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On the Vibrations under Moving Masses of Elastically Supported Plate Resting on Bi-Parametric Foundation with Stiffness Variation

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Abstract

The dynamic behaviour of elastically supported (non-classical boundary condition) rectangular plate carrying moving concentrated masses and resting on bi-parametric (Pasternak) elastic foundation with stiffness variation is considered in this investigation. The governing equation is a fourth order partial differential equation with variable and singular coefficients. In order to solve the governing differential equation, it is reduced to a sequence of coupled second order ordinary differential equations using a technique based on separation of variables. The modified method of Struble is used to simplify the coupled differential equations and the integral transformations are then employed for the solutions of the simplified equations. The analysis of resonance shows that, for the same natural frequency, the critical speed (and the natural frequency) for the moving mass problem is smaller than that of the moving force problem. Thus, resonance is reached earlier in the moving mass system than in the moving force system The results in plotted curves show that as the value of the rotatory inertia correction factor R_o increases, the response amplitudes of the plate decrease. It is also shown that as the value of the shear modulus G_o increases the displacement amplitudes of the plate decrease for fixed foundation modulus F_0. For fixed R_o, F_o and G_o, the transverse deflections of the elastically supported rectangular plates under the actions of moving masses are higher than those when only the force effects of the moving load are considered. This implies that resonance is reached earlier in moving mass problem than in moving force problem. Thus, safety is more guaranteed with the moving mass solution.

Keywords: Bi-Parametric Foundation, Shear deformation, Resonance, Critical Speed, Natural Frequency.

1.0 Introduction

The analysis of flexure of beam resting on a Winkler foundation and under moving load is very common in literature especially when the foundation modulus is constant. It is generally known that the dynamical problems of structures under moving load and resting on a foundation is generally complex, the complexity increases if the foundation stiffness varies along the structure. Aside the problem of singularity brought in by the inclusion of the inertia effects of the moving load, the coefficients of the governing fourth order partial differential equation are no longer constant but variable. Earlier researchers into beam member on variable elastic foundation include Franklin and Scott [1] who presented a closed-form solution to a linear variation of the foundation modulus using contour-integrals. Closely following this, Lentini [2] presented a finite difference method to solve the problem where the foundation stiffness varies along x as a power of x. Much later, Clastornik et al [3] presented a solution for the finite beams resting on a Winkler elastic foundation with stiffness variation that can be presented as a general polynomial of x. Though works in [1,2,3] are useful, the loads acting on the beams are not moving loads. In a more recent development, Oni and Awodola [4] extended the works of these previous authors to investigate the dynamic response to moving concentrated masses of uniform Rayleigh beams resting on variable Winkler elastic foundation.

Recently, many researchers have made efforts in the study of dynamics of structures under moving loads [5-10]. In all of these, considerations have been limited to cases of one-dimensional (beam) problems. Where two-dimensional (plate) problems have been considered, the foundation moduli are taken to be constants. No considerations have been given to the class of dynamical problems in which the foundation is the type with stiffness variation

The foundation model based on Winkler's approximation model is very common in literature, whereas, in such an important Engineering problem as the vibration of plates resting on elastic foundation, a more accurate Two-Parameter (Pasternak) foundation model which in addition to subgrade modulus incorporates the shear effect of the foundation should be used rather

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than the Winkler's approximation model. Eisenberger and Clastornik [11] presented two methods for the solution of beams on variable two-parameter elastic foundation. Also, Gbadeyan and Oni [12] studied the dynamic analysis of an elastic plate continuously supported by an elastic Pasternak foundation traversed by an arbitrary number of concentrated masses. In their work, they assumed that both the foundation modulus and the shear modulus are constants.

In all these investigations, extension of the theory to cover two-dimensional (plate) problem in which the plate is resting on Pasternak elastic foundation with stiffness variation has not been considered. Where this has been considered, it has been exclusively reserved for elastic structures having the classical boundary conditions such as the Clamped edge, Free edge, Simply supported edge and Sliding edge boundary conditions. For practical applications in many cases, it is more realistic to consider non-classical boundary conditions, such as the elastically supported edge condition, because the classical boundary conditions can seldom be realized. This study is however concerned with the behaviour of elastically supported rectangular plate under the action of concentrated moving masses and resting on Pasternak elastic foundation with stiffness variation.

2.0 Governing Equation

The dynamic transverse displacement Z(x,y,t) of a rectangular plate when it is resting on a Pasternak elastic foundation with stiffness variation and traversed by concentrated masses M_i moving with velocity c_i is governed by the fourth order partial differential equation given in [13] as

$$W\nabla^2\left[\nabla^2 Z(x,y,t)\right] + \circ\,\frac{\partial^2 Z(x,y,t)}{\partial t^2} = \circ R_0\left[\frac{\partial^4}{\partial t^2\partial x^2} + \frac{\partial^4}{\partial t^2\partial y^2}\right]Z(x,y,t) - F_0\left[4x - 3x^2 + x^3\right]Z(x,y,t)$$
$$+ G_0[-13 + 12x - 3x^2]\frac{\partial}{\partial x}Z(x,y,t) + G_0[12 - 13x + 6x^2 - x^3]\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]Z(x,y,t) \tag{1}$$
$$+ \sum_{i=1}^{N}\left[M_i g ¢(x - c_i t)¢(y - s) - M_i\left(\frac{\partial^2}{\partial t^2} + 2c_i\frac{\partial^2}{\partial t\partial x} + c_i^2\frac{\partial^2}{\partial x^2}\right)Z(x,y,t)¢(x - c_i t)¢(y - s)\right]$$

Where $W = \dfrac{Eh^2}{12(1-v)}$ (2)

is the bending rigidity of the plate, ∇^2 is the two-dimensional Laplacian operator, h is the plate's thickness, E is the Young's Modulus, v is the Poisson's ratio $(v < 1)$, « is the mass per unit area of the plate, R_0 is the Rotatory inertia correction factor, F_0 is the foundation constant, G_0 is the shear modulus, g is the acceleration due to gravity, δ(.) is the Dirac-Delta function, x and y are respectively the spatial coordinates in x and y directions and t is the time coordinate.

The initial conditions, without any loss of generality, is taken as

$$Z(x,y,t) = 0 = \frac{\partial Z(x,y,t)}{\partial t} \tag{3}$$

3.0 Analytical Approximate Solution

The method of analysis involves expressing the Dirac – Delta function as a Fourier cosine series. In order to solve equation (1), a technique [14] based on separation of variables is used to reduce it to a set of coupled second order ordinary differential equations. Then, the modified asymptotic method of Struble in conjunction with the techniques of integral transformation and convolution theory are employed to obtain the closed form solution of the resulting second order ordinary differential equations.

In the first instance, we consider rectangular plate elastically supported at edges y = 0, y = L_Y with simple support at edges x = 0, x = L_X, the boundary conditions can be written as [10]

$$Z(0,y,t) = 0, \qquad Z(L_X, y,t) = 0 \tag{4}$$

$$\frac{\partial^2 Z(x,0,t)}{\partial y^2} - k_1\frac{\partial Z(x,0,t)}{\partial y} = 0, \quad \frac{\partial^2 Z(x,L_Y,t)}{\partial y^2} - k_1\frac{\partial Z(x,L_Y,t)}{\partial y} = 0 \tag{5}$$

$$\frac{\partial^2 Z(0,y,t)}{\partial x^2} = 0, \qquad \frac{\partial^2 Z(L_X,y,t)}{\partial x^2} = 0 \tag{6}$$

$$\frac{\partial^3 Z(x,0,t)}{\partial y^3} + k_2 Z(x,0,t) = 0, \quad \frac{\partial^3 Z(x,L_Y,t)}{\partial y^3} + k_2 Z(x,L_Y,t) = 0 \tag{7}$$

and for normal modes

$$\Psi_{ni}(0) = 0, \qquad \Psi_{ni}(L_X) = 0 \tag{8}$$

$$\frac{\partial^2 \Psi_{nj}(0)}{\partial y^2} - k_1 \frac{\partial \Psi_{nj}(0)}{\partial y} = 0, \quad \frac{\partial^2 \Psi_{nj}(L_Y)}{\partial y^2} - k_1 \frac{\partial \Psi_{nj}(L_Y)}{\partial y} = 0 \tag{9}$$

$$\frac{\partial^2 \Psi_{ni}(0)}{\partial x^2} = 0, \qquad \frac{\partial^2 \Psi_{ni}(L_X)}{\partial x^2} = 0 \tag{10}$$

$$\frac{\partial^3 \Psi_{nj}(0)}{\partial y^3} + k_2 \Psi_{nj}(0) = 0, \qquad \frac{\partial^3 \Psi_{nj}(L_Y)}{\partial y^3} + k_2 \Psi_{nj}(L_Y) = 0 \tag{11}$$

where k_1 is the stiffness against rotation and k_2 is the stiffness against translation.

