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Comparison of Numerical Solution of Some First Order Differential Equations (FODE's) with Initial Value Problems (IVP's) Using Picard, Euler and Modified Euler Methods (PEMEM)

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Abstract

Differential equations can be solved using many methods that are generally accepted in Mathematics. However, it is believed that one method should be more accurate, efficient, sufficient and unique than the other. Thus; solutions of First order Differential Equations (FOD's) with Initial Value Problems (IVP's) by the three different methods; Picard, Euler and Modified Euler Methods (PEMEM) will be exercised. As such numerical computational algorithm, convergence rate, approximation errors and uniqueness will be investigated.

Keywords: First Order Differential Equation (FODE), Initial Value problem (IVP), Convergence rate, Analytical Solution, Numerical Solution, Error estimate, Picard, Euler and Modified Euler Methods (PEMEM).

1.0 Introduction

Parker and Sochacki theorem on Existence and Uniqueness states that if both $f(x,y)$ and $\frac{\partial}{\partial}$ are continuous in some region around the point (x_0, y_0) then there is a unique solution to the IVP [1]

$$\begin{cases} y' & = f(x,y) \\ y(x_0) & = y_0(I \quad) \end{cases} \tag{1}$$

Valid in some interval around x_0. In other words, if the slope field is sufficiently smooth at each point, then there is unique integral curve passing through any given point. How do we prove such a theorem? There are two methods, but both use a sequence of approximate solutions and prove that these approximations converge at least in a small interval aroundx_0. One method is due to Euler and is quite simple to use in practice: one simply ''connect the dots'' in the slope field. The disadvantage to this method is that it only gives an approximation ''at the dots''. In other words, Euler's method only approximates the values of the solution at a finite list of points. It does not give us formula for an approximate function at every point. However, Euler's method has the advantage that its accuracy can be improved with only minor modifications. Foremost applications some version of an improved Euler method is ideal. A second method is due Picard. The Picard method gives a sequence of functions which converges to the solution. Picard's method is far less efficient computationally than Eulerian methods, but it introduces an important technique that will be useful for the error analysis of Eulerian methods. An approximation method is useless without an estimate of the error. Picard's method begins by transforming the pair of conditions that are the IVP into a single integral equation. Estimates with integrals are fairly straightforward.Parker and Sochacki (2000) showed that a large class of ODE's could be converted to polynomial form using substitutions and using a system of equation. While this class of ODE's is dense in the analytic functions, it does not include all analytic functions. They also showed one can approximate the solution by a polynomial system and the resulting error bound when using these approximations [2]. Parker and Sochacki also showed that if $x_0 \neq 0$, one computes the iteration as if $x_0 = 0$ and then the approximated solution to the ODE is $y^n(x + x_0)$. This algorithm is called the modified Picard method (MPM). While the MPM algorithm easily computes the approximations, since it only depends on calculating derivatives and integrals of the underlying polynomials, it has some limitations. They also showed how to handle the PDE including the initial conditions. However, the method requires the initial conditions in polynomial form. While in some PDE's this is the case, many time one computes a Taylor polynomial that approximates the initial condition to high degree.

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This results in a substantial increase in computational time. For some problems, the initial condition is not explicitly known, but only a digitized form of the data.For example, in image processing, most of the data have already been digitized and we have to interpolate the data using polynomials in other to apply the Modified Euler Method (MEM). If this is done, the resulting polynomial may not effectively approximate the derivatives of the original function. The polynomial approximation might contain large number of oscillations that do not represent the underlying data accurately. Finally, we would also like to handle boundary conditions in a simple manner, but keep the extensibility of the Modified Euler Method (MEM), which does not allow for a boundary condition.Picard's method, sometimes called the method of successive approximations, gives a means of proving the existence of solutions to the method of DE. Emile Picard, a French Mathematician, who developed the method in the early 20th century. It has proven to be so powerful that it has replaced the Cauchy- Lipchitz method that was previously employed for such endeavours.

Picard developed his method while he was a Professor at the University of Paris. It arose out of a study involving the Picard-Lindel of existence theorem that had been formulated at the end of the 19th century. Picard's method is utilized in similar situations as those that employ the Taylor series method. It is a method that converts the differential Equation into an equation involving integrals.Some DE's are difficult to solve, but Picard's method provides a numerical process by which solution can be approximated. The method consists of constructing a sequence of functions that will approach the desired solution upon successive iteration. It is similar to the Taylor series method in that successive iterations also approach the desired solution to a DE. Picard's method allows us to find a series solution about some fixed point. The number of terms or iterations that is required to reach the desired solution depends on how far from the chosen point the solution must apply. The closer the chosen point to the known point, the fewer terms that are needed. It can be shown that the series is convergent and provides a solution to the differential equation of interest although the number of terms will depend upon how rapidly the series converges as well[3].The details of Picard's method involve starting with an initial value problem and expressing it as an integral equation. This is done by integrating both sides with respect to one variable from a defined starting point to a defined termination point, x_0 t x_1. The initial value given is substituted into the resulting integral equation. This yields the simple fraction evaluated at the initial value summed with the remaining integral, after a simple substitution and appropriate arrangements of the limits on the remaining integral, the result can be used to generate successive approximations of a solution to the initial equation. The number of iteration steps is determined by two factors; how quickly the series converges and how far away from the point of interest is the value given in the initial problem [4].The term ''Picard iteration'' occurs in two places in undergraduate mathematics. In numerical analysis it is used when discussing fixed point iteration for finding a numerical approximation to the equation$x = g(x)$. In differential equations, Picard iteration is a constructive procedure for establishing the existence of a solution to a DE $y' = f(x, y)$ that passes through the point(x_0, y_0)[5].Picard iteration is a widely used procedure for solving the nonlinear equation governing flow in variably saturated porous media. The method is simple to code and computationally cheap, but has been known to fail or converge slowly [6].Picard showed that an entire function can omit not more than one finite value without being reduced to a constant function and if there exist at least two values, each of which is taken on only a finite number of times, the function is a polynomial [7]. Otherwise the function takes on every value, other than the exceptional one, an infinite number of times. His beautiful proof of what is known as Picard's Big [8].Picard iteration is a special kind of fixed point iteration. We call x a fixed point of a function if x = f(x).Suppose a sequence is defined by:$x_{n+1} = f(x_n), x_1 = [sc\quad g\quad a\ the\ f\quad p\quad]$. Often you will find that x_n converges to a fixed point of f. The process of taking the successive terms of such a sequence is called *iteration*. We are going to apply this iterative idea to differential equations and we come up with the Picard method. Basically, we are going to apply fixed point iteration to a whole differential equation. The goal here is to use Picard method to find a solution to the given FODE with IVP of the form in (1) ODE frequently occurs as mathematical models in many branches of science, engineering and economy. Unfortunately it is seldom that these equations have solutions that can be expressed in closed form, so it is common to seek approximate solutions by means of numerical methods [10]; nowadays this can usually be achieved very inexpensively to high accuracy and with a reliable bound on the error between the analytical solution and its numerical approximation. In this section we shall be concerned with the construction and the analysis of numerical methods for FODE of the form in (1). For the real – valued function y of the real variable$x$, where $y' = \frac{d}{d}$. In oder to select a particular integral from the infinite family of solution curves that constitute the general solution to (1), the FODE will be considered in tandem with an initial condition: given two real number We seek a solution to (1) for$x > x_0 \ni \ y(x_0) = y_0$. The FODE (1) together with the IVP is called FODE with IVP. In general, even if $f(x, y)$is a continuous function, there is no guarantee that the IVP in (1) possesses a unique solution. Fortunately, under a further mild condition on the function$f$, the existence and uniqueness of a solution to (1) can be ensure: the result is encapsulated in the next theorem[11].

