-1))^2}\right.$$
$$\left.-\frac{\alpha_1 X_9}{(\alpha_1\ +\alpha_1\ +\alpha_1\ (X_8-1)}\right|$$

$$\leq \left|\frac{-2X_5}{(\alpha_1+\alpha_1+\alpha_1(X_8-1)}\right| + \left|\frac{-\alpha_7 X_1}{(\alpha_1+\alpha_1+\alpha_1(X_8-1)}\right| + \left|\frac{\alpha_1 X_5 X_1 \alpha_7}{(\alpha_1+\alpha_1+\alpha_1(X_8-1))^2}\right| + \left|\frac{-\alpha_1 X_5}{(\alpha_1+\alpha_1+\alpha_1(X_8-1)}\right| \leq T_2,$$

$$\left|\frac{\partial f_1}{\partial X_6}\right| = \left|\frac{\alpha_7 X_1}{(\alpha_1+\alpha_1+\alpha_1(X_8-1)} - \frac{2\alpha_1 X_3}{(\alpha_1+\alpha_1+\alpha_1(X_8-1)} - \frac{\alpha_1 X_8}{(\alpha_1+\alpha_1+\alpha_1(X_8-1)}\right|$$

$$\leq \left|\frac{\alpha_7 X_1}{(\alpha_1+\alpha_1+\alpha_1(X_8-1)}\right| + \left|\frac{-2\alpha_1 X_3}{(\alpha_1+\alpha_1+\alpha_1(X_8-1)}\right| + \left|\frac{-\alpha_1 X_8}{(\alpha_1+\alpha_1+\alpha_1(X_8-1)}\right|$$

$$\leq T_2,$$

$$\left|\frac{\partial f_1}{\partial X_7}\right| = \left|\frac{-\alpha_1 X_4}{(\alpha_1+\alpha_1+\alpha_1(X_8-1)} - \frac{2\alpha_1 X_4}{(\alpha_1+\alpha_1+\alpha_1(X_8-1)} + \frac{2\alpha_1 X_8 X_4}{(\alpha_1+\alpha_1+\alpha_1(X_8-1)}\right|$$

$$\leq \left|\frac{-\alpha_1 X_4}{(\alpha_1+\alpha_1+\alpha_1(X_8-1)}\right| + \left|\frac{-2\alpha_1 X_4}{(\alpha_1+\alpha_1+\alpha_1(X_8-1)}\right| + \left|\frac{2\alpha_1 X_8 X_4}{(\alpha_1+\alpha_1+\alpha_1(X_8-1)}\right|$$

$$\leq T_2,$$

$$\left|\frac{\partial f_1}{\partial X_8}\right| = \left|\frac{A_4(-\alpha_1 X_1 \frac{\partial f_5}{\partial X_8} + \alpha_2\alpha_2 X_4^2 + 2\alpha_1 X_4 X_7 - \alpha_1 X_6 - \alpha_1 E_4}{A_4^2}\right|$$

$$\leq \left|\frac{-A_4(-\alpha_1 X_1 \frac{\partial f_5}{\partial X_8}}{(\alpha_1+\alpha_1+\alpha_1(X_8-1)}\right| + \left|\frac{\alpha_2\alpha_2 X_4^2}{(\alpha_1+\alpha_1+\alpha_1(X_8-1)}\right| + \left|\frac{2\alpha_1 X_4 X_7 - \alpha_1 X_6}{(\alpha_1+\alpha_1+\alpha_1(X_8-1)}\right|$$

$$+ \left|\frac{-\alpha_1 X_6}{(\alpha_1+\alpha_1+\alpha_1(X_8-1)}\right| + \left|\frac{-\alpha_1 E_4}{((\alpha_1+\alpha_1+\alpha_1(X_8-1))^2}\right| \leq T_3,$$

where T_3 is a real constant.

