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A New Hybrid Block Method for the Solution of Linear Second Order Ordinary Differential Equations

K.M. Fasasi, A.O. Adesanya and A.A Momoh

Department of Mathematics, Modibbo Adama University of Technology, P.M.B 2076, Yola, Adamawa State, Nigeria.

Abstract

A new block method for the solution of linear second order initial value problems of ordinary differential equations was developed. The method was developed using interpolation and collocation of power series approximate solution to generate a continuous linear multistep method which was evaluated at some selected grid and off grid points to give the discrete linear multistep method. The block method was augmented by the introduction of off grid points so as to circumvent Dahlquist zero stability barrier and upgrade the order of consistency of the methods. The basic properties of the block method were investigated and were found to be zero stable, consistent and convergent. Results from the numerical experiments on linear second order ordinary differential equations revealed that the performance of the developed method is better in terms of error reduction and converges more closely to the exact solution. The results also re-affirmed that hybrid methods gave better approximation especially when the step is lower than higher k step method.

Keywords: consistent, convergent, collocation, hybrid points, interpolation, zero stable.
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1.0 Introduction

Consider the second order initial value problem of the form

$$y'' = f(x, y(x), y'(x)) \quad y(x_0) = y_0,\ y'(x_0) = y'_0, \qquad (1)$$

Where x_0 the initial point and f is continuous within the interval of integration and satisfies the existence and uniqueness conditions.

Many complex problems emanating from physical sciences and Engineering give rise to differential equations. Some of these problems are due to natural phenomenon, they do not have a closed solution, hence analytical approach may not be able appropriate to solve them. The alternative means is to use approximate solution by the application of numerical methods.

The second order ordinary differential equation of the form (1) is solved in the past by first reducing it to a system of first order differential equations and then applying different available methods for solving systems of first order Initial Value Problems (IVPs). This approach had been reported by several scholars [1, 2]. They asserted that the method is quite good but has a lot of problems associated with it. For instance, writing computer program is tedious especially when subroutines are incorporated to supply the starting values required for the method. This usually results to longer computer time and more computational work. It has also been reported that this method does not utilize additional information associated with specific ordinary differential equation, such as the oscillatory nature of the solution.

Researchers have been making efforts to develop a direct method that could solve higher order ordinary differential equations without necessarily reducing them to system of first order before solving them[3-6]. The schemes developed by these scholars can only handle some special ordinary differential equations and the level of accuracy obtained is not high after testing them on some ODEs.

Corresponding author: K.M. Fasasi, E-mail: kolawolefasasi@yahoo.com, Tel.: +2348032090578

In order to cater for these setbacks, several scholars;[2, 7-13] proposed block method which has a lot of advantages in terms of less computational cost, self-starting and can solve both special and general ordinary differential equations. It is also more accurate than some existing methods. In this paper, we propose a new hybrid block method for the solution of linear second order ordinary differential equations; the method was derived through the use of collocation and interpolation of power series approximate solution.

2.0 Methodology

We consider a power series approximate solution in the form:

$$y(x)=\sum_{j=0}^{r+s-1} a_j x^j \qquad (2)$$

where r and s are the numbers of interpolation and collocation points respectively.
The second derivative of (2) gives

$$y''(x)=\sum_{j=2}^{r+s-1} j(j-1)a_j x^{j-2} \qquad (3)$$

Substituting (3) into (1) gives

$$f(x,y,y')=\sum_{j=2}^{r+s-1} j(j-1)a_j x^{j-2} \qquad (4)$$

Interpolating (2) at the $x_{n+r,}\, r=\frac{4}{6},\frac{5}{6}$ and collocating (4) at $x_{n+s,}\, s=0,\frac{2}{6},\frac{4}{6},1$

gives a system of nonlinear systems of equation in the form

$$AX=U \qquad (5)$$

where

$$A=[a_0,a_1,a_2,a_3,a_4,a_5]^T$$

$$U=\left[y_{n+\frac{4}{6}},y_{n+\frac{5}{6}},f_n,f_{n+\frac{1}{4}},f_{n+\frac{2}{6}},f_{n+\frac{4}{6}},f_{n+1}\right]^T$$