2.0 Material and Method

3.0 Picard's Method

This is the first method we shall consider. This deals with successive integration as we progress from one step to another.

The details of Picard's method involve starting with IVP and expressing it as an integral equation. This is done by integrating both sides with respect to one variable from a defined starting point to a defined termination point, x_0 *to* x_1. The initial value given is substituted into the resulting integral equation. This yields the function evaluated at the initial value summed with the remaining integral. After a simple substitution and appropriate arrangements of the limits on the remaining integral; the result can be used to generate successive approximations of a solution to the initial equation. The number of iteration steps is determined by two factors: how quickly the series converges and how far away from the point of interest is the value given in the initial problem[12]. The equations (equation 4) as the equivalent solution to equation (1), will be used for the values of $n = 0, 1, 2..$ in determining the solution to (1).

4.0 Picard's Method of Successive Approximation

Considering the FODE with the IVP in (1), then the solution to equation (1) is equivalently given as the integral equation:

$$y = y_0 + \int_{x_0}^{x} f\big(x, y(x)\big) d \qquad (2a)$$

$$\equiv y^{(n)}(x) = y_0 + \int_{x_0}^{x} f\big(x, y^{(n-1)}\big) d \qquad (2b)$$

Proof

In this method, the value of the dependent variable y is expressed as a function of x.

To show that (2a) and (2b) are the equivalent integral solution of equation (1) it suffices that:

Let $y' = f(x, y)$ be the FODE with IVP $y(x_0) = y_0$ ie from equation (1)

Since $y(x)$ is a differentiable function in some neighbourhood of x_0 then, $f\big(x, y(x)\big)$ is a continuous function of x in some neighbourhood of x_0. Thus; it is integrable in this neighbourhood of x_0. Now, if we integrate (1) between x_0 and x, we get :

ie $\frac{dy}{dx} = f(x, y)$ with the initial condition $x = x_0$, $y(x_0) = y_0$

$$\Rightarrow dy = f(x, y)\, dx$$

$$\Rightarrow \int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y)\, dx$$

$$\Rightarrow [y]_{y_0}^{y} = \int_{x_0}^{x} f(x, y)\, dx$$

$$\Rightarrow y - y_0 = \int_{x_0}^{x} f(x, y)\, dx$$

$$\Rightarrow y(x) = y_0 + \int_{x_0}^{x} f(x, y)\, dx \qquad (3a)$$

This equation satisfies the initial condition in equation (1) as

$$y(x_0) = y_0 + \int_{x_0}^{x} f(x, y)\, d \qquad (3b)$$

The value of y is replaced by y_0 in the RHS of equation(1) and assuming the solution is $y^{(1)}(x)$, then the first approximation of y becomes:

$$y_1(x) = y_0 + \int_{x_0}^{x} f(x, y_0)\, d \qquad (3c)$$

Again $y^{(1)}(x)$ is replaced in (3c) from (3b) and the second approximation $y^{(2)}(x)$ is obtained as:

$$y_2(x) = y_0 + \int_{x_0}^{x} f\big(x, y^{(1)}\big)\, d$$

This way, the following approximations of y are generated.

$$y_3(x) = y_0 + \int_{x_0}^{x} f\big(x, y^{(2)}\big)\, d$$

$$y_4(x) = y_0 + \int_{x_0}^{x} f\big(x, y^{(3)}\big)\, d$$

$$\text{ie: } y_n(x) = y_0 + \int_{x_0}^{x} f\big(x, y_{(n-1)}\big)\, dx \qquad (4)$$

Hence equation (3b) and (4) gives theProof of (2a)and the n^{th}iteration for the Picard's Method (PM) respectively.

Thus a sequence $y_1, y_2 \dots y_n$ of y is generated in terms of x. Equation (4) will be used for $n = 0, 1, 2..$ in determining the solution to any first order DE of the form inequation (1).

Problem 1

Use Picard's Method (PM) to solve the differential equation:

$$\frac{dy}{dx} = x^2 + y \qquad (5)$$

with initial condition

$y(0) = 0$. Also find the values of $y(0.1)$ and $y(0.2)$

Solution

From equation (5):

$$\frac{d}{d} = x^2 + y$$

by the initial condition :

$y(0) = 0.$

ie $y(0) \Rightarrow$ value of y at $x = 0$

ie $y = 0$ when $x = 0$ (Initial Condition (IC)), where

$f(x, y) = x^2 + y$ by equation (1)

So, by equation (4):

ie: $y_n(x) = y_0 + \int_{x_0}^{x} f(x, y_{(n-1)})\, dx$

Suppose n = 1 since $n \in Z^+$, then;

by equation (4) $\Rightarrow y_n(x) = y_0 + \int_{x_0}^{x} f(x, y_{(n-1)})\, dx$ becomes:

ie $y_1(x) = y_0 + \int_{x_0}^{x} f(x, y_{(1-1)})\, dx$

since $y_0 = 0$ and $x_0 = 0$ by the Initial Condition (IC) then;

$$\Rightarrow y_1(x) = 0 + \int_0^x f(x, y_{(0)})\, dx$$

$$\Rightarrow y_1(x) = y_0 + \int_0^x (x^2 + y_0)\, dx$$

$$\Rightarrow y_1(x) = 0 + \int_0^x (x^2 + 0)\, dx, \; y_0 = 0 \text{ and } x_0 = 0$$

$$= \int_0^x x^2\, dx = \left[\frac{x^{2+1}}{2+1}\right]_0^x = \left[\frac{x^3}{3}\right]_0^x$$

evaluating the interval

$$\Rightarrow y_1(x) = \left[\frac{x^3}{3} - \frac{(0)^3}{3}\right]$$

$$= \left[\left(\frac{x^3}{3}\right) - \left(\frac{(0)^3}{3}\right)\right]$$

$$= \left[\frac{x^3}{3} - 0\right]$$

$$= \left[\frac{x^3}{3}\right]$$

$$\text{hence } y_1(x) = \frac{x^3}{3} \qquad (6)$$

again when $n = 2$ then (4) becomes:

$$\Rightarrow y_2(x) = y_0 + \int_{x_0}^{x} f(x, y_{2-1})\, dx$$

$$\Rightarrow y_2(x) = y_0 + \int_{x_0}^{x} f(x, y_1)\, dx; \;\text{ where } f(x, y_1) = x^2 + y_1,$$

$$y_0 = 0 \text{ and } y_1(x) = \frac{x^3}{3} \text{ by (6)}$$

$$\Rightarrow y_2(x) = y_0 + \int_{x_0}^{x} (x^2 + y_1)\,dx$$

$$= 0 + \int_{x_0}^{x} \left(x^2 + \frac{x^3}{3}\right) dx$$

$$= \int_{x_0}^{x} \left(x^2 + \frac{x^3}{3}\right) dx$$

$$= \int_{0}^{x} (x^2)\,dx + \int_{0}^{x} \left(\frac{x^3}{3}\right) dx;\ x_0 = 0$$

$$= \left[\frac{x^{2+1}}{2+1}\right]_0^x + \left[\frac{x^{3+1}}{3(3+1)}\right]_0^x$$

$$\text{ie } y_2(x) = \left[\frac{x^3}{3}\right]_0^x + \left[\frac{x^4}{3.4}\right]_0^x = \left[\frac{x^3}{3} + \frac{x^4}{3.4}\right]_0^x$$