Secondly, we consider an elastic rectangular plate resting on a variable Pasternak elastic foundation and having elastic supports at all its edges, the boundary conditions are given in [10] as

$$\frac{\partial^2 Z(0,y,t)}{\partial x^2} - k_1 \frac{\partial Z(0,y,t)}{\partial x} = 0, \quad \frac{\partial^2 Z(L_X,y,t)}{\partial x^2} - k_1 \frac{\partial Z(L_X,y,t)}{\partial x} = 0 \tag{12}$$

$$\frac{\partial^2 Z(x,0,t)}{\partial y^2} - k_1 \frac{\partial Z(x,0,t)}{\partial y} = 0, \quad \frac{\partial^2 Z(x,L_Y,t)}{\partial y^2} - k_1 \frac{\partial Z(x,L_Y,t)}{\partial y} = 0 \tag{13}$$

$$\frac{\partial^3 Z(0,y,t)}{\partial x^3} + k_2 Z(0,y,t) = 0, \qquad \frac{\partial^3 Z(L_X,y,t)}{\partial x^3} + k_2 Z(L_X,y,t) = 0 \tag{14}$$

$$\frac{\partial^3 Z(x,0,t)}{\partial y^3} + k_2 Z(x,0,t) = 0, \qquad \frac{\partial^3 Z(x,L_Y,t)}{\partial y^3} + k_2 Z(x,L_Y,t) = 0 \tag{15}$$

and for normal modes

$$\frac{\partial^2 \Psi_{ni}(0)}{\partial x^2} - k_1 \frac{\partial \Psi_{ni}(0)}{\partial x} = 0, \quad \frac{\partial^2 \Psi_{ni}(L_Y)}{\partial x^2} - k_1 \frac{\partial \Psi_{ni}(L_Y)}{\partial x} = 0 \tag{16}$$

$$\frac{\partial^2 \Psi_{nj}(0)}{\partial y^2} - k_1 \frac{\partial \Psi_{nj}(0)}{\partial y} = 0, \quad \frac{\partial^2 \Psi_{nj}(L_Y)}{\partial y^2} - k_1 \frac{\partial \Psi_{nj}(L_Y)}{\partial y} = 0 \tag{17}$$

$$\frac{\partial^3 \Psi_{ni}(0)}{\partial x^3} + k_2 \Psi_{ni}(0) = 0, \qquad \frac{\partial^3 \Psi_{ni}(L_Y)}{\partial x^3} + k_2 \Psi_{ni}(L_Y) = 0 \tag{18}$$

$$\frac{\partial^3 \Psi_{nj}(0)}{\partial y^3} + k_2 \Psi_{nj}(0) = 0, \qquad \frac{\partial^3 \Psi_{nj}(L_Y)}{\partial y^3} + k_2 \Psi_{nj}(L_Y) = 0 \tag{19}$$

where k_1 and k_2 are the stiffness against rotation and the stiffness against translation respectively.

In order to solve equation (1), in the first instance, the deflection is written in the form [14]

$$Z(x,y,t) = \sum_{n=1}^{\infty} \phi_n(x,y) T_n(t) \tag{20}$$

where ϕ_n are the known eigenfunctions of the plate with the same boundary conditions. The ϕ_n have the form of

$$\nabla^4 \phi_n - \lambda_n^4 \phi_n = 0 \tag{21}$$

where $$\lambda_n^4 = \frac{\Omega_n^2 \mu}{W} \tag{22}$$

Ω_n, n = 1, 2, 3, … , are the natural frequencies of the dynamical system and $T_n(t)$ are amplitude functions which have to be calculated.

Next, the equation (1) is rewritten as

$$\begin{aligned} &\frac{W}{\mu}\nabla^4 Z(x,y,t) + \frac{\partial^2 Z(x,y,t)}{\partial t^2} = R_0\left[\frac{\partial^4}{\partial t^2 \partial x^2} + \frac{\partial^4}{\partial t^2 \partial y^2}\right] Z(x,y,t) - \frac{F_0}{\mu}\left[4x - 3x^2 + x^3\right] Z(x,y,t) \\ &+ \frac{G_0}{\mu}[-13 + 12x - 3x^2]\frac{\partial}{\partial x} Z(x,y,t) + \frac{G_0}{\mu}[12 - 13x + 6x^2 - x^3]\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] Z(x,y,t) \\ &+ \sum_{i=1}^{N}\left[\frac{M_i g}{\mu}\delta(x - c_i t)\delta(y - s) - \frac{M_i}{\mu}\left(\frac{\partial^2}{\partial t^2} + 2c_i \frac{\partial^2}{\partial t \partial x} + c_i^2 \frac{\partial^2}{\partial x^2}\right) Z(x,y,t)\delta(x - c_i t)\delta(y - s)\right] \end{aligned} \tag{23}$$

The right hand side of equation (23) is written in the form of a series to have

$$R_0\left[\frac{\partial^4}{\partial t^2\partial x^2}+\frac{\partial^4}{\partial t^2\partial y^2}\right]Z(x,y,t)-\frac{F_0}{∘}\left[4x-3x^2+x^3\right]Z(x,y,t)$$
$$+\frac{G_0}{∘}\left[\left(-13+12x-3x^2\right)\frac{\partial}{\partial x}Z(x,y,t)+\left(12-13x+6x^2-x^3\right)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)Z(x,y,t)\right] \quad (24)$$
$$+\sum_{i=1}^{N}\left[\frac{M_i g}{∘}¢(x-c_it)¢(y-s)-\frac{M_i}{∘}\left(\frac{\partial^2}{\partial t^2}+2c_i\frac{\partial^2}{\partial t\partial x}+c_i^2\frac{\partial^2}{\partial x^2}\right)Z(x,y,t)¢(x-c_it)¢(y-s)\right]=\sum_{n=1}^{\infty}¤_n(x,y)¥_n(t)$$

Substituting equation (20) into equation (24) we have

$$\sum_{n=1}^{\infty}\Big\{R_0\left[¨_{n,xx}(x,y)±_{n,tt}(t)+¨_{n,yy}(x,y)±_{n,tt}(t)\right]-\frac{F_0}{∘}\left[4x-3x^2+x^3\right]¨_n(x,y)±_n(t)$$
$$+\frac{G_0}{∘}\left[\left(-13+12x-3x^2\right)¨_{n,x}(x,y)±_n(t)+\left(12-13x+6x^2-x^3\right)\left(¨_{n,xx}(x,y)±_n(t)+¨_{n,yy}(x,y)±_n(t)\right)\right]$$
$$+\sum_{i=1}^{N}\Big[\ \frac{M_i g}{∘}¢(x-c_it)¢(y-s)-\frac{M_i}{∘}\Big(¨_n(x,y)±_{n,tt}(t)+2c_i¨_{n,x}(x,y)±_{n,t}(t)$$
$$+c_i^2¨_{n,xx}(x,y)±_n(t)\ \Big)¢(x-c_it)¢(y-s)\ \Big]\ \Big\}=\sum_{n=1}^{\infty}¨_n(x,y)¥_n(t) \quad (25)$$

$$\sum_{n=1}^{\infty}\Big\{R_0\left[¤_{n,xx}(x,y)±_{n,tt}(t)+¤_{n,yy}(x,y)±_{n,tt}(t)\right]-\frac{F_0}{∘}\left[4x-3x^2+x^3\right]¤_n(x,y)±_n(t)$$
$$+\frac{G_0}{∘}\left[\left(-13+12x-3x^2\right)¤_{n,x}(x,y)±_n(t)+\left(12-13x+6x^2-x^3\right)\left(¤_{n,xx}(x,y)±_n(t)+¤_{n,yy}(x,y)±_n(t)\right)\right]$$
$$+\sum_{i=1}^{N}\Big[\ \frac{M_i g}{∘}¢(x-c_it)¢(y-s)-\frac{M_i}{∘}\Big(¤_n(x,y)±_{n,tt}(t)+2c_i¤_{n,x}(x,y)±_{n,t}(t)$$
$$+c_i^2¤_{n,xx}(x,y)±_n(t)\ \Big)¢(x-c_it)¢(y-s)\ \Big]\ \Big\}=\sum_{n=1}^{\infty}¤_n(x,y)¥_n(t) \quad (25)$$

where

$$¤_{n,x}(x,y)\ implies\ \frac{\partial ¤_n(x,y)}{\partial x},\quad ¤_{n,xx}(x,y)\ implies\ \frac{\partial^2 ¤_n(x,y)}{\partial x^2},$$
$$¤_{n,y}(x,y)\ implies\ \frac{\partial ¤_n(x,y)}{\partial y},\quad ¤_{n,yy}(x,y)\ implies\ \frac{\partial^2 ¤_n(x,y)}{\partial y^2}, \quad (26)$$
$$±_{n,t}(t)\ implies\ \frac{d±_n(t)}{dt}\ and\ ±_{n,tt}(t)\ implies\ \frac{d^2±_n(t)}{dt^2}$$