$$X=\begin{bmatrix}1 & x_{n+\frac{2}{3}} & x^2_{n+\frac{2}{3}} & x^3_{n+\frac{2}{3}} & x^4_{n+\frac{2}{3}} & x^5_{n+\frac{2}{3}}\\ 1 & x_{n+\frac{5}{6}} & x^2_{n+\frac{5}{6}} & x^3_{n+\frac{5}{6}} & x^4_{n+\frac{5}{6}} & x^5_{n+\frac{5}{6}}\\ 0 & 0 & 2 & 6x_n & 12x^2_n & 20x^3_n\\ 0 & 0 & 2 & 6x_{n+\frac{1}{3}} & 12x^2_{n+\frac{1}{3}} & 20x^3_{n+\frac{1}{3}}\\ 0 & 0 & 2 & 6x_{n+\frac{2}{3}} & 12x^2_{n+\frac{2}{3}} & 20x^3_{n+\frac{2}{3}}\\ 0 & 0 & 2 & 6x_{n+1} & 12x^2_{n+1} & 20x^3_{n+1}\end{bmatrix}$$

Solving (5) for the unknown constants $a_j's$ and substituting into(2) gives a continuous linear multistep method which when solved for the independent solution at the grid points gives a continuous block formula of the form.

$$y(x)=\sum_{m=0}^{1}\frac{(jh)^{(m)}}{m!}y_n^{(m)}+h^2\sum_{j=0}^{1}\pm_j f_{n+j}+\pm_{\frac{1}{3}} f_{n+\frac{1}{3}}+\pm_{\frac{2}{3}} f_{n+\frac{2}{3}} \qquad (6)$$

the coefficient of f_{n+j} are given by

$$\sim_0=\frac{-1}{120}(27t^5-90t^4+110t^3-60t^2$$

$$\sim_{\frac{1}{3}}=\frac{1}{40}(27t^5-75t^4+60t^3)$$

$$\sim_{\frac{2}{3}} = \frac{-1}{40}(27t^5 - 60t^4 + 30t^3)$$

$$\sim_1 = \frac{1}{120}(27t^5 - 45t^4 + 20t^3)$$

evaluating (6) at $t = 0, \frac{1}{3}, \frac{2}{3}, 1$ gives a discrete block formula of the form

$$A^{(0)}Y_m^{(i)} = \sum_i^1 e_i y_n^{(i)} + h^{2-i}\left[df(y_n) + bF(Y_m)\right]$$

Where

i is the power of the derivatives

$A^0 = 6 \times 6$ identity matrix

$$Y_m = \left[y_{n+\frac{1}{6}}, y_{n+\frac{1}{3}}, y_{n+\frac{1}{2}}, y_{n+\frac{2}{3}}, y_{n+\frac{5}{6}}, y_{n+1} \right]^T$$

$$y_n^{(i)} = \left[y_{n-\frac{1}{6}}, y_{n-\frac{1}{3}}, y_{n-\frac{1}{2}}, y_{n-\frac{2}{3}}, y_{n-\frac{5}{6}}, y_n \right]^T$$

$$F(Y_m) = \left[f_n, f_{n+\frac{1}{6}}, f_{n+\frac{1}{3}}, f_{n+\frac{1}{2}}, f_{n+\frac{2}{3}}, f_{n+\frac{5}{6}}, f_{n+1} \right]^T$$

$$f(y_n) = \left[f_n, f_{n-\frac{1}{6}}, f_{n-\frac{1}{3}}, f_{n-\frac{1}{2}}, f_{n-\frac{2}{3}}, f_{n-\frac{5}{6}}, f_{n-1} \right]^T$$