$$= \left[\frac{x^3}{3} + \frac{x^4}{3.4}\right]_0^x = \left[\frac{x^3}{3} + \frac{x^4}{3.4}\right] - \left[\frac{(0)^3}{3} + \frac{(0)^4}{3.4}\right]$$

$$= \left[\frac{x^3}{3} + \frac{x^4}{3.4}\right] - [0+0] = \left[\frac{x^3}{3} + \frac{x^4}{3.4}\right] - [0] = \left[\frac{x^3}{3} + \frac{x^4}{3.4}\right]$$

$$\text{hence } y_2(x) = \left[\frac{x^3}{3} + \frac{x^4}{3.4}\right] \qquad (7)$$

again when $n = 3$ then (4) becomes:

$$\Rightarrow y_3(x) = y_0 + \int_{x_0}^{x} f(x, y_{3-1})\,dx$$

$$\Rightarrow y_3(x) = y_0 + \int_{x_0}^{x} f(x, y_2)\,dx$$

$$\Rightarrow y_3(x) = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx;\ \text{ where } f(x, y_2) = x^2 + y_2,$$

$$y_0 = 0 \text{ and } y_2(x) = \frac{x^3}{3} + \frac{x^4}{3.4} \text{ by (7)}$$

$$\Rightarrow y_3(x) = 0 + \int_{x_0}^{x} (x^2 + y_2)\,dx$$

$$= 0 + \int_{x_0}^{x} \left(x^2 + \frac{x^3}{3} + \frac{x^4}{3.4}\right) dx$$

$$= \int_{x_0}^{x} \left(x^2 + \frac{x^3}{3} + \frac{x^4}{3.4}\right) dx$$

$$= \int_{0}^{x} (x^2)\,dx + \int_{0}^{x} \left(\frac{x^3}{3}\right) dx + \int_{0}^{x} \left(\frac{x^4}{3.4}\right) dx;\ x_0 = 0$$

$$= \left[\frac{x^{2+1}}{2+1}\right]_0^x + \left[\frac{x^{3+1}}{3(3+1)}\right]_0^x + \left[\frac{x^{4+1}}{3.4(4+1)}\right]_0^x$$

$$\text{ie } y_3(x) = \left[\frac{x^3}{3}\right]_0^x + \left[\frac{x^4}{3.4}\right]_0^x + \left[\frac{x^5}{3.4.5}\right]_0^x = \left[\frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{3.4.5}\right]_0^x$$

$$= \left[\frac{x^3}{3} + \frac{x^4}{3.4} + \frac{x^5}{3.4.5}\right]_0^x$$

$$= \left[\frac{x^3}{3} + \frac{x^4}{3.4} + \frac{x^5}{3.4.5}\right] - \left[\frac{(0)^3}{3} + \frac{(0)^4}{3.4} + \frac{(0)^5}{3.4.5}\right]$$

$$= \left[\frac{x^3}{3} + \frac{x^4}{3.4} + \frac{x^5}{3.4.5}\right] - [0+0+0] = \left[\frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5}\right] - [0]$$

$$= \left[\frac{x^3}{3} + \frac{x^4}{3.4} + \frac{x^5}{3.4.5}\right]$$

hence $y_3(x) = \frac{x^3}{3} + \frac{x^4}{3.4} + \frac{x^5}{3.4.5} \quad \ldots$ (8)

Remark

Equation (8) gives y_3 expressed as a power series in x. By choosing a desired value for x = 0.1 into (6), (7) and (8) then :

now; by evaluating equation (6) for x = 0.1 as:

ie when x = 0.1

$$\Rightarrow y_1(0.1) = \frac{(0.1)^3}{3} = \frac{0.001}{3} = 3.333333333 \times 10^{-4}$$

$\therefore \; y_1(0.1) \cong 0.0003$ (9)

again when x = 0.2 then (3) becomes:

by (6): $y_1(x) = \left[\frac{x^3}{3}\right]$

$$\Rightarrow y_1(0.2) = \frac{(0.2)^3}{3}$$

$$= \frac{0.008}{3}$$

$$= 2.666666666 \times 10^{-4}$$

$\therefore \; y_1(0.2) \cong 0.0027$ (10)

si b (7): $y_2(x) = \left[\frac{x^3}{3} + \frac{x^4}{3.4}\right]$, a $x = 0.1$

$$y_2(x) = y_2(0.1) = \left[\frac{(0.1)^3}{3} + \frac{(0.1)^4}{3.4}\right] = \left[\frac{0.001}{3} + \frac{0.0001}{3}\right]$$

$$= [3.333333333 \times 10^{-4} + 3.333333333 \times 10^{-5}]$$

ie $y_2(0.1) = 3.666666667 \times 10^{-4}$

hence $y_2(0.1) = 0.0004$ (11)

similarly; by (7) *at* $x = 0.2$ *becomes* :

$$ie \quad y_2(x) = y_2(0.2) = \left[\frac{(0.2)^3}{3} + \frac{(0.2)^4}{3.4}\right] = \left[\frac{(0.008)}{3} + \frac{(0.0016)}{12}\right] = \left[2.666666667 \times 10^{-3} + 1.333333333 \times 10^{-4}\right]$$

thus; $y_2(0.2) = 0.0028$ (12)

also equation (7) *when* $x = 0.1$ *becomes* :

$$ie \; y_3(x) = \left[\frac{x^3}{3} + \frac{x^4}{3.4} + \frac{x^5}{3.4.5}\right] = y_3(0.1) = \left[\frac{(0.1)^3}{3} + \frac{(0.1)^4}{3.4} + \frac{(0.1)^5}{3.4.5}\right] = \left[\frac{0.001}{3} + \frac{0.0001}{12} + \frac{0.00001}{60}\right]$$

$$= 3.333333333 \times 10^{-4} + 8.333333333 \times 10^{-6} + 1.666666667 \times 10^{-7} = 3.418333333 \times 10^{-4}$$

hence $y_3(0.1) \cong 0.0003$ (13)

Moreso; *when* $x = 0.2$ *equation* (8) *becomes* :

$$ie : y_3(x) = \left[\frac{x^3}{3} + \frac{x^4}{3.4} + \frac{x^5}{3.4.5}\right] = y_3(0.2) = \left[\frac{(0.2)^3}{3} + \frac{(0.2)^4}{3.4} + \frac{(0.2)^5}{3.4.5}\right] = \left[\frac{0.008}{3} + \frac{0.0016}{12} + \frac{0.00032}{60}\right]$$

$$= 2.666666667 \times 10^{-3} + 1.3333 \quad 33333 \times 10^{-4} + 5.333333333 \times 10^{-6} = 2.805333333 \times 10^{-6}$$

Thus; $y_3(0.2) \cong 0.0028 \quad \vdots$ (14)

Table 1 : Result generated From Picard Method (PM) for the step size h =0.05, for the values of $x = 0.1$ a 0.2

n	y_n a $x = 0.1$	y_n a $x = 0.2$
1	0.0003	0.0027
2	0.0004	0.0028
3	0.0003	0.0028
Analytical Solution	0.0003	0.0028
Associated Error	0.0000	0.0000

5.0 The Euler's Method (EM)

This is the most simple but crude method to solve differential equation of the form in (1). Considering the FODE with the IVP in (1), then the solution to (1) is equivalently given as finding solution to the integral equation:

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 (IVP) \end{cases} \quad (15a)$$

$$y_{n+1} = y_n + hf(x_n, y_n), n = 0,1,2, \ldots \quad (15b)$$

Proof

To show that equation (15b) is the equivalent solution to any first order DE of the form in equation (1) by Euler Method (EM) also suffices that:

Let $y' = f(x, y)$ be the FODE with IVP $y(x_0) = y_0$ i f $(15a)$

$$\Rightarrow \frac{d}{d} = f(x,y)$$

$$d = f(x,y)\, d$$

Let $x_1 = x_0 + h$, where h is small. Then by Taylor's series

$y_1 = y(x_0 + h) = y_0 + h\left(\frac{d}{d}\right)_{x_0} + \frac{h^2}{2}\left(\frac{d}{d}\right)_{c_1}$, where c_1 lies between x_0 a x

$$y_0 + hf(x_0, y_0) + \frac{h^2}{2} y''(c_1)$$

If the step size h is chosen small enough, then the second-Order term may be neglected and hence y_1 is given by:

$\Rightarrow y_1 = y_0 + hf(x_0, y_0)$

$\Rightarrow y_2 = y_1 + hf(x_1, y_1)$

$\Rightarrow y_3 = y_2 + hf(x_2, y_2)$

And so on

In general,

$$y_{n+1} = y_n + hf(x_n, y_n), n = 0,1,2, \ldots \quad (15c)$$

$$\text{where } x_k = x_0 + kh \quad (15d)$$

Thus: equation (15a) gives the$(n + 1)$th iteration , hence the Proof of Euler Method (EM).

This method is very slow. To get a reasonable accuracy with Euler's method, the value of h should be taken as small.It may be noted that the Euler's method is a single-step explicit method. According to Atkinson *et al*., (1989) '' Euler method is a first-order numerical procedure proposed by Leonhard Euler for solving ODE's with IVP's''. It is the most basic explicit method for numerical integration of ODE's and is considered as the simplified Runge-Kutta method

Problem 2

Find the values of $y(0.1)$ and $y(0.2)$ from the following differential equation

$$\frac{d}{d} = x^2 + y$$

with initial condition

$y(0) = 0.$ A f the v o $y(0.1)$ a $y(0.2)$

Solution 2

let $h = 0.05, x_0 = 0;\ y_0 = 0$ by the ivp then;

by $x_k = x_{k-1} + h$ where $k = 1,2,3, \ldots,$ (16)

when $k = 1$

$\Rightarrow x_k = x_1 = x_{1-1} + h = x_0 + h$

$\Rightarrow x_1 = x_0 + h$ where $x_0 = 0$ and $h = 0.05$

ie $x_1 = 0 + 0.05 = 0.05$

$\therefore$ by equation (15a)when $n = 0$

$y_{n+1} = y_n + hf(x_n, y_n), n = 0,1,2, \ldots$

ie $y_{0+1} = y_0 + hf(x_0, y_0), n = 0$

ie $y_1 = y_0 + hf(x_0, y_0), n = 0$
ie $y_1 = y(0.05) = y_0 + hf(x_0, y_0), n = 0$
ie $y_1 = y(0.05) = y_0 + h(x_0^2 + y_0), n = 0$
$= 0 + (0.05)((0)^2 + 0)$
hence; $y_1 = 0$ (17)
again by (15b) $x_k = x_{k-1} + h$ where $k = 1,2,3, \ldots,$
when $k = 2$
$\Rightarrow x_k = x_2 = x_{2-1} + h = x_1 + h$
$\Rightarrow x_2 = x_1 + h$ where $x_1 = 0.05$ and $h = 0.05$
ie $x_2 = 0.05 + 0.05 = 0.1$
$\therefore$ by equation (15a)when $n = 1$
$y_{n+1} = y_n + hf(x_k, y_n), n = 0,1,2, \ldots$ and $k = 1,2,3, \ldots.$
ie $y_{1+1} = y_1 + hf(x_1, y_1), n = 1, k = 2$
ie $y_2 = y_1 + hf(x_2, y_1)$
ie $y_2 = y(0.1) = y_1 + hf(x_2, y_1)$
ie $y(0.1) = y_1 + h(x_1^2 + y_1)$, where $x_2 = 0.1, y_1 = 0$ and $h = 0.05$
$= 0 + (0.05)((0.1)^2 + 0)$
$= (0.05)(0.01)$
$= (0.0005)$
hence; $y_2 = 0.0005$ (18)
similarly; by (15b) $x_k = x_{k-1} + h$ where $k = 1,2,3, \ldots,$
when $k = 3$
$\Rightarrow x_k = x_3 = x_{3-1} + h = x_2 + h$
$\Rightarrow x_3 = x_2 + h$ where $x_2 = 0.1$ and $h = 0.05$
ie $x_3 = 0.1 + 0.05 = 0.15$
$\therefore$ by equation (15a)when $n = 2$
$y_{n+1} = y_n + hf(x_k, y_n), n = 0,1,2, \ldots , k = 1,2,3, \ldots.,$
ie $y_{2+1} = y_2 + hf(x_3, y_2), n = 2 , k = 3$
ie $y_3 = y_2 + hf(x_3, y_2)$
ie $y_3 = y(0.15) = y_2 + hf(x_3, y_2)$
ie $y(0.15) = y_2 + h(x_3^2 + y_2)$, where $x_3 = 0.15, y_2 = 0.0005$ and $h = 0.05$
$= 0.0005 + (0.05)((0.15)^2 + 0.0005)$
$= 0.0005 + (0.05)(0.0225 + 0.0005)$
$= 0.0005 + (0.05)(0.023)$
$= 0.0005 + 0.00115$
$= 0.00165$
hence; $y_3 = 0.0017$ (19)
similarly; by (15b) $x_k = x_{k-1} + h$ where $k = 1,2,3, \ldots,$
when $k = 4$
$\Rightarrow x_k = x_4 = x_{4-1} + h = x_3 + h$
$\Rightarrow x_4 = x_3 + h$ where $x_3 = 0.2$ and $h = 0.05$
ie $x_4 = 0.15 + 0.05 = 0.2$
$\therefore$ by equation (15a)when $n = 3$
$y_{n+1} = y_n + hf(x_k, y_n), n = 0,1,2, \ldots , k = 1,2,3, \ldots.,$
ie $y_{3+1} = y_3 + hf(x_4, y_3), n = 3 , k = 4$
ie $y_4 = y_3 + hf(x_4, y_3)$
ie $y_4 = y(0.2) = y_3 + hf(x_4, y_3)$
ie $y(0.15) = y_2 + h(x_4^2 + y_3)$, where $x_4 = 0.2, y_3 = 0.00165$ and $h = 0.05$
$= 0.00165 + (0.05)((0.2)^2 + 0.000165)$
$= 0.00165 + (0.05)(0.04 + 0.00165)$
$= 0.00165 + (0.05)(0.04165)$
$= 0.00165 + 0.0020825$
$= 0.0037325$
hence; $y_4 = 0.0037$ (20)

Table 2 : Result generated From Euler Method (EM) for the step size h =0.05

n	x_n	y_n	Analytical Method(AM)	Associated Error (AE)
1	0.05	0.0000	0.0000	0.0000
2	0.1	0.0005	0.0003	0.0002
3	0.15	0.0017	0.0012	0.0005
4	0.2	0.0037	0.0028	0.0009

6.0 Modified Euler Method (Heun Method). (MEM)

Considering the FODE with the IVP in (1), then the solution to (1) is equivalently as finding solution equation in (15a) for the Modified Euler Method given as:

$$y_{((i+1),n)} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f\left(x_{i+1}, y_{((i+1),n)}\right)\right] \qquad (21a)$$

Proof

To show that equation (21a) is the equivalent solution to first order DE of the form in equation (1) which suffices Heun Method:

Let $y' = f(x,y)$ be the FODE with IVP $y(x_0) = y_0$ ie from (1)

$$\Rightarrow \frac{dy}{dx} = f(x,y)$$

$$\Rightarrow dy = f(x,y)\, dx$$

$$\Rightarrow \int_{y_0}^{y} dy = \int_{x_0}^{x} f(x,y)\, dx$$

$$\Rightarrow [y]_{y_0}^{y} = \int_{x_0}^{x} f(x,y)\, dx$$

$$\Rightarrow y - y_0 = \int_{x_0}^{x} f(x,y)\, dx$$

$$\Rightarrow y_1 = y_1 + \int_{x_0}^{x} f(x,y)\, dx$$

The integration of the RHS can be done using any numerical method. If the trapezoidal rule is used with step size $h(= x_1 - x_0)$ then the above integration becomes:

$$y(x) = y(x_0) + \frac{h}{2}\left[f\left(x_0, y(x_0)\right) + f\left(x_1, y(x_1)\right)\right] \qquad (21b)$$

Remark

The RHS of (21) involve an unknown quantit$y(x_1)$. This value can be determined by the Euler Method (EM). Denoting this value by $y_1^{(0)}(x_0)$. Then the resulting formula for finding $y_1$is :

$$y_1(x_1) = y(x_0) + \frac{h}{2}\left[f\left(x_0, y(x_0)\right) + f\left(x_1, y_1^{(0)}(x_1)\right)\right]$$

$$\Rightarrow y_1^{(0)} = y_0 + hf(x_0, y_0)$$

$$\text{ie } y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^0)] \qquad (22)$$

Equation (22) gives the first approximation of y_1.

The second approximation is:

$$\text{ie } y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^1)]$$

The $(n+1)$th approximation of y is:

$$\text{ie } y_1^{(k+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^k\right)\right]$$

Generally,

$$\Rightarrow y_{i+1}^{(0)} = y_i + hf(x_i, y_i)$$

$$\text{ie rewritten as } y_{((i+1),n)} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f\left(x_{i+1}, y_{((i+1),n)}\right)\right] \qquad (23)$$

for $i = n = 1,2,3, \ldots$

$$y_{((i+1),n)} = y_i + hf(x_i, y_i), \text{for } i = n = 0. \qquad (24)$$

Where $i = n = 0,1,2,3, \ldots$

The iterations are continued until two successive approximations $y_{i+1}^{(n)}$ and $y_{i+1}^{(n+1)}$ coincide to the desired accuracy. The iterations converge rapidly for sufficiently small spacing

7.0 Modified Euler Method (MEM)

In this method, problem of the form in equation (1) will be solved using Modified Euler Method. Thus; Considering same problem solved in problem (1) and problem (2) for Picard and Euler Methods respectively using Modified Euler Method inproblem (3)

Problem 3

Find the values of $y(0.1)$ and $y(0.2)$ from the following differential equation

$$\frac{dy}{dx} = x^2 + y$$

with initial condition

$y(0) = 0$. Also find the values of $y(0.1)$ and $y(0.2)$

Solution 3

From the given general term of Modified Euler Method in equation ((7) and (8)):

ie $y_{((i+1),n)} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f\left(x_{i+1}, y_{((i+1),n)}\right)\right]$ f0r $i = n = 1,2,3, \ldots$

$y_{((i+1),n)} = y_i + hf(x_i, y_i)$, for $i = n = 0$

Using a desired step size of h= 0.05

By the IVP:

$y(0) \Rightarrow$ value of y at $x = 0$

ie $y = 0$ when $x = 0$ (Initial Condition (IC)),

where $f(x, y) = x^2 + y$ by equation (1)

at $x_0 = 0, y_0 = 0, h = 0.05, i = n = 0$, and $f(x_0, y_0) = x_0^2 + y_0$

$y_{((1),0)} = y_0 + hf(x_0, y_0)$

$y_{((1),0)} = y_0 + h(x_0^2 + y_0)$

$= 0 + (0.05)[(0)^2 + 0]$

hence $y_{((1),0)} = 0.00000 = 0$

ie: $y_{((1),0)} = 0$ (25)

at $x_0 = 0, x_1 = x_0 + h = 0 + 0.05 = 0.05, y_0 = 0, h = 0.05, i = 0 =, n = 1,$

$f\left(x_1, y_{((1),0)}\right) = x_1^2 + y_{((1),0)}, y_{((1),0)} = 0$and $f(x_0, y_0) = x_0^2 + y_0$

$$y_{((0+1),1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_{0+1}, y_{((0+1),0)}\right)\right]$$

$$y_{((1),1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_{((1),0)}\right)\right]$$

$$= y_0 + \frac{h}{2}\left[(x_0^2 + y_0) + \left(x_1^2 + y_{((1),0)}\right)\right]$$

$$y_{((1),1)} = 0 + \left(\frac{0.05}{2}\right)[(0) + ((0.05)^2 + 0)]$$

$= (0.025)[(0.0025)] = 0.0000625$

ie $y_{((1),1)} = 0.0000625$

hence $y_{((1),1)} \cong 0.0001$ (26)

at $x_0 = 0, y_0 = 0, h = 0.05, i = 0, n = 2$, and $f(x_0, y_0) = x_0^2 + y_0$

$f\left(x_1, y_{((1),2)}\right) = x_1^2 + y_{((1),1)}, y_{((1),1)} = 0.0000625$, $f(x_0, y_0) = x_0^2 + y_0 = 0$

$x_2 = x_1 + h = 0.05 + 0.05 = 0.1$. ie $x_1 = 0.05$ and $x_2 = 0.1$

$$y_{((0+1),2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_{((1),1)}\right)\right]$$

$$y_{((1),2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_{((1),1)}\right)\right]$$

$$= y_0 + \frac{h}{2}\left[(x_0^2 + y_0) + \left(x_1^2 + y_{((1),1)}\right)\right]$$