Multiplying both sides of equation (25) by $\phi_p(x,y)$ and integrating on area A of the plate, we have

$$\sum_{n=1}^{\infty}\int_A\ \Big\{R_0\left[¤_{n,xx}(x,y)¤_p(x,y)±_{n,tt}(t)+¤_{n,yy}(x,y)¤_p(x,y)±_{n,tt}(t)\right]$$
$$-\frac{F_0}{∘}\left[4x-3x^2+x^3\right]¤_n(x,y)¤_p(x,y)±_n(t)+\frac{G_0}{∘}\Big[\left(-13+12x-3x^2\right)¤_{n,x}(x,y)¤_p(x,y)±_n(t)$$
$$+\left(12-13x+6x^2-x^3\right)\left(¤_{n,xx}(x,y)¤_p(x,y)±_n(t)+¤_{n,yy}(x,y)¤_p(x,y)±_n(t)\right)\Big] \quad (27)$$
$$+\sum_{i=1}^{N}\Big[\ \frac{M_i g}{∘}¤_p(x,y)¢(x-c_it)¢(y-s)-\frac{M_i}{«}\Big(¤_n(x,y)¤_p(x,y)±_{n,tt}(t)+2c_i¤_{n,x}(x,y)¤_p(x,y)±_{n,t}(t)$$
$$+c_i^2¤_{n,xx}(x,y)¤_p(x,y)±_n(t)\ \Big)¢(x-c_it)¢(y-s)\ \Big]\ \Big\}dA=\sum_{n=1}^{\infty}\int_A ¤_n(x,y)¤_p(x,y)¥_n(t)dA$$

Considering the orthogonality of $\phi_n(x,y)$, we have

$$¥_n(t) = \frac{1}{«}\sum_{n=1}^{\infty}\int_A \Big\{R_0\big[¤_{n,xx}(x,y)¤_p(x,y)±_{n,tt}(t) + ¤_{n,yy}(x,y)¤_p(x,y)±_{n,tt}(t)\big]$$
$$-\frac{F_0}{°}\big[4x-3x^2+x^3\big]¤_n(x,y)¤_p(x,y)±_n(t) + \frac{G_0}{°}\Big[\big(-13+12x-3x^2\big)¤_{n,x}(x,y)¤_p(x,y)±_n(t)$$
$$+\big(12-13x+6x^2-x^3\big)\big(¤_{n,xx}(x,y)¤_p(x,y)±_n(t) + ¤_{n,yy}(x,y)¤_p(x,y)±_n(t)\big)\Big] \quad (28)$$
$$+\sum_{i=1}^{N}\Big[\ \frac{M_i g}{°}¤_p(x,y)¢(x-c_i t)¢(y-s) - \frac{M_i}{°}\big(¤_n(x,y)¤_p(x,y)±_{n,tt}(t)$$
$$+2c_i¤_{n,x}(x,y)¤_p(x,y)±_{n,t}(t) + c_i^2¤_{n,xx}(x,y)¤_p(x,y)±_n(t)\ \big)¢(x-c_i t)¢(y-s)\ \Big]\Big\}dA$$

where $« = \int_A ¤_p^2 dA$

Using (28), equation (23), taking into account (20) and (21), can be written as

$$¤_n(x,y)\left[\frac{Wµ_n^4}{°}±_n(t) + ±_{n,tt}(t)\right] = \frac{¤_n(x,y)}{«}\sum_{q=1}^{\infty}\int_A \Big\{R_0\big[¤_{q,xx}(x,y)¤_p(x,y)±_{q,tt}(t) + ¤_{q,yy}(x,y)¤_p(x,y)±_{q,tt}(t)\big]$$
$$-\frac{F_0}{°}\big[4x-3x^2+x^3\big]¤_q(x,y)¤_p(x,y)±_q(t) + \frac{G_0}{°}\Big[\big(-13+12x-3x^2\big)¤_{q,x}(x,y)¤_p(x,y)±_q(t) \quad (29)$$
$$+\big(12-13x+6x^2-x^3\big)\big(¤_{q,xx}(x,y)¤_p(x,y)±_q(t) + ¤_{q,yy}(x,y)¤_p(x,y)±_q(t)\big)\Big] + \sum_{i=1}^{N}\Big[\ \frac{M_i g}{°}¤_p(x,y)¢(x-c_i t)¢(y-s)$$
$$-\frac{M_i}{°}\big(¤_q(x,y)¤_p(x,y)±_{q,tt}(t) + 2c_i¤_{q,x}(x,y)¤_p(x,y)±_{q,t}(t) + c_i^2¤_{q,xx}(x,y)¤_p(x,y)±_q(t)\ \big)¢(x-c_i t)¢(y-s)\ \Big]\Big\}dA$$

Equation (29) must be satisfied for arbitrary x, y and this is possible only when

$$±_{n,tt}(t) + \frac{Wµ_n^4}{°}±_n(t) = \frac{1}{«}\sum_{q=1}^{\infty}\int_A \Big\{R_0\big[¤_{q,xx}(x,y)¤_p(x,y)±_{q,tt}(t) + ¤_{q,yy}(x,y)¤_p(x,y)±_{q,tt}(t)\big]$$
$$-\frac{F_0}{°}\big[4x-3x^2+x^3\big]¤_q(x,y)¤_p(x,y)±_q(t) + \frac{G_0}{°}\Big[\big(-13+12x-3x^2\big)¤_{q,x}(x,y)¤_p(x,y)±_q(t)$$
$$+\big(12-13x+6x^2-x^3\big)\big(¤_{q,xx}(x,y)¤_p(x,y)±_q(t) + ¤_{q,yy}(x,y)¤_p(x,y)±_q(t)\big)\Big] \quad (30)$$
$$+\sum_{i=1}^{N}\Big[\ \frac{M_i g}{°}¤_p(x,y)¢(x-c_i t)¢(y-s) - \frac{M_i}{°}\big(¤_q(x,y)¤_p(x,y)±_{q,tt}(t)$$
$$+2c_i¤_{q,x}(x,y)¤_p(x,y)±_{q,t}(t) + c_i^2¤_{q,xx}(x,y)¤_p(x,y)±_q(t)\ \big)¢(x-c_i t)¢(y-s)\ \Big]\Big\}dA$$

The system in equation (30) is a set of coupled ordinary differential equations.
Considering the property of the Dirac-Delta function and expressing it in the Fourier cosine series as

$$¢(x-c_i t) = \frac{1}{L_X}\left[1 + 2\sum_{j=1}^{\infty}\cos\frac{j®c_i t}{L_X}\cos\frac{j®x}{L_X}\right] \quad (31)$$

and

$$¢(y-s) = \frac{1}{L_Y}\left[1 + 2\sum_{k=1}^{\infty}\cos\frac{k®s}{L_Y}\cos\frac{k®y}{L_Y}\right] \quad (32)$$

equation (30) becomes

$$\frac{d^2±_n(t)}{dt^2} + \sim_n^2±_n(t) - \frac{1}{«}\sum_{q=1}^{\infty}\left\{R_0P_1^*\frac{d^2±_q(t)}{dt^2} - \left[\frac{F_0}{°}P_{2A}^* - \frac{G_0}{°}P_{2B}^*\right]±_q(t) - \sum_{i=1}^{N}\frac{M_i}{L_X L_Y °}\left[2\left(\frac{P_3^*}{2} + \sum_{k=1}^{\infty}\cos\frac{k®s}{L_Y}P_3^{**}(k)\right.\right.\right.$$
$$\left.+\sum_{j=1}^{\infty}\cos\frac{j®c_i t}{L_X}P_3^{***}(j) + 2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j®c_i t}{L_X}\cos\frac{k®s}{L_Y}P_3^{****}(j,k)\right)\frac{d^2±_q(t)}{dt^2} + 4c_i\left(\frac{P_4^*}{2} + \sum_{k=1}^{\infty}\cos\frac{k®s}{L_Y}P_4^{**}(k)\right. \quad (33)$$
$$\left.+\sum_{j=1}^{\infty}\cos\frac{j®c_i t}{L_X}P_4^{***}(j) + 2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j®c_i t}{L_X}\cos\frac{k®s}{L_Y}P_4^{****}(j,k)\right)\frac{d±_q(t)}{dt} + 2c_i^2\left(\frac{P_5^*}{2} + \sum_{k=1}^{\infty}\cos\frac{k®s}{L_Y}P_5^{**}(k)\right.$$
$$\left.\left.\left.+\sum_{j=1}^{\infty}\cos\frac{j®c_i t}{L_X}P_5^{***}(j) + 2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j®c_i t}{L_X}\cos\frac{k®s}{L_Y}P_5^{****}(j,k)\right)±_q(t)\right]\right\} = \sum_{i=1}^{N}\frac{M_i g}{«°}¤_p(c_i t, s)$$

where $\sim_n^2 = \frac{W\mu_n^4}{\circ}$,

$$P_1^* = \int_0^{L_X}\int_0^{L_Y}\left[\phi_{n,xx}(x,y)+\phi_{n,yy}(x,y)\right]\phi_p(x,y)\,dy\,dx,\quad P_2^* = \int_0^{L_X}\int_0^{L_Y}\left[4x-3x^2+x^3\right]\phi_n(x,y)\phi_p(x,y)\,dy\,dx,$$

$$P_3^* = \int_0^{L_X}\int_0^{L_Y}\phi_n(x,y)\phi_p(x,y)\,dy\,dx,\quad P_3^{**}(k) = \int_0^{L_X}\int_0^{L_Y}\cos\frac{k\pi y}{L_Y}\phi_n(x,y)\phi_p(x,y)\,dy\,dx,$$

$$P_3^{***}(j) = \int_0^{L_X}\int_0^{L_Y}\cos\frac{j\pi x}{L_X}\phi_n(x,y)\phi_p(x,y)\,dy\,dx,\quad P_3^{****}(j,k) = \int_0^{L_X}\int_0^{L_Y}\cos\frac{j\pi x}{L_X}\cos\frac{k\pi y}{L_Y}\phi_n(x,y)\phi_p(x,y)\,dy\,dx,$$

