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Further on the Evolution of the Group Velocity for Water Waves

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Abstract

In the present study the application of inhomogeneous quasi-linear kinematic wave equation is provided. Different functional forms taken as the source characteristics were considered. The initial Cauchy data introduced in each model is either in the form of smooth function or parameterized curves.

In the case of a source that is constant in space and time, it is interestingly unexpected to obtain solution that exhibits catastrophic behavior. The other sources model with initial data provide equal unexpected solutions.

1.0 Introduction

The quasi-linear kinematic wave equation already deduced governs the processes of group velocity evolutions [1].The extreme characteristic behavior of the solutions were clearly illustrated. In the theory, the kinematic wave equation was regarded as a homogeneous quasi-linear equation in the context of water wave theory.

The concept of group velocity is very fundamental to the theory of linear wave group [2]. Its role in the energy focusing involving the intercrossing of monochromatic wave groups is fundamental in the related analysis [3,4].

In the present considerations, inhomogeneous forms of quasi-linear kinematic wave equation with varying source forcing functions will be analyzed.

The Cauchy initial data may be modeled with smooth functions or parameterized curves in space and time.

2.0 The Linear Source Function

This approach will involve the inhomogeneous case. In this case, the evolutional process is assumed to be induced by a given forcing function. In this regard, the forcing function will be assumed to be a linear function in x-coordinate. Thus, with group velocity

$C_g = C_g(x, t)$, then

$$\frac{\partial C_g}{d} + C_y \frac{\partial C_g}{d} = x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (1)$$

Subject to the initial data

$$C_{y\,(x,0)} = x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

The usual equation of the characteristics[3] may be put in the form for equation (1)

$$\frac{d}{1} = \frac{dx}{C_y} = \frac{dC_y}{x} = k \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Where k is a constant that is non-dimensional.

It follows from (3) that

$$d \;\; = kC_y, dC_y = x \quad a \qquad d(x + C_y) = k(x + C_y)$$

$$\frac{d(x + C_y)}{x + C_y} = k = \frac{d}{1}.$$

Consequently,

$$h\ \left(\frac{x + C_y}{C_1}\right) = t$$

$$(x + C_y)e^{-t} = C_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

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Further, from (3)
$x = C_y\, dC_y,\ thu$,
$x^2 - C_y^2 = C_2$……………………………………………………………………(5)
Using the initial data $C_y(x,0) = x$, equation (4) gives
$2x = C_1$
Using (5), $C_2 = 0$
Thus, in (4)
$(x + C_y)e^{-t} = 2x$………………………………………………………… (6)
It follows that:
$C_y = x(2e^t - 1)$ …………………………………………………………… (6a)
Equation (6a)gives the space/time evolution of the group velocity profile.
Differentiation of (6a) with respect to space and time suggests that C_y has no extreme point for t≥ 0.
The case of spatial oscillatory initial data as the details of the above is as follows: In this consideration, the initial data will be assumed to be oscillatory with wave parameter k. Thus, we state that
$C_y(x,0) = \sin k$, $0 < x < \infty$ … … … … … … . . … … … … … … … … … … … … (7)
Using equations (4),(5)and (7), we obtain
$x^2 - s\ {}^2 k = C_2$……………………………………………………………………(8)
$x + \sin k = C_1$ … … … … … . . … … … … … … … … … … … … … … … … … … . (9)
EliminatingC_2 a C_1 from (8) and (9), then,
$x + s = (x + C_y)e^{-t}$ …. (10)
$x(1 - e^{-t}) + \sin k = e^{-t}C_y$
$C_y = e^t(x + \sin k\ ) - x$ … . . (11)
$C_y(x,0) = \sin k$ … (12)
agrees with (7)
We investigate the optimal values of $C_y(x,t)$. In this case, let,x_0 a $t_0$be the space and time location of the optimum value of $C_y(x,t)$. The point$(x_0,\ t_0)$ is obtained from the solution of the simultaneous equations.
$\frac{\partial C_g}{d}(x_0, t_0) = 0$, $\frac{\partial C_g}{\partial_t}(x_0, t_0)$from which
$e^{t_0}(1 + k\cos kx_0) = 1$ … . (13)
$e^{t_0}(x_0 + \sin kx_0) = 0$ … (14)
$e^{t_0} = 0$only if $t_0 = -\infty$ from (14)
Thus, $C_y(x,t)$ has no optimum value in finite time whilst from equ (13)
$x_0 = c^{-1}\left[\frac{e^{-t_0-1}}{k}\right]$ ……………………………………………………………………………(15)

3.0 Constant Source Model

The case where the source function is constant and equal to unity but initial data is still a linear function of x, may be put in the form:
$\frac{\partial C_y}{d} + C_g\frac{\partial C_y}{\partial} = 1$ … (16)
$C_y(x,0) = a$ ……………………………………………………………………………………. (17)
a is a constant and non-dimensional . In general, all the quantities in this study are dimensionalised[5,6].