$$y_{((1),2)} = 0 + \left(\frac{0.05}{2}\right)[(0) + ((0.05)^2 + 0.0000625)]$$

$= (0.025)[(0.0025625)] = 6.40625 \times 10^{-5}$
ie $y_{((1),2)} = 6.40625 \times 10 -^{-5}$
hence $y_{((1),2)} \cong 0.0001$ (27)
at $x_0 = 0, y_0 = 0, h = 0.05, i = 1, n = 0$, and $f\left(x_1, y_{((1),2)}\right) = x_1^2 + y_{((1),2)}$
$y_{((1),2)} = 6.40625 \times 10 -^{-5}$
$x_1 = x_0 + h = 0 + 0.05 = 0.05.$ ie $x_1 = 0.05$
$y_{((2),0)} = y_{((1),2)} + hf\left(x_1, y_{((1),2)}\right)$
$y_{((2),0)} = y_{((1),2)} + h\left(x_1^2 + y_{((1),2)}\right)$
$= 6.40625 \times 10^{-5} + (0.05)[(0.05)^2 + 6.40625 \times 10^{-5}]$
$= 6.40625 \times 10^{-5} + (0.05)[(2.5640625 \times 10^{-3})]$
$= 6.40625 \times 10^{-5} + (1.28203125 \times 10^{-4})$
$= 1.92265625 \times 10^{-4}$
thus; $y_{((2),0)} \cong 0.0002$ (28)
at $x_0 = 0, y_0 = 0, h = 0.05, i = n = 1, f\left(x_1, y_{((1),2)}\right) = x_1^2 + y_{((1),2)}$,
$f\left(x_2, y_{((2),0)}\right) = x_2^2 + y_{((2),0)}, x_1^2 + y_{((1),2)} = 2.5640625 \times 10^{-3}$
$, y_{((2),0)} = 1.92265625 \times 10^{-4}, x_2 = x_1 + h = 0.05 + 0.05 = 0.1.$
ie : $x_1 = 0.05$, $x_2 = 0.1$ and $y_{((1),2)} = 6.40625 \times 10^{-5}$.
$y_{((2),1)} = y_{((1),2)} + \frac{h}{2}\left[f\left(x_1, y_{((1),2)}\right) + f\left(x_2, y_{((2),0)}\right)\right]$
$y_{((2),1)} = y_{((1),2)} + \frac{h}{2}\left[\left(x_1^2 + y_{((1),2)}\right) + \left(x_2^2 + y_{((2),0)}\right)\right]$
$y_{((2),1)} = 6.40625 \times 10^{-5} + \left(\frac{0.05}{2}\right)[((0.05)^2 + 6.40625 \times 10^{-5}) + ((0.1)^2 + 1.92265625 \times 10^{-4})]$
$= 6.40625 \times 10^{-5} + (0.025)[(0.0025 + 6.40625 \times 10^{-5}) + (0.01 + 1.92265625 \times 10^{-4})]$
$= 6.40625 \times 10^{-5} + (0.025)(1.275632813 \times 10^{-2})$
$= 6.40625 \times 10^{-5} + 3.189082031 \times 10^{-4}$
ie $y_{((2),1)} = 3.829707031 \times 10^{-4}$
thus; $y_{((2),1)} \cong 0.0004$ (29)
at $h = 0.05, i = 1, n = 2, f\left(x_1, y_{((1),2)}\right) = x_1^2 + y_{((1),2)}$,
$f\left(x_2, y_{((2),1)}\right) = x_2^2 + y_{((2),1)}, x_1^2 + y_{((1),2)} = 2.5640625 \times 10^{-3}$
$, y_{((2),1)} = 3.829707031 \times 10 -^{-4}, x_2 = x_1 + h = 0.05 + 0.05 = 0.1.$
ie : $x_1 = 0.05$, $x_2 = 0.1$ and $y_{((1),2)} = 6.40625 \times 10^{-5}$.
$y_{((2),2)} = y_{((1),2)} + \frac{h}{2}\left[f\left(x_1, y_{((1),2)}\right) + f\left(x_2, y_{((2),1)}\right)\right]$
$y_{((2),2)} = y_{((1),2)} + \frac{h}{2}\left[\left(x_1^2 + y_{((1),2)}\right) + \left(x_2^2 + y_{((2),1)}\right)\right]$

$y_{((2),2)} = 6.40625 \times 10^{-5} + \left(\frac{0.05}{2}\right)[((0.05)^2 + 6.40625 \times 10^{-5}) + ((0.1)^2 + 3.829707031 \times 10^{-4})]$
$= 6.40625 \times 10^{-5} + (0.025)[(0.0025 + 6.40625 \times 10^{-5}) + (0.01 + 3.829707031 \times 10^{-4})]$
$= 6.40625 \times 10^{-5} + (0.025)(1.29470332 \times 10^{-2})$
$= 6.40625 \times 10^{-5} + 3.236758301 \times 10^{-4}$
ie $y_{((2),2)} = 3.877383301 \times 10^{-4}$
thus; $y_{((2),2)} \cong 0.0004$ (30)
at $h = 0.05, i = 2, n = 0, f\left(x_2, y_{((2),2)}\right) = x_2^2 + y_{((2),2)}$,
$y_{((2),2)} = 3.877383301 \times 10^{-4}, x_2 = x_1 + h = 0.05 + 0.05 = 0.1,$
ie $x_2 = 0.1$
$y_{((3),0)} = y_{((2),2)} + hf\left(x_2, y_{(2,2)}\right)$

$y_{((3),0)} = y_{((2),2)} + h\left(x_2^2 + y_{((2),2)}\right)$

$= 3.877383301 \times 10^{-4} + (0.05)[(0.1)^2 + 3.877383301 \times 10^{-4}]$

$= 3.877383301 \times 10^{-4} + (0.05)[(1.038773833 \times 10^{-2})]$

$= 3.877383301 \times 10^{-4} + 5.193869165 \times 10^{-4}$

$= 9.071252466 \times 10^{-4}$

thus; $y_{((3),0)} \cong 0.0009$ (31)

at $h = 0.05, i = 2, n = 1, f\left(x_2, y_{((2),2)}\right) = x_2^2 + y_{((2),2)}$,

$f\left(x_3, y_{((3),0)}\right) = x_3^2 + y_{((3),0)}, x_2^2 + y_{((2),2)} = 1.038773833 \times 10^{-2}$

$y_{((2),2)} = 3.877383301 \times 10^{-4}, y_{((3),0)} = 9.071252466 \times 10^{-4}$

$x_2 = x_1 + h = 0.05 + 0.05 = 0.1, x_3 = x_2 + h = 0.1 + 0.05 = 0.15$

ie $x_3 = 0.15$ and $x_2 = 0.1$.

$$y_{((3),1)} = y_{((2),2)} + \frac{h}{2}\left[f\left(x_2, y_{((2),2)}\right) + f\left(x_3, y_{((3),0)}\right)\right]$$

$$y_{((3),1)} = y_{((2),2)} + \frac{h}{2}\left[\left(x_2^2 + y_{((2),2)}\right) + \left(x_3^2 + y_{((3),0)}\right)\right]$$

$$y_{((3),1)} = 3.877383301 \times 10^{-4} + \left(\frac{0.05}{2}\right)[(1.038773833 \times 10^{-2}) + ((0.15)^2 + 9.071252466 \times 10^{-4})]$$

$= 3.877383301 \times 10^{-4} + (0.025)[(1.038773833 \times 10^{-2}) + (0.0225 + 9.071252466 \times 10^{-4})]$

$= 3.877383301 \times 10^{-4} + (0.025)(3.379486358 \times 10^{-2})$

$= 3.877383301 \times 10^{-4} + 8.448715894 \times 10^{-4}$

ie $y_{((3),1)} = 1.23260992 \times 10^{-3}$

thus; $y_{((3),1)} \cong 0.0012$ (32)

at $h = 0.05, i = 2, n = 2, f\left(x_2, y_{((2),2)}\right) = x_2^2 + y_{((2),2)}$,

$f\left(x_3, y_{((3),1)}\right) = x_3^2 + y_{((3),1)}, x_2^2 + y_{((2),2)} = 1.038773833 \times 10^{-2}$

$y_{((2),2)} = 3.877383301 \times 10^{-4}, y_{((3),1)} = 1.23260992 \times 10^{-3}$

$x_2 = x_1 + h = 0.05 + 0.05 = 0.1, x_3 = x_2 + h = 0.1 + 0.05 = 0.15$

ie $x_3 = 0.15$ and $x_2 = 0.1$.

$$y_{((3),2)} = y_{((2),2)} + \frac{h}{2}\left[f\left(x_2, y_{((2),2)}\right) + f\left(x_3, y_{((3),1)}\right)\right]$$

$$y_{((3),2)} = y_{((2),2)} + \frac{h}{2}\left[\left(x_2^2 + y_{((2),2)}\right) + \left(x_3^2 + y_{((3),1)}\right)\right]$$

$$y_{((3),2)} = 3.877383301 \times 10^{-4} + \left(\frac{0.05}{2}\right)[(1.038773833 \times 10^{-2}) + ((0.15)^2 + 1.23260992 \times 10^{-3})]$$