$$P_4^* = \int_0^{L_X}\int_0^{L_Y}\phi_{n,x}(x,y)\phi_p(x,y)\,dy\,dx,\quad P_4^{**}(k) = \int_0^{L_X}\int_0^{L_Y}\cos\frac{k\pi y}{L_Y}\phi_{n,x}(x,y)\phi_p(x,y)\,dy\,dx,$$

$$P_4^{***}(j) = \int_0^{L_X}\int_0^{L_Y}\cos\frac{j\pi x}{L_X}\phi_{n,x}(x,y)\phi_p(x,y)\,dy\,dx,\quad P_4^{****}(j,k) = \int_0^{L_X}\int_0^{L_Y}\cos\frac{j\pi x}{L_X}\cos\frac{k\pi y}{L_Y}\phi_{n,x}(x,y)\phi_p(x,y)\,dy\,dx,$$

$$P_5^* = \int_0^{L_X}\int_0^{L_Y}\phi_{n,xx}(x,y)\phi_p(x,y)\,dy\,dx,\quad P_5^{**}(k) = \int_0^{L_X}\int_0^{L_Y}\cos\frac{k\pi y}{L_Y}\phi_{n,xx}(x,y)\phi_p(x,y)\,dy\,dx,$$

$$P_5^{***}(j) = \int_0^{L_X}\int_0^{L_Y}\cos\frac{j\pi x}{L_X}\phi_{n,xx}(x,y)\phi_p(x,y)\,dy\,dx,\quad P_5^{****}(j,k) = \int_0^{L_X}\int_0^{L_Y}\cos\frac{j\pi x}{L_X}\cos\frac{k\pi y}{L_Y}\phi_{n,xx}(x,y)\phi_p(x,y)\,dy\,dx,$$

$$P_{2A}^* = 4h_1 - 3h_2 + h_3,\; P_{2B}^* = -13h_4 + 12h_5 - 3h_6 + 12(h_7+h_8) - 13(h_9+h_{10}) + 6(h_{11}+h_{12}) - (h_{13}+h_{14})$$

$$h_1 = \int_0^{L_Y}\int_0^{L_X} x\phi_n(x,y)\phi_p(x,y)dxdy,\; h_2 = \int_0^{L_Y}\int_0^{L_X} x^2\phi_n(x,y)\phi_p(x,y)dxdy,\; h_3 = \int_0^{L_Y}\int_0^{L_X} x^3\phi_n(x,y)\phi_p(x,y)dxdy$$

$$h_4 = \int_0^{L_Y}\int_0^{L_X} \phi_{n,x}(x,y)\phi_p(x,y)dxdy,\; h_5 = \int_0^{L_Y}\int_0^{L_X} x\phi_{n,x}(x,y)\phi_p(x,y)dxdy,\; h_6 = \int_0^{L_Y}\int_0^{L_X} x^2\phi_{n,x}(x,y)\phi_p(x,y)dxdy$$

$$h_7 = \int_0^{L_Y}\int_0^{L_X} \phi_{n,xx}(x,y)\phi_p(x,y)dxdy,\; h_8 = \int_0^{L_Y}\int_0^{L_X} \phi_{n,yy}(x,y)\phi_p(x,y)dxdy,\; h_9 = \int_0^{L_Y}\int_0^{L_X} x\phi_{n,xx}(x,y)\phi_p(x,y)dxdy$$

$$h_{10} = \int_0^{L_Y}\int_0^{L_X} x\phi_{n,yy}(x,y)\phi_p(x,y)dxdy,\; h_{11} = \int_0^{L_Y}\int_0^{L_X} x^2\phi_{n,xx}(x,y)\phi_p(x,y)dxdy,\; h_{12} = \int_0^{L_Y}\int_0^{L_X} x^2\phi_{n,yy}(x,y)\phi_p(x,y)dxdy$$

$$h_{13} = \int_0^{L_Y}\int_0^{L_X} x^3\phi_{n,xx}(x,y)\phi_p(x,y)dxdy \quad and \quad h_{14} = \int_0^{L_Y}\int_0^{L_X} x^3\phi_{n,yy}(x,y)\phi_p(x,y)dxdy$$

The second order coupled differential equation (33) is the transformed equation governing the problem of a rectangular plate on a Pasternak elastic foundation with stiffness variation.

$\phi_n(x,y)$ are assumed to be the products of the functions $\psi_{ni}(x)$ and $\psi_{nj}(y)$ which are the beam functions in the directions of x and y axes respectively [15, 16]. That is

$$\phi_n(x,y) = \psi_{ni}(x)\psi_{nj}(y) \quad (34)$$

these beam functions can be defined respectively, as

$$\psi_{ni}(x) = \sin\frac{\Omega_{ni}x}{L_X} + A_{ni}\cos\frac{\Omega_{ni}x}{L_X} + B_{ni}\sinh\frac{\Omega_{ni}x}{L_X} + C_{ni}\cosh\frac{\Omega_{ni}x}{L_X} \quad (35)$$

and

$$\psi_{nj}(x) = \sin\frac{\Omega_{nj}y}{L_Y} + A_{nj}\cos\frac{\Omega_{nj}y}{L_Y} + B_{nj}\sinh\frac{\Omega_{nj}y}{L_Y} + C_{nj}\cosh\frac{\Omega_{nj}y}{L_Y} \quad (36)$$

where A_{ni}, A_{nj}, B_{ni}, B_{nj}, C_{ni} and C_{nj} are constants determined by the boundary conditions. Ω_{ni} and Ω_{nj} are called the mode frequencies.

In order to solve equation (33) we shall consider only one mass M traveling with uniform velocity c along the line y = s. Thus for the single mass M equation (33) reduces to

$$\frac{d^2\pm_n(t)}{dt^2}+\sim_n^2\pm_n(t)-\frac{1}{\text{«}}\sum_{q=1}^{\infty}\Bigg\{R_0P_1^*\frac{d^2\pm_q(t)}{dt^2}-\frac{G_0}{\circ}\left[\frac{F_0}{G_0}P_{2A}^*-P_{2B}^*\right]\pm_q(t)-\Gamma^0\Bigg[2\Bigg(\frac{P_3^*}{2}+\sum_{k=1}^{\infty}\cos\frac{k\text{®}s}{L_Y}P_3^{**}(k)$$

$$+\sum_{j=1}^{\infty}\cos\frac{j\text{®}ct}{L_X}P_3^{***}(j)+2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\text{®}ct}{L_X}\cos\frac{k\text{®}s}{L_Y}P_3^{****}(j,k)\Bigg)\frac{d^2\pm_q(t)}{dt^2}+4c\Bigg(\frac{P_4^*}{2}+\sum_{k=1}^{\infty}\cos\frac{k\text{®}s}{L_Y}P_4^{**}(k) \qquad (37)$$

$$+\sum_{j=1}^{\infty}\cos\frac{j\text{®}ct}{L_X}P_4^{***}(j)+2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\text{®}ct}{L_X}\cos\frac{k\text{®}s}{L_Y}P_4^{****}(j,k)\Bigg)\frac{d\pm_q(t)}{dt}+2c^2\Bigg(\frac{P_5^*}{2}+\sum_{k=1}^{\infty}\cos\frac{k\text{®}s}{L_Y}P_5^{**}(k)$$

$$+\sum_{j=1}^{\infty}\cos\frac{j\text{®}ct}{L_X}P_5^{***}(j)+2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\text{®}ct}{L_X}\cos\frac{k\text{®}s}{L_Y}P_5^{****}(j,k)\Bigg)\pm_q(t)\Bigg]\Bigg\}=\frac{Mg}{\text{«}\circ}\Psi_{pi}(ct)\Psi_{pj}(s)$$

where $\Gamma^0=\frac{M}{L_XL_Y\circ}$ (38)

Equation (37) is the fundamental equation of our problem. We shall discuss two cases of the equation (37) namely; the **moving force** and the **moving mass** problems.

Case I: Moving Force problem

An approximate model of the differential equation describing the response of a rectangular plate resting on a variable Bi-Parametric (Pasternak) elastic foundation and traversed by a moving force would be obtained from equation (37) by setting $\Gamma^0 = 0$.

Thus, setting $\Gamma^0 = 0$, equation (37) reduces to

$$\frac{d^2\pm_n(t)}{dt^2}+\sim_n^2\pm_n(t)-R_0\sum_{q=1}^{\infty}\frac{P_1^*}{\text{«}}\frac{d^2\pm_q(t)}{dt^2}-\frac{G_0}{\circ}\sum_{q=1}^{\infty}\frac{1}{\text{«}}\left[P_{2B}^*-\frac{F_0}{G_0}P_{2A}^*\right]\pm_q(t)=\frac{Mg}{\text{«}\circ}\Psi_{pi}(ct)\Psi_{pj}(s) \qquad (39)$$

An exact analytical solution to equation (39) is evidently not possible. Consequently, the approximate analytical solution technique, which is a modification of the asymptotic method of Struble [16] shall be used.