$$\left|\frac{\partial f_1}{\partial X_9}\right| = \left|\frac{-2X_5}{(\alpha_1+\alpha_1+\alpha_1(X_8-1)} + \frac{\alpha_1 \alpha_7 X_1 X_5}{((\alpha_1+\alpha_1+\alpha_1(X_8-1))^2} + \frac{\alpha_1 X_1 \alpha_1 X_2}{((\alpha_1+\alpha_1+\alpha_1(X_8-1))^2}\right.$$

$$\left. - \frac{\alpha_1 X_5}{(\alpha_1+\alpha_1+\alpha_1(X_8-1)} + \frac{2\alpha_1 \alpha_1 X_1 X_9}{((\alpha_1+\alpha_1+\alpha_1(X_8-1))^2} - \frac{2\alpha_1 X_1}{(\alpha_1+\alpha_1+\alpha_1(X_8-1)}\right|$$

$$\leq \left|\frac{-2X_5}{(\alpha_1+\alpha_{10}+\alpha_1(X_8-1)}\right| + \left|\frac{\alpha_1 \alpha_7 X_1 X_5}{((\alpha_1+\alpha_1+\alpha_1(X_8-1))^2}\right| + \left|\frac{\alpha_1 X_1 \alpha_1 X_2}{((\alpha_1+\alpha_1+\alpha_1(X_8-1))^2}\right|$$

$$+ \left|\frac{-\alpha_1 X_5}{(\alpha_1+\alpha_1+\alpha_1(X_8-1)}\right| + \left|\frac{2\alpha_1 \alpha_1 X_1 X_9}{((\alpha_1+\alpha_1+\alpha_1(X_8-1))^2}\right| + \left|\frac{-2\alpha_1 X_1}{(\alpha_1+\alpha_1+\alpha_1(X_8-1)}\right| \leq T_3,$$

$$\left|\frac{\partial f_1}{\partial X_1}\right| = \left|\frac{2X_3}{(\alpha_1+\alpha_1+\alpha_1(X_8-1)} + \frac{\alpha_7 X_8}{(\alpha_1+\alpha_1+\alpha_1(X_8-1)}\right.$$

$$- \frac{\alpha_1 \left[2\alpha_7 X_2 X_1 - X_5 X_5 \alpha_7 - \alpha_1 X_5^2 - \alpha_1 X_3^2 - \alpha_1 \left(\alpha_1 + \alpha_2(1-X_8)\right) X_4^2 - \alpha_1 X_2 X_8 - \alpha_1 X_9 X_2\right]}{((\alpha_1+\alpha_1+\alpha_1(X_8-1))^2}$$

$$\left. + \frac{\alpha_1 X_4^2}{(\alpha_1+\alpha_1+\alpha_1(X_8-1)} - \frac{\alpha_1 X_3}{(\alpha_1+\alpha_1+\alpha_1(X_8-1)}\right| \leq T_3,$$

$$\left|\frac{\partial f_1}{\partial X_1}\right| = \left|\frac{-\alpha_7 X_5}{(\alpha_1+\alpha_1+\alpha_1(X_8-1)} - \frac{2\alpha_1 X_9}{(\alpha_1+\alpha_1+\alpha_1(X_8-1)} - \frac{\alpha_1 X_2}{(\alpha_1+\alpha_1+\alpha_1(X_8-1)}\right|$$

$$\leq \left|\frac{-\alpha_7 X_5}{(\alpha_1+\alpha_1+\alpha_1(X_8-1)}\right| + \left|\frac{-2\alpha_1 X_9}{(\alpha_1+\alpha_1+\alpha_1(X_8-1)}\right| + \left|\frac{\alpha_1 X_2}{(\alpha_1+\alpha_1+\alpha_1(X_8-1)}\right| \leq T_3. \quad (3.14)$$

Then there exists a number (Lipchitz constant) K, $0 \leq K < \infty$, such that

$$K = m \left\{\left|\frac{\partial f_i}{\partial}\right|\right\}, i = 1, \dots, 11. \quad (3.15)$$

Consequently, $\frac{\partial f_i}{\partial}$ for each $i = 1, \dots, 11$ is bounded and $f_i(X_1, X_2, X_3, X_4, X_5, X_6, \dots, X_1)$ for each $i = 1, \dots, 11$ is Lipchitzcontinuous. Hence there exists a unique solution of the system of coupled non-linear ordinary differential equations (2.28)-(2.31) subject to (2.32).