when $i = 0$

$$e_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad e_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{5}{6} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$d_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1057}{103680} \\ 0 & 0 & 0 & 0 & 0 & \frac{97}{3240} \\ 0 & 0 & 0 & 0 & 0 & \frac{193}{3840} \\ 0 & 0 & 0 & 0 & 0 & \frac{28}{405} \\ 0 & 0 & 0 & 0 & 0 & \frac{1825}{20736} \\ 0 & 0 & 0 & 0 & 0 & \frac{13}{120} \end{bmatrix}, \quad b_0 = \begin{bmatrix} 0 & \frac{193}{34560} & 0 & \frac{-83}{34560} & 0 & \frac{53}{106680} \\ 0 & \frac{19}{540} & 0 & \frac{-13}{1080} & 0 & \frac{1}{405} \\ 0 & \frac{117}{1280} & 0 & \frac{-27}{1280} & 0 & \frac{17}{3840} \\ 0 & \frac{22}{135} & 0 & \frac{-2}{135} & 0 & \frac{-2}{405} \\ 0 & \frac{1625}{6912} & 0 & \frac{125}{6912} & 0 & \frac{125}{20736} \\ 0 & \frac{3}{10} & 0 & \frac{3}{40} & 0 & \frac{1}{60} \end{bmatrix},$$

When $i=1$

$$e_1=\begin{bmatrix}0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\end{bmatrix},\quad d_1=\begin{bmatrix}0&0&0&0&0&\frac{119}{1152}\\0&0&0&0&0&\frac{1}{8}\\0&0&0&0&0&\frac{15}{128}\\0&0&0&0&0&\frac{1}{9}\\0&0&0&0&0&\frac{15}{128}\\0&0&0&0&0&\frac{1}{8}\end{bmatrix},\quad b_1=\begin{bmatrix}0&\frac{107}{1152}&0&\frac{-43}{1152}&0&\frac{1}{128}\\0&\frac{19}{72}&0&\frac{-5}{72}&0&\frac{1}{72}\\0&\frac{51}{128}&0&\frac{-3}{128}&0&\frac{1}{128}\\0&\frac{4}{9}&0&\frac{1}{9}&0&0\\0&\frac{475}{1152}&0&\frac{325}{1152}&0&\frac{25}{1152}\\0&\frac{3}{8}&0&\frac{3}{8}&0&\frac{1}{8}\end{bmatrix}$$

3.0 Analysis of Our New Schemes

4.0 Order of the block

Let the linear operator associated with the block method be defined as

$$\mathrm{L}\{y(x):h\} = A^0 y_m^{(i)} - \sum_{i=0}^{1-i} h^i e_i y_n^{(i)} - h^{2-i}\left[df(y_n)+bF(y_m)\right] \qquad (7)$$

Expanding y_{n+j} and f_{n+j} in Taylor series expansion and comparing the coefficient of h gives

$$L\{y(x):h\} = c_0 y(x) + c_1 y'(x) + ... + c_p h^p y^p(x) + c_{p+1}h^{p+1}y^{p+1}(x) + c_{p+2}h^{p+2}y^{p+2}(x) + ...$$

Definition1

The linear operator L and associated block method are said to be of order p if $c_0 = c_1 = c_2 = ... = c_{p+1} = 0,$ $c_{p+2} \neq 0$ c_{p+2} is called the error constant and implies that the truncation error is given by $t_{n+k} = c_{p+2}h^{p+2}y^{p+2}(x) + O(h^{p+3}).$ Comparing the coefficient of h, the order of the method is five with error constant $c_0 = c_1 = c_2 = c_3 = c_4 = c_5 = 0,$

$$c_6 = \left[\frac{-49}{11197440}, \frac{-7}{349920}, \frac{-1}{27640}, \frac{-1}{21870}, \frac{-125}{2239488}, \frac{-1}{12960}\right],$$

5.0 Consistency

A method is said to be consistent, if it has order greater than one. From the above analysis, it is obvious that our method is consistent.

6.0 Zero Stability

A block method is said to be zero stable as $h \to 0$ the root $r_j, j = 1(1)k$ of the first characteristics polynomial $\circ(r) = 0$ that is $\left|\sum A^0 \Re^{K-1}\right| \leq 1$ for those root with $|\Re| = 1,$ must be simple.