First, we neglect the rotatory inertial term and rearrange the equation (39) to take the form

$$\frac{d^2\pm_n(t)}{dt^2}+\left[\sim_n^2-\Gamma^*\left(P_{2B}^*-\frac{F_0}{G_0}P_{2A}^*\right)\right]\pm_n(t)-\Gamma^*\sum_{\substack{q=1\\q\neq n}}^{\infty}\left[P_{2B}^*-\frac{F_0}{G_0}P_{2A}^*\right]\pm_q(t)=\frac{Mg}{\text{«}\circ}\Psi_{pi}(ct)\Psi_{pj}(s) \qquad (40)$$

where $\Gamma^*=\frac{G_0}{\circ\text{«}}$ (41)

By means of this technique, one seeks the modified frequency corresponding to the frequency of the free system due to the presence of the shear modulus G_0. An equivalent free system operator defined by the modified frequency then replaces equation (40). Thus, we set the right hand side of (40) to zero and consider a parameter $\lambda^* < 1$ for any arbitrary ratio Γ^* defined as

$$\text{a}^*=\frac{\Gamma^*}{1+\Gamma^*} \qquad (42)$$

so that $\Gamma^*=\text{a}^*+o(\text{a}^{*2})$ (43)

Thus, the homogeneous part of equation (40) can be replaced with

$$\frac{d^2\pm_n(t)}{dt^2}+\text{¥}_s^2\pm_n(t)=0 \qquad (44)$$

where

$$\text{¥}_s=\sim_n-\frac{\text{a}^*\left(P_{2B}^*-\frac{F_0}{G_0}P_{2A}^*\right)}{2\sim_n} \qquad (45)$$

is the modified frequency due to the effect of the shear modulus of the foundation. It is observed that when $\lambda^* = 0$, we recover the frequency of the moving force problem when the shear modulus effect of the foundation is neglected

Using equation (44), equation (39) can be written as

$$\frac{d^2T_n(t)}{dt^2}+\gamma_s^2T_n(t)-\alpha_pP_1^*\frac{d^2T_n(t)}{dt^2}-\alpha_pP_1^*\sum_{\substack{q=1\\q\neq n}}^{\infty}\frac{d^2T_q(t)}{dt^2}=\frac{Mg}{\mu}\Psi_{pi}(ct)\Psi_{pj}(s) \qquad (46)$$

where $\alpha_p=\dfrac{R_0}{P^*}$

We then seek the modified frequency corresponding to the frequency of the free system due to the presence of the effect of rotatory inertia correction factor R_0. An equivalent free system operator defined by the modified frequency then replaces equation (46). To this end, the homogenous part of equation (46) is rearranged to take the form

$$\frac{d^2T_n(t)}{dt^2}+\frac{\gamma_s^2}{1-\alpha_pP_1^*}T_n(t)-\frac{\alpha_pP_1^*}{1-\alpha_pP_1^*}\sum_{\substack{q=1\\q\neq n}}^{\infty}\frac{d^2T_q(t)}{dt^2}=0 \qquad (47)$$

Consider the parameter $\varepsilon^* < 1$ for any arbitrary ratio defined as

$$\varepsilon^*=\frac{\alpha_p}{1+\alpha_p} \qquad (48)$$

It can be shown that

$$\alpha_p=\varepsilon^*+o(\varepsilon^{*2}) \qquad (49)$$

Following the same argument, equation (47) can be replaced with

$$\frac{d^2T_n(t)}{dt^2}+\gamma_{sf}^2T_n(t)=0 \qquad (50)$$

where $\gamma_{sf}=\gamma_s\left[1+\dfrac{\varepsilon^*P_1^*}{2}\right]$ (51)

represents the modified frequency corresponding to the frequency of the free system due to the presence of the rotatory inertia. It is observed that when $\varepsilon^* = 0$, we recover the frequency of the moving force problem when the rotatory inertia effect is neglected.

In order to solve the non-homogenous equation (46), the differential operator which acts on $T_n(t)$ is replaced by the equivalent free system operator defined by the modified frequency γ_{sf}. Thus the moving force problem (39) is reduced to the non-homogeneous ordinary differential equation given as

$$\frac{d^2T_n(t)}{dt^2}+\gamma_{sf}^2T_n(t)=K_m\Psi_{pi}(ct)\Psi_{pj}(s) \qquad (52)$$

where

$$K_m=\frac{Mg}{\mu} \qquad (53)$$

Using (35), equation (52) can be written as

$$\frac{d^2T_n(t)}{dt^2}+\gamma_{sf}^2T_n(t)=K_m\Psi_{pj}(s)\left[\sin\omega_{pi}t+A_{pi}\cos\omega_{pi}t+B_{pi}\sinh\omega_{pi}t+C_{pi}\cosh\omega_{pi}t\right] \qquad (54)$$

where

$$\omega_{pi}=\frac{\Omega_{pi}c}{L_X} \qquad (55)$$

When equation (54) is solved in conjunction with the initial conditions (3), one obtains expression for $\sigma_n(t)$. Thus, in view of equation (20), one obtains

$$Z(x,y,t)=\sum_{ni=1}^{\infty}\sum_{nj=1}^{\infty}\frac{K_m\Psi_{pj}(s)}{¥_{sf}[¥_{sf}^4-\sim_{pi}^4]}\{[¥_{sf}^2-\sim_{pi}^2][C_{pi}¥_{sf}(\cosh\sim_{pi}t-\cos¥_{sf}t)$$
$$+B_{pi}(¥_{sf}\sinh\sim_{pi}t-\sim_{pi}\sin¥_{sf}t)]+[¥_{sf}^2+\sim_{pi}^2][A_{pi}¥_{sf}(\cos\sim_{pi}t-\cos¥_{sf}t) \quad (56)$$
$$-(\sim_{pi}\sin¥_{sf}t-¥_{sf}\sin\sim_{pi}t)]\}[\sin\frac{\Omega_{ni}x}{L_X}+A_{ni}\cos\frac{\Omega_{ni}x}{L_X}+B_{ni}\sinh\frac{\Omega_{ni}x}{L_X}$$
$$+C_{ni}\cosh\frac{\Omega_{ni}x}{L_X}][\sin\frac{\Omega_{nj}y}{L_Y}+A_{nj}\cos\frac{\Omega_{nj}y}{L_Y}+B_{nj}\sinh\frac{\Omega_{nj}y}{L_Y}+C_{nj}\cosh\frac{\Omega_{nj}y}{L_Y}]$$

as the transverse displacement response to a moving force of a rectangular plate resting on variable non-Winkler (Pasternak) elastic foundation.

Case II: Moving Mass problem

In this section, we seek the solution to the entire equation (37) when no term of the coupled differential equation is neglected. Evidently, an exact analytical solution to equation (37) does not exist; an analytical approximate method is therefore desirable. To this end, the approximate analytical solution method of Struble that has been used to tackle this form of coupled differential equation shall be employed to treat equation (37). We take note that, neglecting the terms representing the inertia effect of the moving mass we obtain equation (46) and then equation (52). The homogeneous part of this equation can be replaced by a free system operator defined by the modified frequency γ_{sf}, due to the presence of the effects of rotatory inertia and the shear modulus of the foundation. Thus, equation (37) can be rewritten in the form

$$\left[1+\frac{2£_s}{P^*}\left(\frac{P_3^*}{2}+\sum_{k=1}^{\infty}\cos\frac{k®s}{L_Y}P_3^{**}(k)+\sum_{j=1}^{\infty}\cos\frac{j®ct}{L_X}P_3^{***}(j)+2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j®ct}{L_X}\cos\frac{k®s}{L_Y}P_3^{****}(j,k)\right)\right]\frac{d^2±_n(t)}{dt^2}$$
$$+\frac{4£_sc}{P^*}\left(\frac{P_4^*}{2}+\sum_{k=1}^{\infty}\cos\frac{k®s}{L_Y}P_4^{**}(k)+\sum_{j=1}^{\infty}\cos\frac{j®ct}{L_X}P_4^{***}(j)+2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j®ct}{L_X}\cos\frac{k®s}{L_Y}P_4^{****}(j,k)\right)\frac{d±_n(t)}{dt}$$
$$+\left[¥_{sf}^2+\frac{2£_sc^2}{P^*}\left(\frac{P_5^*}{2}+\sum_{k=1}^{\infty}\cos\frac{k®s}{L_Y}P_5^{**}(k)+\sum_{j=1}^{\infty}\cos\frac{j®ct}{L_X}P_5^{***}(j)+2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j®ct}{L_X}\cos\frac{k®s}{L_Y}P_5^{****}(j,k)\right)\right]±_n(t)$$
$$+\frac{£_s}{P^*}\sum_{\substack{q=1\\q\neq n}}\left[2\left(\frac{P_3^*}{2}+\sum_{k=1}^{\infty}\cos\frac{k®s}{L_Y}P_3^{**}(k)+\sum_{j=1}^{\infty}\cos\frac{j®ct}{L_X}P_3^{***}(j)+2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j®ct}{L_X}\cos\frac{k®s}{L_Y}P_3^{****}(j,k)\right)\frac{d^2±_q(t)}{dt^2}\right.$$
$$4c\left(\frac{P_4^*}{2}+\sum_{k=1}^{\infty}\cos\frac{k®s}{L_Y}P_4^{**}(k)+\sum_{j=1}^{\infty}\cos\frac{j®ct}{L_X}P_4^{***}(j)+2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j®ct}{L_X}\cos\frac{k®s}{L_Y}P_4^{****}(j,k)\right)\frac{d±_q(t)}{dt}$$
$$\left.+2c^2\left(\frac{P_5^*}{2}+\sum_{k=1}^{\infty}\cos\frac{k®s}{L_Y}P_5^{**}(k)+\sum_{j=1}^{\infty}\cos\frac{j®ct}{L_X}P_5^{***}(j)+2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j®ct}{L_X}\cos\frac{k®s}{L_Y}P_5^{****}(j,k)\right)±_q(t)\right]$$
$$=\frac{£_sgL_XL_Y}{«}\Psi_{pi}(ct)\Psi_{pj}(s) \quad (57)$$

where $£_s=\dfrac{M}{L_XL_Y°}$ (58)