3.2 Stability Analysis

The concepts of equilibrium point and stability are motivated by the desire to keep a dynamical system in, or at least close to, some desirable state. The term equilibrium or equilibrium point of a dynamical system, is used for a state of the system that does not change in the course of time, i.e. if the system is in an equilibrium at time t_0, then it will stay there for all times

$t \geq t_0$. A real system is always subject to some fluctuations in the state. There are some external factors that are unpredictable and cannot be modeled; some dynamics that have very little impact on the behavior of the system are neglected in the modeling, etc. Even if the mathematical model of a physical system would be perfect, which is impossible, the system state would still be subject to quantum mechanical fluctuations. The concept of stability in the theory of dynamical systems is motivated by the desire, that the system state stays at least
close to an equilibrium point after small fluctuations in the state. Stability analysis tells us about the convergence property of the differential equations. A steady state point is stable, if the differential equation converges to that point. Other-wise, it is unstable.

3.3 Definition

Assume that $X = 0$ is an equilibrium point of (2.33)-(2.42) and let $\|.\|$be an arbitrary norm on $\mathbb{R}^n$. The equilibrium point X or critical point is said to be stable, if and only if for every $R > 0$ there is an $r > 0$, such that$\|\phi(t,\xi)\| \leq R$ for all $\|\xi\| \leq r$ and all $t \geq 0$, where $\phi$is the solution of the system.
If the equilibrium $X = 0$ is not stable in this sense, then there is $R > 0$such that any fluctuation in the state from zero, no matter how small, can lead to a state x with$\|\xi\| \geq R$. Such equilibrium is called unstable. The set of those points in the state-space of a dynamical system, which are attracted to an equilibrium point by the dynamics of the system, is of great importance. It is called the region of attraction of the equilibrium.

3.4 Theorems on stability

Theorem 3.4a

Consider the autonomous first order system of differential equations;

$$X' = a_1 X + a_1 Y, \quad (3.16)$$
$$Y' = a_2 X + a_2 Y.$$

Where $a_{i,j}$ are real constants.Equation (3.16) can be rewritten in matrix form as

$$\begin{bmatrix} X' \\ Y' \end{bmatrix} = \begin{bmatrix} a_1 & a_1 \\ a_2 & a_2 \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix}. \quad (3.17)$$

Thus we obtain an auxiliary equation$|A - \lambda I| = 0$,where
A = $\begin{bmatrix} a_1 & a_1 \\ a_2 & a_2 \end{bmatrix}$, $I$is the identity matrix and λ is an eigenvalue. Therefore, simplifying the auxiliary equation above we get

$$\lambda^2 - (a_1 + a_1)\lambda + (a_1 a_2 - a_2 a_1) = 0 \quad (3.18)$$

such that $a_1 a_2 - a_1 a_2 \neq 0$ and the point (X_0, Y_0) is the only critical point. Let λ_1 and λ_2 be the two roots of the auxiliary equation above. Then,the critical point is said to be

(i) Stable if λ_1 and λ_2 are purely imaginary.
(ii) Asymptotically stable if $R\ \lambda_1 < 0$ and $R\ \lambda_2 < 0$
(iii) Unstable in all other cases.

Consequently, we formulate theorem on the stability of critical points of system of non-linear equations (2.28)-(2.31) as follows:

Theorem 3.4b

Let $X = (X_1, X_2, X_3, \ldots X_1)$ be a critical point of non-linear system $X' = H$ such that the auxiliary equation $|H - \lambda I| = 0$ $h\emptyset$, where H is 10×10 matrix, I and λ_i are identity and eigenvalues respectively $\forall\ i = 1,2, \ldots .10$. Therefore the critical point is unstable.