For our method

$$\circ(r) = \left| r\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix} - \begin{bmatrix}0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\end{bmatrix}\right| = 0 \;;$$

$$r^5(r-1) = 0$$

Hence our method gives roots [0, 0, 0, 0, 0, 1] therefore it is zero stable.

7.0 Convergence

Definition: The necessary and sufficient conditions for a linear multistep method to be convergent are that it must be consistent and zero stable. Hence our method is convergent**.**

8.0 Numerical Examples

In this section, we test the efficiency of our method on some numerical examples

Problem 1. $y'' = -y \;\; y(0) = 1, y'(0) = 0$

Exact solution: $y(x) = \cos x$

Source**: [** 14]

Problem 2. $y'' = y \;\; y(0) = 1, y'(0) = -1$

Exact solution**:** $y(x) = \exp(-x), x \geq 0$

Source: **[** 14]

Error=|Exact result- Computed result|

NM=Error in New Method

Table 1: Comparison of Absolute Errors for Problem 1

x	Y_{exact}	$h = 0.2$ Error in [14]	NM	$h = 0.1$ Error in[14]	NM	$h = 0.05$ Errorin[14]	NM
0.2	0.980067	2.008×10^{-2}	6.346×10^{-11}	1.500×10^{-2}	2.505×10^{-12}	1.255×10^{-2}	5.085×10^{-14}
0.4	0.921061	8.029×10^{-2}	6.362×10^{-10}	6.729×10^{-2}	1.285×10^{-11}	7.489×10^{-2}	2.234×10^{-13}
0.6	0.825336	1.696×10^{-1}	1.680×10^{-09}	1.605×10^{-1}	3.028×10^{-11}	1.624×10^{-1}	5.036×10^{-13}
0.8	0.696707	2.958×10^{-1}	3.119×10^{-09}	2.916×10^{-1}	5.343×10^{-11}	3.168×10^{-1}	8.696×10^{-13}
1.0	0.540302	4.552×10^{-1}	4.842×10^{-09}	4.538×10^{-1}	8.044×10^{-11}	3.168×10^{-1}	1.292×10^{-12}

Table 2: Comparison ofAbsolute Errors for Problem 2

x	Y_{exact}	$h = 0.2$ Error in [14]	NM	$h = 0.1$ Error in[14]	NM	$h = 0.05$ Errorin[14]	NM
0.2	0.818731	3.000×10^{-5}	2.243×10^{-11}	1.000×10^{-5}	1.839×10^{-12}	8.000×10^{-7}	3.930×10^{-14}
0.4	0.670320	6.000×10^{-5}	5.401×10^{-10}	1.000×10^{-5}	1.092×10^{-11}	1.400×10^{-6}	1.881×10^{-13}
0.6	0.548811	1.100×10^{-4}	1.484×10^{-09}	2.000×10^{-5}	2.630×10^{-11}	1.600×10^{-6}	4.330×10^{-13}
0.8	0.449329	1.840×10^{-3}	2.819×10^{-09}	5.000×10^{-5}	4.753×10^{-11}	3.900×10^{-5}	7.669×10^{-13}
1.0	0.367879	1.842×10^{-3}	4.538×10^{-09}	3.390×10^{-3}	7.461×10^{-11}	5.394×10^{-4}	1.191×10^{-12}

9.0 Discussion of Result

We have considered two numerical examples to test the efficiency of our method. Problems 1 and 2 were solved in [14]. They proposed Nystrom Method for direct solution of initial value problem of second order ordinary differential equation to solve the two problems we considered. Nystrom method is a single step method for solving initial value problem for second order ordinary differential equation [6]. Our method gave better approximation as shown in Tables 1 and 2 despite the higher order methods they proposed.

10.0 Conclusion

We have proposed a new hybrid block method for the solution of linear second order initial value problems which was implemented in continuous block method.Continuous block method has advantage of evaluation at all selected points within the interval of integration. The results show that our method gave better approximation than the existing methods that we compared our results with.

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