Furthermore, for our plate model, resting on variable non-Winkler elastic foundation and traversed by a moving mass, we rearrange equation (57) to take the form

$$\frac{d^2±_n(t)}{dt^2}+\frac{£_sR_2(t)}{1+£_sR_1(t)}\frac{d±_n(t)}{dt}+\frac{¥_{sf}^2+£_sR_3(t)}{1+£_sR_1(t)}±_n(t)+\frac{£_s}{1+£_sR_1(t)}\sum_{\substack{q=1\\q\neq n}}^{\infty}\left[R_1(t)\frac{d^2±_q(t)}{dt^2}+R_2(t)\frac{d±_q(t)}{dt}\right. \quad (59)$$
$$\left.+R_3(t)±_q(t)\right]=\frac{£_sgL_XL_Y}{[1+£_sR_1(t)]«}\Psi_{pi}(ct)\Psi_{pj}(s)$$

where

$$R_1(t)=\frac{2}{P^*}\left[\frac{P_3^*}{2}+\sum_{k=1}^{\infty}\cos\frac{k®s}{L_Y}P_3^{**}(k)+\sum_{j=1}^{\infty}\cos\frac{j®ct}{L_X}P_3^{***}(j)+2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j®ct}{L_X}\cos\frac{k®s}{L_Y}P_3^{****}(j,k)\right] \quad (60)$$

$$R_2(t)=\frac{2c}{P^*}\left[\frac{P_4^*}{2}+\sum_{k=1}^{\infty}\cos\frac{k®s}{L_Y}P_4^{**}(k)+\sum_{j=1}^{\infty}\cos\frac{j®ct}{L_X}P_4^{***}(j)+2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j®ct}{L_X}\cos\frac{k®s}{L_Y}P_4^{****}(j,k)\right] \quad (61)$$

$$R_3(t)=\frac{2c^2}{P^*}\left[\frac{P_5^*}{2}+\sum_{k=1}^{\infty}\cos\frac{k®s}{L_Y}P_5^{**}(k)+\sum_{j=1}^{\infty}\cos\frac{j®ct}{L_X}P_5^{***}(j)+2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j®ct}{L_X}\cos\frac{k®s}{L_Y}P_5^{****}(j,k)\right] \quad (62)$$

Going through the same arguments and analysis as in the previous case, considering the homogeneous part of equation (59), the modified frequency corresponding to the frequency of the free system due to the presence of the moving mass M is obtained and an equivalent free system operator defined by the modified frequency then replaces equation (59).
Thus, equation (59) becomes

$$\frac{d^2 \pm_n(t)}{dt^2} + \beta_{sf}^2 \pm_n(t) = \frac{\mu_0 g L_X L_Y}{\mu} \Psi_{pi}(ct)\Psi_{pj}(s) \tag{63}$$

where ε_s has been written as a function of the mass ratio μ_0 and

$$\beta_{sf} = \gamma_{sf}\left[1 - \frac{\mu_0}{2}\left(R_1 - \frac{R_3}{\gamma_{sf}^2}\right)\right] \tag{64}$$

is the modified frequency corresponding to the frequency of the free system due to the presence of moving mass. Here, it is remarked that this modified frequency has in it the effects of the shear modulus of the foundation and rotatory inertia. It is observed that when $\mu_0 = 0$ in equation (64), we recover the frequency of the moving force problem of the same dynamical system.
Using (35), equation (63) becomes

$$\frac{d^2 \pm_n(t)}{dt^2} + \beta_{sf}^2 \pm_n(t) = G_g \Psi_{pj}(s)[\sin \sim_{pi} t + A_{pi}\cos \sim_{pi} t + B_{pi}\sinh \sim_{pi} t + C_{pi}\cosh \sim_{pi} t] \tag{65}$$

where $G_g = \frac{\mu_0 g L_X L_Y}{\mu}$ (66)

It is noticed that equation (65) is analogous to equation (54) with β_{sf} and G_g replacing γ_{sf} and K_m respectively. Therefore, one obtains

$$\begin{aligned} Z(x,y,t) = \sum_{ni=1}^{\infty}\sum_{nj=1}^{\infty} & \frac{G_g \Psi_{pj}(s)}{\beta_{sf}[\beta_{sf}^4 - \sim_{pi}^4]} \{[\beta_{sf}^2 - \sim_{pi}^2][C_{pi}\beta_{sf}(\cosh \sim_{pi} t - \cos\beta_{sf} t) \\ & + B_{pi}(\beta_{sf}\sinh \sim_{pi} t - \sim_{pi}\sin\beta_{sf} t)] + [\beta_{sf}^2 + \sim_{pi}^2][A_{pi}\beta_{sf}(\cos \sim_{pi} t - \cos\beta_{sf} t) \\ & - (\sim_{pi}\sin\beta_{sf} t - \beta_{sf}\sin \sim_{pi} t)]\}[\sin\frac{\Omega_{ni}x}{L_X} + A_{ni}\cos\frac{\Omega_{ni}x}{L_X} + B_{ni}\sinh\frac{\Omega_{ni}x}{L_X} \\ & + C_{ni}\cosh\frac{\Omega_{ni}x}{L_X}][\sin\frac{\Omega_{nj}y}{L_Y} + A_{nj}\cos\frac{\Omega_{nj}y}{L_Y} + B_{nj}\sinh\frac{\Omega_{nj}y}{L_Y} + C_{nj}\cosh\frac{\Omega_{nj}y}{L_Y}] \end{aligned} \tag{67}$$

Equation (67) is the transverse displacement response to a moving mass of a rectangular plate resting on variable Pasternak elastic foundation and having arbitrary edge supports. The constants A_{ni}, A_{pi}, A_{nj}, A_{pj}, B_{ni}, B_{pi}, B_{nj}, B_{pj}, C_{ni}, C_{pi}, C_{nj} and C_{pj} are to be determined from the choice of the end support condition.

4.0 Discussion of the Analytical Solutions

Here, we shall examine the phenomenon of resonance. From equation (56), the rectangular plate on a variable Pasternak elastic foundation and traversed by a moving force encounters a resonance effect when

$$\gamma_{sf} = \frac{\Omega_{pi} c}{L_X} \tag{68}$$

while equation (67) reveals that the same plate under the action of a moving mass reaches the state of resonance whenever

$$\beta_{sf} = \frac{\Omega_{pi} c}{L_X} \tag{69}$$

where $\beta_{sf} = \gamma_{sf}\left[1 - \frac{\mu_0}{2}\left(R_1 - \frac{R_3}{\gamma_{sf}^2}\right)\right]$ (70)

Equations (68) and (69) imply

$$\gamma_{sf}\left[1 - \frac{\mu_0}{2}\left(R_1 - \frac{R_3}{\gamma_{sf}^2}\right)\right] = \frac{\Omega_{pi} c}{L_X} \tag{71}$$

Consequently, for the same natural frequency, the critical speed (and the natural frequency) for the moving mass problem is smaller than that of the moving force problem. Thus, resonance is reached earlier in the moving mass system than in the moving force system.

5.0 Illustrative Examples

a. Rectangular plate elastically supported at edges y = 0, y = L_Y with simple support at edges x = 0, x = L_X.

At x = 0 and x = L_X, the plate is taken to be simply supported and at the edges y = 0 and y = L_Y, it is taken to be elastically supported.