Proof:

Considering the system of equations

$$X_1 = f, \quad (3.19)$$
$$X_1' = f' = X_2, \quad (3.20)$$
$$X_2' = f'' = X_3, \quad (3.21)$$
$$X_3' = f''' = \frac{1}{[a_1 + a_2(a_1 - X_7)]}\{a_7[(X_2X_5 - X_3X_4)] + a_2X_3X_8 - a_1X_3X_1 - a_3X_2^2 - a_4 - a_5X_7 - a_6X_2\}, \quad (3.22)$$
$$X_4' = G' = X_5, \quad (3.23)$$
$$X_5' = G'' = X_6, \quad (3.24)$$
$$X_6' = G''' = \frac{1}{[a_1 + a_2(a_1 - X_7)]}[2(X_2X_5 - X_3X_4) + a_7(X_5^2 - X_5X_6) + a_2X_9[B] + a_2X_3X_1 + a_2X_6X_8 - a_1X_3X_4 - a_1X_1X_6 - a_1 X_2^2 - 2a_3(X_2X_5 - a_2 - a_5X_9 - a_2 X_7 - a_6X_5 - a_2 X_2], ,$$
$$X_7 = \theta,$$
$$X_8' = \theta' = X_8, \quad (3.25)$$
$$X_8' = \theta'' = \frac{1}{\{a_1 + a_1 + a_1 (X_7 - 1)\}}[a_7(X_2X_9 - X_8X_4) - a_1 X_8^2 - a_1 X_2^2 - a_1 (a_1 + a_2(a_1 - X_7))X_5^2X_2X_7 - a_1X_8X_1], \quad (3.26)$$
$$X_9 = \varphi,$$

$X_9' = \varphi' = X_1$, (3.27)

$X_1\ ' = \varphi'' = \frac{1}{\{\alpha_1 + \alpha_1 + \alpha_1\ (X_7 - 1)\}}[2[X_2X_9 - X_8X_4] + \alpha_7\{X_9X_5 - X_4X_1\ \} - \alpha_1\ X_9C - 2\alpha_1\ X_8X_1\ - 2\alpha_1\ (X_2X_5 - \alpha_2\ X_2^2 - 2\alpha_1\ X_3X_6 - \alpha_1\alpha_2\ X_3^2 + \alpha_2\alpha_2\ X_7X_3^2 - 2\alpha_1\ X_3X_6 - \alpha_2\alpha_2\ X_3^2 + 2\alpha_1\ X_7X_3X_6 + \alpha_1\ X_3^2X_9 - \alpha_1\ X_2X_9 - \alpha_1\ X_7X_5 - \alpha_1X_1X_1\ - \alpha_1X_8X_4]$. (3.28)

Expressing the system of equation above in matrix form,we have

$$\begin{bmatrix} X_1' \\ X_2' \\ X_3' \\ X_4' \\ X_5' \\ X_6' \\ X_7' \\ X_8' \\ X_9' \\ X_{10}' \end{bmatrix} = \begin{bmatrix} \frac{\partial X_1'}{\partial X_1} & \frac{\partial X_1'}{\partial X_2} & \frac{\partial X_1'}{\partial X_3} & \cdots & \frac{\partial X_1'}{\partial X_{10}} \\ \frac{\partial X_2'}{\partial X_1} & \frac{\partial X_2'}{\partial X_2} & \frac{\partial X_2'}{\partial X_3} & \cdots & \frac{\partial X_2'}{\partial X_{10}} \\ \frac{\partial X_3'}{\partial X_1} & \frac{\partial X_3'}{\partial X_2} & \frac{\partial X_3'}{\partial X_3} & \cdots & \frac{\partial X_3'}{\partial X_{10}} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ \frac{\partial X_{10}'}{\partial X_1} & \frac{\partial X_{10}'}{\partial X_2} & \frac{\partial X_{10}'}{\partial X_3} & \cdots & \frac{\partial X_{10}'}{\partial X_{10}} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \\ X_7 \\ X_8 \\ X_9 \\ X_{10} \end{bmatrix}$$

Hence the system becomes

$X' = H$.