The characteristic equations associated with (16) is as follows
$\frac{d}{1} = \frac{d}{C_g} = \frac{dC_g}{1} = k$ …………………………………………………………………………..(18)
From (18), the following are derived
$x - Cy_{/_2} = C_1$ … . (19)
$t - C_y = C_2$ … (20)
From (17) and (19)
$x - \frac{a^2\ x^2}{2} = C_1$ … (21)

And from (20)

$-a \quad = C_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots . (22)$

Consequently, $2C_1 = C_2\left(\frac{2}{u} - C_2\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (23)$

$2x - C_y^2 = \left(t - C_y\right)\left(\frac{2}{a} - t + C_y\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots . (24)$

Thus, from (24) $C_y = \frac{1}{2}\left\{\frac{2u(x-t)+ut^2}{a\ -1}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (25)$

Solution (25) behaves catastrophically, if $t = \frac{1}{u}$

$C_y = 0, \quad i: \ at^2 - 2t + 2a \ = 0. \ tha \ i: \ t = t_1 a \quad t_2$

$whe \quad t_1 = \frac{1}{a} + \frac{1}{a}\sqrt{(1 - 2a\ )}, \ t_2 = \frac{1}{a} - \frac{1}{a}\sqrt{(1 - 2a\ )} \ldots\ldots\ldots\ldots\ldots\ldots\ldots . (26)$

$f \quad the. \ , \ C_y \ c \quad t - a \quad p \quad x < \frac{1}{2u}$

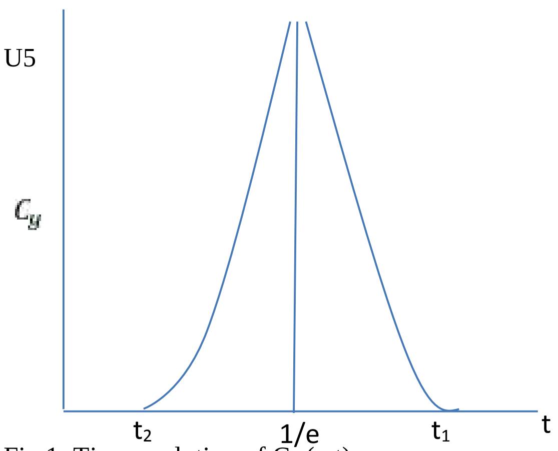

Fig 1: Time evolution of C_g (x,t)

Finally, a simple form of $C_y\,(x, t)$ is obtained if we parameterized the initially with s, τ as follows:

$x_t(s, o) = {s^2}/{2}, \qquad t_0\,(s, o) = s\ a \qquad C_y(s, o) = s \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots . (27)$

But from (18), $\frac{d}{d} = C_y = \tau + C_{y(s,0)}$, $\frac{d}{d}$=1 , $\frac{dC_y}{d} = 1$

Thus,

$x(\tau, s) = {\tau^2}/{2} + \tau C_y(s, o) + x_0(s) = \tau + t_0(s, 0), C_y(s, \tau) = \tau + C_y(s, 0) \ldots (28)$

Using the parameterized curve,

$x(\tau, s) = {\tau^2}/{2} + \tau. \ + {s^2}/{2} \ , t = \tau + s, C_y(s, \tau) = \tau + s \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (29)$

Thus from (29), we eliminate $\tau \ a \quad s \ a \ f$

$2x = \tau^2 + 2\tau. \ + s^2 = (\tau + s)^2 \ = tC_y(x, t)$

Thus $C_y(x, t) = {2x}/{t} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots . (30)$

(30) Suggest that $C_y(x, t)$ does not touch t-axis (horizontal) but vanishes along $x - a$ (vertical)

4.0 Conclusion

In the previous paper [1], the process governed by the kinematic equation which is homogeneous was considered. However, the present study is interested in the process controlled by forcing function as the source. Various forms of Cauchy initial data were imposed. Each set of initial datae describe different wave forms for the group velocity. For example, with non-dimensionalised parameters, a unit source provides the profile which vanishes along x–axis (vertical) but does not touch t-axis (horizontal). However, introducing a parameterized initial curve, a singulanty appears in the form of catastrophy observed in previous homogeneous kinematic equation already considered [1].

5.0 References

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