Using the conditions (4-11) in equations (35) and (36), the following values of the constants and the frequency equation are obtained for the elastic edges.

$$C_{nj} = \frac{\left[\frac{\Omega_{nj}}{L_Y} - k_1 r_2\right]\sin\Omega_{nj} + \left[k_1 + \frac{r_2\Omega_{nj}}{L_Y}\right]\cos\Omega_{nj} - \frac{r_1\Omega_{nj}}{L_Y}\sinh\Omega_{nj} + k_1 r_1\cosh\Omega_{nj}}{k_1 r_1\sin\Omega_{nj} - \frac{r_1\Omega_{nj}}{L_Y}\cos\Omega_{nj} + \left[\frac{r_3\Omega_{nj}}{L_Y} - k_1\right]\sinh\Omega_{nj} + \left[\frac{\Omega_{nj}}{L_Y} - k_1 r_3\right]\cosh\Omega_{nj}}$$

$$= \frac{-\left[\frac{r_2\Omega_{nj}^3}{L_Y^3} + k_2\right]\sin\Omega_{nj} + \left[\frac{\Omega_{nj}^3}{L_Y^3} - k_2 r_2\right]\cos\Omega_{nj} - k_2 r_1\sinh\Omega_{nj} - \frac{r_1\Omega_{nj}^3}{L_Y^3}\cosh\Omega_{nj}}{\frac{r_1\Omega_{nj}^3}{L_Y^3}\sin\Omega_{nj} + k_2 r_1\cos\Omega_{nj} + \left[\frac{\Omega_{nj}^3}{L_Y^3} + k_2 r_3\right]\sinh\Omega_{nj} + \left[\frac{r_3\Omega_{nj}^3}{L_Y^3} + k_2\right]\cosh\Omega_{nj}}, \quad (72)$$

$$A_{nj} = r_1 C_{nj} + r_2 \;\; and \;\; B_{nj} = r_3 C_{nj} + r_1 \quad (73)$$

where

$$r_1 = \frac{\frac{\Omega_{nj}^4}{L_Y^4} + k_1 k_2}{\frac{\Omega_{nj}^4}{L_Y^4} - k_1 k_2}; \quad r_2 = \frac{-\frac{2k_1\Omega_{nj}^3}{L_Y^3}}{\frac{\Omega_{nj}^4}{L_Y^4} - k_1 k_2} \;\; and \;\; r_3 = \frac{\frac{-2k_2\Omega_{nj}}{L_Y}}{\frac{\Omega_{nj}^4}{L_Y^4} - k_1 k_2}.$$

Equation (72) when simplified yields

$$\tan\Omega_{nj} = \tanh\Omega_{nj} \quad (74)$$

which is termed the frequency equation for the elastic edge, such that

$$\Omega_1 = 3.927, \; \Omega_2 = 7.069, \; \Omega_3 = 10.210, \ldots \quad (75)$$

For the simple edges, it can be shown that

$A_{ni} = 0$, $B_{ni} = 0$, $C_{ni} = 0$, and $\Omega_{ni} = n_i\pi$ (76)

Similarly, $A_{pi} = 0$, $B_{pi} = 0$, $C_{pi} = 0$, and $\Omega_{pi} = p_i\pi$ (77)

Using (72), (73), (75), (76) and (77) in equations (56) and (67) one obtains the displacement response respectively to a moving force and a moving mass of a simple-elastic rectangular plate resting on a variable Pasternak elastic foundation.

b. Elastic support at all edges.

Using the conditions (12-19) in equations (35) and (36), one obtains

$$C_{ni} = \frac{\left[\frac{\Omega_{ni}}{L_X} - k_1 r_2(i)\right]\sin\Omega_{ni} + \left[k_1 + \frac{r_2(i)\Omega_{ni}}{L_X}\right]\cos\Omega_{ni} - \frac{r_1(i)\Omega_{ni}}{L_X}\sinh\Omega_{ni} + k_1 r_1(i)\cosh\Omega_{ni}}{k_1 r_1(i)\sin\Omega_{ni} - \frac{r_1(i)\Omega_{ni}}{L_X}\cos\Omega_{ni} + \left[\frac{r_3(i)\Omega_{ni}}{L_X} - k_1\right]\sinh\Omega_{ni} + \left[\frac{\Omega_{ni}}{L_X} - k_1 r_3(i)\right]\cosh\Omega_{ni}}$$

$$= \frac{-\left[\frac{r_2(i)\Omega_{ni}^3}{L_X^3} + k_2\right]\sin\Omega_{ni} + \left[\frac{\Omega_{ni}^3}{L_X^3} - k_2 r_2(i)\right]\cos\Omega_{ni} - k_2 r_1(i)\sinh\Omega_{ni} - \frac{r_1(i)\Omega_{ni}^3}{L_X^3}\cosh\Omega_{ni}}{\frac{r_1(i)\Omega_{ni}^3}{L_X^3}\sin\Omega_{ni} + k_2 r_1(i)\cos\Omega_{ni} + \left[\frac{\Omega_{ni}^3}{L_X^3} + k_2 r_3(i)\right]\sinh\Omega_{ni} + \left[\frac{r_3(i)\Omega_{ni}^3}{L_X^3} + k_2\right]\cosh\Omega_{ni}}, \quad (78)$$

$$A_{ni} = r_1(i) C_{ni} + r_2(i) \;\; and \;\; B_{ni} = r_3(i) C_{ni} + r_1(i) \quad (79)$$

Where

$$r_1(i) = \frac{\frac{\Omega_{ni}^4}{L_X^4} + k_1 k_2}{\frac{\Omega_{ni}^4}{L_X^4} - k_1 k_2}; \quad r_2(i) = \frac{-\frac{2k_1\Omega_{ni}^3}{L_X^3}}{\frac{\Omega_{ni}^4}{L_X^4} - k_1 k_2} \ \ and \ \ r_3(i) = \frac{\frac{-2k_2\Omega_{ni}}{L_X}}{\frac{\Omega_{ni}^4}{L_X^4} - k_1 k_2}.$$

and

$$C_{nj} = \frac{\left[\frac{\Omega_{nj}}{L_Y} - k_1 r_2(j)\right]\sin\Omega_{nj} + \left[k_1 + \frac{r_2(j)\Omega_{nj}}{L_Y}\right]\cos\Omega_{nj} - \frac{r_1(j)\Omega_{nj}}{L_Y}\sinh\Omega_{nj} + k_1 r_1(j)\cosh\Omega_{nj}}{k_1 r_1(j)\sin\Omega_{nj} - \frac{r_1(j)\Omega_{nj}}{L_Y}\cos\Omega_{nj} + \left[\frac{r_3(j)\Omega_{nj}}{L_Y} - k_1\right]\sinh\Omega_{nj} + \left[\frac{\Omega_{nj}}{L_Y} - k_1 r_3(j)\right]\cosh\Omega_{nj}} \tag{80}$$

$$= \frac{-\left[\frac{r_2(j)\Omega_{nj}^3}{L_Y^3} + k_2\right]\sin\Omega_{nj} + \left[\frac{\Omega_{nj}^3}{L_Y^3} - k_2 r_2(j)\right]\cos\Omega_{nj} - k_2 r_1(j)\sinh\Omega_{nj} - \frac{r_1(j)\Omega_{nj}^3}{L_Y^3}\cosh\Omega_{nj}}{\frac{r_1(j)\Omega_{nj}^3}{L_Y^3}\sin\Omega_{nj} + k_2 r_1(j)\cos\Omega_{nj} + \left[\frac{\Omega_{nj}^3}{L_Y^3} + k_2 r_3(j)\right]\sinh\Omega_{nj} + \left[\frac{r_3(j)\Omega_{nj}^3}{L_Y^3} + k_2\right]\cosh\Omega_{nj}},$$

$$A_{nj} = r_1(j)C_{nj} + r_2(j) \ \ and \ \ B_{nj} = r_3(j)C_{nj} + r_1(j) \tag{81}$$

where

$$r_1(j) = \frac{\frac{\Omega_{nj}^4}{L_Y^4} + k_1 k_2}{\frac{\Omega_{nj}^4}{L_Y^4} - k_1 k_2}; \quad r_2(j) = \frac{-\frac{2k_1\Omega_{nj}^3}{L_Y^3}}{\frac{\Omega_{nj}^4}{L_Y^4} - k_1 k_2} \ \ and \ \ r_3(j) = \frac{\frac{-2k_2\Omega_{nj}}{L_Y}}{\frac{\Omega_{nj}^4}{L_Y^4} - k_1 k_2}.$$

Equations (78) and (80) when simplified yield

$$\tan\Omega_{ni} = \tanh\Omega_{ni} \tag{82}$$

and

$$\tan\Omega_{nj} = \tanh\Omega_{nj} \tag{83}$$

Using (78), (79), (80), (81), (82) and (83) in equations (56) and (67) one obtains the transverse-displacement response respectively to a moving force and a moving mass of an elastically supported rectangular plate resting on a variable Pasternak elastic foundation.

6.0 Numerical Calculations and Discussion of Results

For the calculations of practical interests in dynamics of structures and engineering design for the elastically supported plate resting on variable Pasternak elastic foundation, a rectangular plate of length $L_Y = 0.914$m and breadth $L_X = 0.457$m is considered. It is assumed that the mass travels at the constant velocity 0.8123m/s. The values for E, S and Γ are chosen to be $2.109\text{x}10^9$kg/m^2, 0.4m and 0.2 respectively. For various values of the foundation modulus F_0, Shear modulus G_0 and the rotatory inertia correction factor R_0, the deflections of the elastically supported plate are calculated and plotted against time t.

a. Simple – elastic rectangular plate on variable Pasternak foundation.

Figures 6.1 – 6.3 present the responses of the plate simply supported at the edges x = 0 and x = L_X and elastically supported at the edges y = 0 and y = L_Y.

Figure 6.1 displays the effect of rotatory inertia correction factor R_0 on the transverse deflection of moving force for simple–elastic rectangular plate, while Figure 6.2 displays the effect of Shear modulus G_0 on the transverse displacement of moving mass for simple–elastic plate. It is shown that as both R_0 and G_0 increase the amplitude of the deflection decreases respectively for the simple-elastic rectangular plate resting on variable Pasternak elastic foundation.