Obtaining the partial derivative of the matrix above using the equation (3.19)-(3.28), H becomes

$$H = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & D_1 & D_2 & D_3 & D_4 & 0 & D_5 & D_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ D_7 & D_8 & D_9 & D_{10} & D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{33} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ D_{16} & D_{17} & D_{18} & D_{19} & 0 & 0 & D_{20} & D_{21} & D_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ D_{23} & D_{24} & D_{25} & D_{26} & D_{27} & D_{28} & D_{29} & D_{30} & D_{31} & D_{32} \end{pmatrix}.$$

Where $A = \frac{A_7}{A_5}$, $A_5 = \frac{1}{[\alpha_1 + \alpha_2(\alpha_1 - X_7)]}$, $A_6 = \frac{1}{\{\alpha_1 + \alpha_1 + \alpha_1\ (X_7 - 1)\}}$,

$A_7 = \alpha_7[(X_2X_5 - X_3X_4)] + \alpha_2X_3X_8 - \alpha_1X_3X_1 - \alpha_3X_3^2 - \alpha_4 - \alpha_5X_7 - \alpha_6X_2$,

$A_8 = [2(X_2X_5 - X_3X_4) + \alpha_7(X_5^2 - X_5X_6) + \alpha_2X_9[A] + \alpha_2X_3X_1\ + \alpha_2X_6X_8 - \alpha_1X_3X_4 - \alpha_1X_1X_6 - \alpha_1\ X_2^2 - 2\alpha_3(X_2X_5 - \alpha_2\ - \alpha_5X_9 - \alpha_2\ X_7 - \alpha_6X_5 - \alpha_2\ X_2]$,

$A_9 = \left|\alpha_7(X_2X_9 - X_8X_4) - \alpha_1\ X_8^2 - \alpha_1\ X_2^2 - \alpha_1\ \left(\alpha_1 + \alpha_2(\alpha_1 - X_7)\right)X_3^2X_2X_7 - \alpha_1X_8X_1\right|$,

$A_1\ = 2[X_2X_9 - X_8X_4] + \alpha_7\{X_9X_5 - X_4X_1\ \} - \alpha_1\ X_9C - 2\alpha_1\ X_8X_1\ - 2\alpha_1\ (X_2X_5 - \alpha_2\ X_2^2 - 2\alpha_1\ X_3X_6 - \alpha_1\alpha_2\ X_3^2 + \alpha_2\alpha_2\ X_7X_3^2 - 2\alpha_1\ X_3X_6 - \alpha_2\alpha_2\ X_3^2 + 2\alpha_1\ X_7X_3X_6 + \alpha_1\ X_3^2X_9 - \alpha_1\ X_2X_9 - \alpha_1\ X_7X_5 - \alpha_1X_1X_1\ - \alpha_1X_8X_4$,

$B = \frac{A_5}{A_6}$, $B_1 = \alpha_7X_5 - 2\alpha_3X_2 - X_6$, $B_1\ = \alpha_2X_8 - \alpha_7X_5 - \alpha_1X_1$, $B_1\ = -\alpha_2X_9\alpha_5A_5 + \alpha_2^2A_7X_9$,

$B_1\ = \alpha_7X_9 - 2\alpha_1\ X_2 - \alpha_1\ \alpha_7$, $B_1\ = -2\alpha_1\ X_3A_5$, $B_1\ = A_6\alpha_1\ \alpha_2X_3^2 - A_6\alpha_1\ X_2 - A_9\alpha_1$,

$B_1\ = -\alpha_7X_4 - 2\alpha_1\ X_8 - \alpha_1X_1$, $B_1\ = 2\alpha_1\ \alpha_1\ X_9X_2 + \alpha_1\ X_9X_7\alpha_1\ - \alpha_7X_3^2\alpha_1$,

$B_1\ = 2X_9 - 2\alpha_1\ X_5 - 2\alpha_2\ X_2 - \alpha_1\ X_9$, $B_1\ = 2\alpha_1\ \alpha_1\ X_9X_3A_5$,

$B_1\ = 2\alpha_1\ X_3X_9 + 2\alpha_1\ X_7X_6 - 2\alpha_2\alpha_2\ X_3 - 2\alpha_1\ X_6 + 2\alpha_2\alpha_2\ X_7X_3 - 2\alpha_1\alpha_2\ X_3 - 2\alpha_1\ X_6$,

$B_2\ = 2X_8 + \alpha_1X_8 + \alpha_7X_1$, $B_2\ = \alpha_7X_9 - 2\alpha_1\ X_2 - \alpha_1\ X_7$, $B_2\ = 2\alpha_1\ X_7X_3 - 2\alpha_1\ X_3 - 2\alpha_1\ X_3$,

$B_2\ = \frac{B_1}{A_6^2}$, $B_2\ = -\alpha_1\ X_9B_2\ + \alpha_2\alpha_2\ X_3^2 + 2\alpha_1\ X_6 - \alpha_1\ X_5$, $B_2\ = \alpha_7X_4 + 2\alpha_1\ X_8 + \alpha_1X_1$,

$B_2\ = 2X_2 + \alpha_7X_5 + \alpha_1\ X_3^2 - \alpha_1\ X_2$, $B_2\ = \alpha_7X_2X_9 + A_9$, $B_2\ = \alpha_7X_4 + 2\alpha_1\ X_8 + \alpha_1X_1$,

$B_2 = -\alpha_7X_4 + \alpha_2X_8 - \alpha_1X_1$, $B_3 = -\alpha_5A_5 + \alpha_2A_7$, $B_4 = 2X_5 - 2\alpha_1\ X_2 - \alpha_2\ - 2\alpha_3X_5$,

$B_5 = \alpha_2\alpha_7X_5X_9 - 2\alpha_2\alpha_3X_9X_2 - \alpha_6\alpha_2X_9$, $B_6 = -2X_4 + \alpha_2X_1\ - \alpha_1X_4$,

$B_7 = \alpha_2^2 X_8 X_9 - \alpha_2\alpha_7 X_4 X_9 - \alpha_1\alpha_2 X_1 X_9$, $B_8 = -2X_3 - \alpha_1 X_3$, $B_9 = 2X_2 + 2\alpha_7 X_5 - \alpha_7 X_6 - 2\alpha_3 X_2 - \alpha_6$

$D_1 = \frac{B_1}{A_5}$, $D_2 = \frac{B_2}{A_5}$, $D_3 = \frac{-\alpha_7 X_3}{A_5}$, $D_4 = \frac{\alpha_7 X_2}{A_5}$, $D_5 = \frac{B_3}{A_5^2}$, $D_6 = \frac{\alpha_2 X_3}{A_5}$,

$D_7 = \frac{\alpha_1\alpha_2 X_9 X_3}{A_5^2} - \frac{\alpha_1 X_6}{A_5}$, $D_8 = \frac{B_4}{A_5} + \frac{B_5}{A_5^2}$, $D_9 = \frac{B_6}{A_5} + \frac{B_7}{A_5^2}$, $D_1 = \frac{B_8}{A_5} - \frac{\alpha_2\alpha_7 X_3 X_9}{A_5^2}$,

$D_1 = \frac{B_9}{A_5} + \frac{\alpha_2\alpha_7 X_2 X_9}{A_5^2}$,

$D_1 = \frac{B_1}{A_5}$, $D_1 = \frac{B_1}{A_5^3} - \frac{\alpha_2\alpha_{72}}{A_5} + \frac{\alpha_2\alpha_{72}A_8}{A_5^2}$, $D_1 = \frac{\alpha_2^2 X_3 X_9}{A_5^2} + \frac{\alpha_2 X_6}{A_5}$, $D_1 = \frac{\alpha_2 A - \alpha_5}{A_5}$,