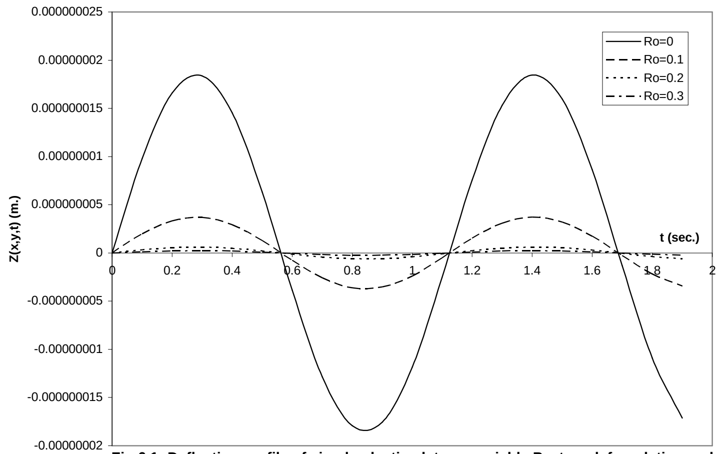

Fig.6.1: Deflection profile of simple-elastic plate on variable Pasternak foundation and traversed by moving force for various values of Ro.

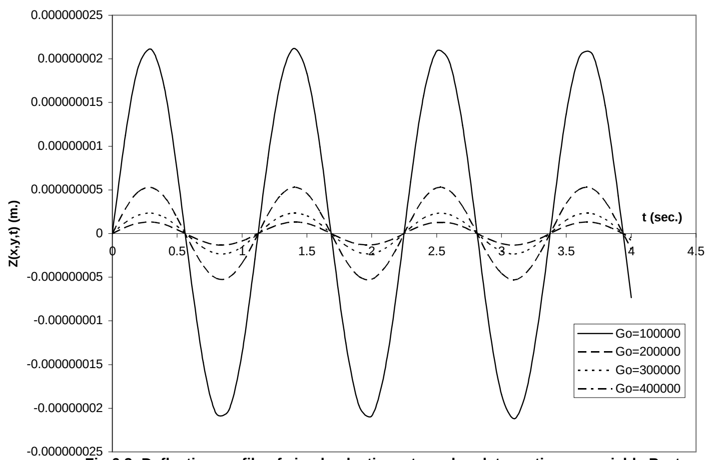

Fig.6.2: Deflection profile of simple-elastic rectangular plate resting on variable Pasternak foundation and traversed by moving mass for various values of Go.

For the purpose of comparison, Figure 6.3 compares the displacement curves of moving force and moving mass for the simple – elastic plate for fixed F_0, G_0 and R_0. It is evident from the graph that the response amplitude of a moving mass is greater than that of a moving force problem.

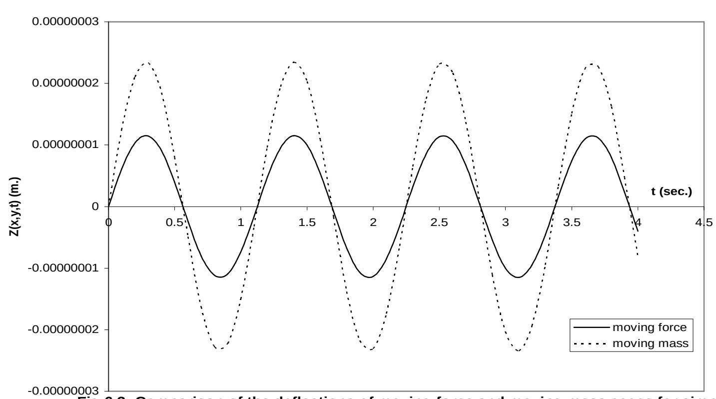

Fig.6.3: Comparison of the deflections of moving force and moving mass cases for simple-elastic rectangular plate resting on variable Pasternak foundation.

b. Elastically supported rectangular plate on variable Pasternak foundation.

The responses of the plate elastically supported at all its edges are presented in Figures 6.4 – 6.6. It is observed in Figures 6.4 and 6.5 that as the values of G_0 and R_0 increase the deflection amplitude of the plate decreases for both cases of moving force and moving mass respectively. Figure 6.6 compares the displacement response of the moving force and moving mass for an elastically supported rectangular plate for fixed values of F_0, G_0 and R_0. It is evident that the displacement response of the moving mass problem is greater than that of the moving force problem.

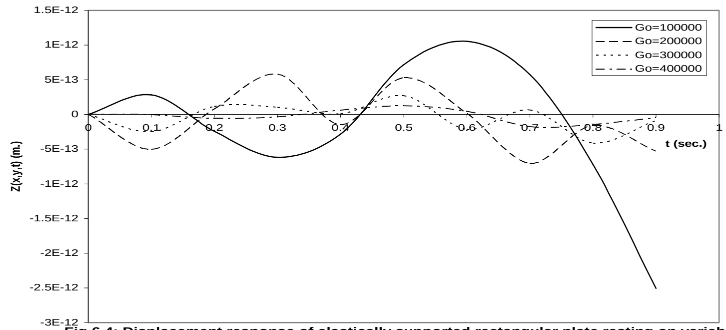

Fig.6.4: Displacement response of elastically supported rectangular plate resting on variable Pasternak foundation and traversed by moving force for various values of Go.

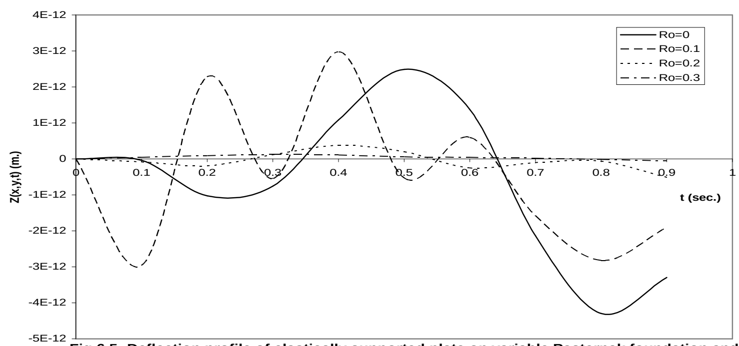

Fig.6.5: Deflection profile of elastically supported plate on variable Pasternak foundation and traversed by moving mass for various values of Ro.

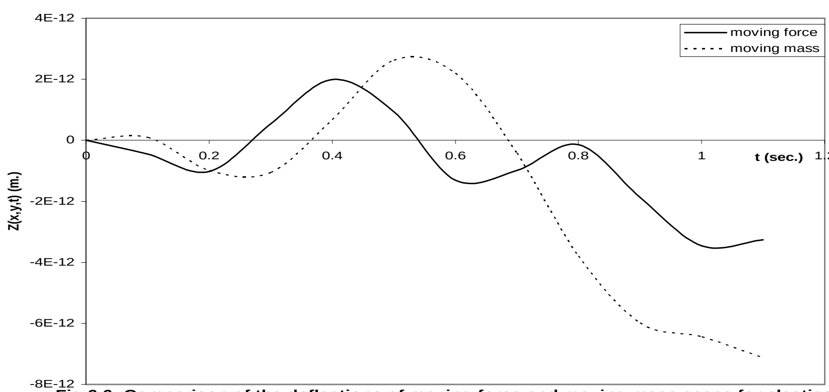

Fig.6.6: Comparison of the deflections of moving force and moving mass cases for elastically supported rectangular plate resting on variable Pasternak foundation.

7.0 Conclusion

In this work, the problem of the dynamic response to moving concentrated masses of elastically supported rectangular plates on variable Pasternak elastic foundations has been studied. The closed form solutions of the fourth order partial differential equations with variable and singular coefficients governing the rectangular plate is obtained for both cases of moving force and moving mass using a solution technique that is based on the separation of variables which was used to remove the singularity in the governing fourth order partial differential equation and to reduce it to a sequence of coupled second order differential equations. The modified Struble's asymptotic technique and the methods of integral transformation are then employed to obtain the analytical solution of the two-dimensional dynamical problem. These solutions are analyzed and resonance conditions are obtained for the problem.

The analyses carried out show that, for the same natural frequency, the critical speed (and the natural frequency) for the moving mass problem is smaller than that of the moving force problem. Thus, resonance is reached earlier in the moving mass system than in the moving force system. Thus, the moving force solution is not an upper bound for the accurate solution of the moving mass problem.

The results in plotted curves show that as the rotatory inertia correction factor increases, the response amplitudes of the plates decrease for both cases of moving force and moving mass problem. When the rotatory inertia correction factor is fixed, the displacements of the elastically supported rectangular plates resting on the variable Pasternak elastic foundations decrease as the shear modulus increases. The effect of shear modulus is more noticeable than that of the foundation modulus.

It is shown further from the results that, for fixed values of rotatory inertia correction factor, foundation modulus and shear modulus, the response amplitude for the moving mass problem is greater than that of the moving force problem implying that resonance is reached earlier in moving mass problem than in moving force problem of the elastically supported rectangular plate resting on variable Pasternak elastic foundation. Also, an increase in the shear modulus results in an increase in the critical speed of the moving load; this shows that risk is reduced when the shear modulus increases. The same result obtains for an increase in both foundation modulus and rotatory inertial correction factor.

8.0 References

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