$D_1 = \frac{-\alpha_1 X_8}{A_6}$,

$D_1 = \frac{B_1}{A_6}$, $D_1 = \frac{B_1}{A_6}$, $D_1 = \frac{-\alpha_7 X_8}{A_6}$, $D_2 = \frac{B_1}{A_6^2}$, $D_2 = \frac{B_1}{A_6}$, $D_2 = \frac{\alpha_7 X_2}{A_6}$,

$D_2 = \frac{\alpha_1\ \alpha_1 X_9 X_8}{A_6^2} - \frac{\alpha_1 X_1}{A_6}$, $D_2 = \frac{B_1}{A_6^2} + \frac{B_1}{A_6}$, $D_2 = \frac{B_1}{A_6^2} + \frac{B_1}{A_6}$,

$D_2 = \frac{\alpha_1\ \alpha_7 X_9 X_8}{A_6^2} - \frac{B_2}{A_6}$, $D_2 = \frac{B_2}{A_6}$, $D_2 = \frac{B_2}{A_6}$, $D_2 = \frac{B_2}{A_6} - \frac{\alpha_1\ A_1}{A_6^2}$,

$D_3 = \frac{-2X_4 - 2\alpha_1\ X_1\ - \alpha_1 X_4}{A_6} - \frac{\alpha_1\ X_9 B_2}{A_6^2}$, $D_3 = \frac{B_2}{A_6} - \frac{\alpha_1\ B_2}{A_6^2}$,

$D_3 = \frac{-B_2}{A_6}$, $D_3 = \frac{s_2 X_3}{A_5}$.

Hence,by substituting the critical points $X = (X_1, \dots, X_1\ ) = (1,0,0,0.5,0,0,-0.3,0,0.1,0)$ obtained from the system of equation (3.19)-(3.28), the matrix H becomes:

$$H=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1.6716 & 0 & 0 & 0 & -2.9289 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & -0.5 & -0.8579 & 0 & -2.9289 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0.1387 & 0 & 0 & 0 & 0 & 0 & -2.3880 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0.2367 & 0 & 0 & 0.1387 & 0 & 0 & -4.2857 & 0 & -2.3880 \end{pmatrix}$$

Therefore, we obtain an auxiliary equation

$|H - \lambda\ | = 0$ as

$$\begin{vmatrix} -a & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -a & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1.6716-a & 0 & 0 & 0 & -2.9289 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & -0.5-a & -0.8579 & 0 & -2.9289 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -a & 1 & 0 & 0 \\ 0.1387 & 0 & 0 & 0 & 0 & 0 & 0 & -2.3880-a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a & 1 \\ 0 & 0.2367 & 0 & 0 & 0.1387 & 0 & 0 & -4.2857 & 0 & -2.3880-a \end{vmatrix} = 0.$$

Such thatthe eigenvalues corresponding to the auxiliary equation are obtained as follows:

$\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = -2.3481$, $\lambda_4 = 2.2489$, $\lambda_5 = -1.9027$, $\lambda_6 = -0.8596$,

$\lambda_7 = 0.1599\ +\ 0.419i$, $\lambda_8 = 0.1599 - 0.4191i$, $\lambda_9 = 0.0460\ +\ 0.3046i$,

$\lambda_1\ = 0.0460\ -\ 0.3046i$.

Hence, by invoking theorem (3.4b), we see that the critical point obtained from the system of equation (3.19)-(3.28) asunstable. This further explained the fact that Magnetohydrodynamic forced convection flow through porous medium is unstable as a result of particles coarseness, rough wall surface etc.

4.0 Conclusion

The problem of MHD forced convective flow over horizontal non-isothermal circular cylinder in fluid saturated porous medium was extended to include more realistic case where the viscosity and thermal conductivity of the fluid changes with respect temperature in the presence of thermal radiation. The mathematical flow model involves a system of highly coupled non-linear partial differential equations that is transformed using local non-similarity techniques. The qualitative properties of solutions which validate system of governing equations representing the physical model were analyzed by formulating theorems and establishing the proof respectively. The findings of the study revealed that the solution of the flow model exist, unique and unstable.

5.0 References

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