$= 3.877383301 \times 10^{-4} + (0.025)[(1.038773833 \times 10^{-2}) + (0.0225 + 1.23260992 \times 10^{-3})]$

$= 3.877383301 \times 10^{-4} + (0.025)(3.412034825 \times 10^{-2})$

$= 3.877383301 \times 10^{-4} + 8.530087062 \times 10^{-4}$

ie $y_{((3),2)} = 1.240747036 \times 10^{-3}$

thus; $y_{((3),2)} \cong 0.0012$ (33)

at $h = 0.05, i = 3, n = 0, f\left(x_3, y_{((3),2)}\right) = x_3^2 + y_{((3),2)}$,

$x_3 = 0.15$ and $x_2 = 0.1, y_{((3),2)} = 1.240747036 \times 10^{-3}$

$y_{((4),0)} = y_{((3),2)} + hf\left(x_3, y_{(3,2)}\right)$

$y_{((4),0)} = y_{((3),2)} + h\left(x_3^2 + y_{((3),2)}\right)$

$= 1.240747036 \times 10^{-3} + (0.05)[(0.15)^2 + 1.240747036 \times 10^{-3}]$

$= 1.240747036 \times 10^{-3} + (0.05)[(0.0225 + 1.240747036 \times 10^{-3})]$

$= 1.240747036 \times 10^{-3} + (0.05)[(2.374074704 \times 10^{-2})]$

$= 1.240747036 \times 10^{-3} + 1.187037352 \times 10^{-4}$

$= 2.427784388 \times 10^{-3}$

thus; $y_{((4),0)} \cong 0.0024$ (34)

at $h = 0.05, i = 3, n = 1, f\left(x_3, y_{((3),2)}\right) = x_3^2 + y_{((3),2)},$

$f\left(x_4, y_{((4),0)}\right) = x_4^2 + y_{((4),0)}, x_3^2 + y_{((3),2)} = 2.374074704 \times 10^{-2}$

$y_{((4),0)} = 2.427784388 \times 10^{-3}, y_{((3),2)} = 1.240747036 \times 10^{-3}$

$x_2 = x_1 + h = 0.05 + 0.05 = 0.1, x_3 = x_2 + h = 0.1 + 0.05 = 0.15$

$x_4 = x_3 + h = 0.15 + 0.05 = 0.2$ ie $x_3 = 0.15$ and $x_4 = 0.2.$

$$y_{((4),1)} = y_{((3),2)} + \frac{h}{2}\left[f\left(x_3, y_{((3),2)}\right) + f\left(x_4, y_{((4),0)}\right)\right]$$

ie: $y_{((4),1)} = y_{((3),2)} + \frac{h}{2}\left[\left(x_3^2 + y_{((3),2)}\right) + \left(x_4^2 + y_{((4),0)}\right)\right]$

$$= 1.240747036 \times 10^{-3} + \left(\frac{0.05}{2}\right)[((0.15)^2 + 1.240747036 \times 10^{-3}) + ((0.2)^2 + 2.427784388 \times 10^{-3})]$$

$$= 1.240747036 \times 10^{-3} + (0.025)[(0.0225 + 1.240747036 \times 10^{-3}) + (0.04 + 2.427784388 \times 10^{-3})]$$

$$= 1.240747036 \times 10^{-3} + (0.025)(6.616853142 \times 10^{-2})$$

$$= 1.240747036 \times 10^{-3} + 1.654213286 \times 10^{-3}$$

ie $y_{((4),1)} = 2.90663972 \times 10^{-3}$

thus; $y_{((4),1)} \cong 0.0029$ (35)

at $h = 0.05, i = 3, n = 2, f\left(x_3, y_{((3),2)}\right) = x_3^2 + y_{((3),2)},$

$f\left(x_4, y_{((4),1)}\right) = x_4^2 + y_{((4),1)}, x_3^2 + y_{((3),2)} = 2.374074704 \times 10^{-2}$

$y_{((4),1)} = 2.90663972 \times 10 -^{-3}, y_{((3),2)} = 1.240747036 \times 10^{-3}$

$x_2 = x_1 + h = 0.05 + 0.05 = 0.1, x_3 = x_2 + h = 0.1 + 0.05 = 0.15$

$x_4 = x_3 + h = 0.15 + 0.05 = 0.2$ ie $x_3 = 0.15$ and $x_4 = 0.2.$

$$y_{((4),2)} = y_{((3),2)} + \frac{h}{2}\left[f\left(x_3, y_{((3),2)}\right) + f\left(x_4, y_{((4),1)}\right)\right]$$

$$y_{((4),2)} = y_{((3),2)} + \frac{h}{2}\left[\left(x_3^2 + y_{((3),2)}\right) + \left(x_4^2 + y_{((4),1)}\right)\right]$$

$$y_{((4),2)} = 1.240747036 \times 10^{-3} + \left(\frac{0.05}{2}\right)[((0.15)^2 + 1.240747036 \times 10^{-3}) + ((0.2)^2 + 2.90663972 \times 10^{-3})]$$

$$= 1.240747036 \times 10^{-3} + (0.025)[(0.0225 + 1.240747036 \times 10^{-3}) + (0.04 + 2.90663972 \times 10^{-3})]$$

$$= 1.240747036 \times 10^{-3} + (0.025)(6.664738676 \times 10^{-2})$$

$$= 1.240747036 \times 10^{-3} + 1.66184669 \times 10^{-3}$$

ie $y_{((4),2)} = 2.906931705 \times 10^{-3}$

thus; $y_{((4),2)} \cong 0.0029$ (36)

Table 3: Result generated From Modified Euler Method (MEM) for the step size h=0.05

n	x_n	y_n	Analytical Solution	Associated Error (AE)
1	0.05	0.0001	0.0000	0.0001
2	0.1	0.0004	0.0003	0.0001
3	0.15	0.0012	0.0012	0.0000
4	0.2	0.0029	0.0028	0.0001

8.0 Analytical Solution of the Problem

The equation considered in this scope can also be solved through the analytical method using the method of integrating factor as follows:

By the equation described inproblem (1, 2 and 3):

Problem 4

Find the values of $y(0.1)$ and $y(0.2)$ from the given differential equation below:

$$\frac{dy}{dx} = x^2 + y$$

with initial condition

$y(0) = 0.$ Also find the values of $y(0.1)$ and $y(0.2)$

Solution 4

Given that $\frac{d}{d} = x^2 + y2$

Using the method of integrating factor the solution to the given problem as described in (3.1) is given below:

ie: $\frac{dy}{dx} = x^2 + y \equiv y' - y = x^2$ (37)

$\Rightarrow \frac{dy}{dx} - y = x^2 \equiv y' - y = x^2$, where $y' = \frac{dy}{dx}$, $p(x) = (-1)$

$q(x) = x^2$ and integrating factor $(I.F) = e^{\int p(x)dx}$

by the $I.F = e^{\int p(x)dx}$

$= e^{\int(-1)dx}$

$= e^{-\int dx}$

$= e^{-x}$

$\therefore$ $I.F = e^{-x}$ (38)

now; multiplying equation (37) by equation (38)

ie: (37) becomes; $y'e^{-x} - ye^{-x} = x^2e^{-x}$

$\therefore$ $y'e^{-x} - ye^{-x} = \frac{d(ye^{-x})}{dx} = x^2e^{-x}$

ie: $\frac{d(ye^{-x})}{dx}dx = x^2e^{-x}dx$

ie: $d(ye^{-x}) = (x^2e^{-x})dx$

integratung both sides:

$\int d(ye^{-x}) = \int (x^2e^{-x})dx$

$\therefore$ $ye^{-x} = \int (x^2e^{-x})dx$ (39)

applying method of integration by part to the R.H.S

$\int udv = uv - \int vdu,$ (40)

where u = function to be differentiated and

v = function to be integrated

ie: $\int (x^2e^{-x})dx = RHS$ (41)

where $dv = e^{-x}dx$ and $v = \int dv = v$

ie: $v = \int e^{-x}dx = [e^{-x}] \div \frac{d(-x)}{dx} = \frac{e^{-x}}{-1} = -e^{-x}$

hence; $v = -e^{-x}$

again by $u = x^2$ $\therefore$ $\frac{du}{dx} = \frac{d(x^2)}{dx}$ $\therefore$ $du = (x^2)dx$

$\therefore$ $du = 2(x^{2-1}) = 2xdx$

$\therefore$ $v = -e^{-x}, u = x^2, dv = e^{-x}$ and $du = 2xdx$ (42)

substituting equation (42) into (40) to give the point process integral solution of (39)

ie: $\int udv = uv - \int vdu$

$\therefore$ $\int (x^2e^{-x})dx = -x^2e^{-x} - \int (2x)(-e^{-x})dx$

$= -x^2e^{-x} - (-)\int (2x)(e^{-x})dx$

$= -x^2e^{-x} + \int (2x)(e^{-x})dx$

$= -x^2e^{-x} + \int 2xe^{-x}dx$ (43)

$$\left.\begin{aligned} &\text{again; } u = 2x, dv = e^{-x}dx \\ &\text{ie: } \frac{du}{dx} = \frac{d(2x)}{dx} \ \square\ du = (2x)dx \\ &\quad \square\ du = 1.(2x^{1-1}) \\ &\qquad = 1.2x^{0} \\ &\qquad = 1 \times 2 \times 1dx \\ &\qquad = 2.dx \\ &\qquad = 2dx \end{aligned}\right\} \qquad (44)$$

$\square\ v = -e^{-x}, u = x^{2}, dv = e^{-x}$and $du = 2xdx$ (45)

using equation (40)

ie: $\int udv = uv - \int vdu,$

$\square$ equation (43) becomes:

$$= -x^{2}e^{-x} + \left(uv - \int vdu\right) \qquad (46)$$

where $v = -e^{-x}, u = 2x, dv = e^{-x}$and $du = 2dx$ (47)

$\square$ by substituting equation (47) into equation (46)we obtain equation (48):

$$\square \int udv = \int 2xe^{-x}dx = -2xe^{-x} - \int (2dx)(-e^{-x})$$

$$\square \int (x^{2}e^{-x})dx = -x^{2}e^{-x} + \left(-2xe^{-x} - \int (2dx)(-e^{-x})\right)$$

$$\square \int (x^{2}e^{-x})dx = -x^{2}e^{-x} - xe^{-x} - (-)2\int (e^{-x})\,dx$$

$$= -x^{2}e^{-x} - 2xe^{-x} + 2\int (e^{-x})\,dx$$

$$= -x^{2}e^{-x} - 2xe^{-x} + 2\left([e^{-x}] \div \frac{d(-x)}{dx}\right) + C$$

$$= -x^{2}e^{-x} - 2xe^{-x} + 2\left(\frac{e^{-x}}{-1}\right) + C$$

$$= -x^{2}e^{-x} - 2xe^{-x} + 2(-e^{-x}) + C$$

$$= -x^{2}e^{-x} - 2xe^{-x} - 2e^{-x} + C$$

$$= -(x^{2}e^{-x} + 2xe^{-x} + 2e^{-x}) + C$$

thus; $\int (x^{2}e^{-x})dx = -(x^{2}e^{-x} + 2xe^{-x} + 2e^{-x}) + C$

$\square$ by (3.31): $ye^{-x} = \int (x^{2}e^{-x})dx = -(x^{2}e^{-x} + 2xe^{-x} + 2e^{-x}) + C$

ie: $ye^{-x} = -(x^{2}e^{-x} + 2xe^{-x} + 2e^{-x}) + C$

ie: $\frac{ye^{-x}}{e^{-x}} = \frac{-(x^{2}e^{-x} + 2xe^{-x} + 2e^{-x}) + C}{e^{-x}}$, ie dividing both side by (e^{-x})

$$\square\ y = \frac{-(x^{2}e^{-x} + 2xe^{-x} + 2e^{-x}) + C}{e^{-x}} = -\left(\frac{x^{2}e^{-x}}{e^{-x}} + \frac{2xe^{-x}}{e^{-x}} + \frac{2e^{-x}}{e^{-x}}\right) + \frac{C}{e^{-x}}$$

$\square\ y(x) = Ce^{x} - (x^{2} + 2x + 2)$ (48)

Equation (48) gives the equivalence analytical solution of problem (1,2 and 3)

But by the given IVP; ie $y(0) = 0$, $\square$ $y = 0$ when $x = 0$

now substituting the IVP into equation (48)

to obtain the value of the constant term of integration (C)

$\square\ y(0) = -((0)^{2} + 2(0) + 2) + Ce^{0} = -2 + C \times 1 = 0$

ie: $C - 2 = 0$, $\square$ $C + 2 - 2 = 0 + 2$, $\square$ $C = 2$

hence $C = 2$ (49)

$\square$ equation (36) becomes $\square$

$y(x) = -(x^{2} + 2x + 2) + 2e^{x}$

thus: $y(x) = 2e^{x} - (x^{2} + 2x + 2)$ (50)

REMARK

Equation (50) gives the general non-numerical solution of problem (1,2 and 3) for any given value of x

Below are the analytical computation of the equivelence unknown sc g b ec (54)

ie : $y(x) = 2e^{x} - (x^{2} + 2x + 2)$
so when x = 0.05
ie : $y(0.05) = 2e^{(0.0\ )} - ((0.05)^{2} + 2(0.05) + 2)$
$= 2(1.051271096) - (0.0025 + 0.1 + 2)$
$= 2.102542192 - (2.1025)$
$= 4.2192 \times 10^{-5}$
he $y(0.05)$ □ 0 (51)
whe $x = 0.1$
ie b : $y(x) = 2e^{x} - (x^{2} + 2x + 2)$
□ $y(0.05) = 2e^{(0.1)} - ((0.1)^{2} + 2(0.1) + 2)$
$= 2(1.105170918) - (0.01 + 0.2 + 2)$
$= 2.210341836 - (2.21)$
$= 3.41836 \times 10^{-4}$
he $y(0.1)$ □ 0.0003 (52)
whe $x = 0.15$
ie b : $y(x) = 2e^{x} - (x^{2} + 2x + 2)$
□ $y(0.15) = 2e^{(0.1\ )} - ((0.15)^{2} + 2(0.15) + 2)$
$= 2(1.161834243) - (0.0225 + 0.3 + 2)$
$= 2.323668486 - (2.3225)$
$= 1.168486 \times 10^{-3}$
he $y(0.15)$ □ 0.0012 (53)
whe $x = 0.2$
ie b : $y(x) = 2e^{x} - (x^{2} + 2x + 2)$
□ $y(0.15) = 2e^{(0.2)} - ((0.2)^{2} + 2(0.2) + 2)$
$= 2(1.221402758) - (0.04 + 0.4 + 2)$
$= 2.442805516 - (2.44)$
$= 2.805516 \times 10^{-3}$
hence y(0.2) □ 0.0028 (54)

Table 4. Result generated From Analytical (AM) for the step size h =0.05

n	x_n	Analytical Solution(AS)
1	0.05	0.0000
2	0.1	0.0003
3	0.15	0.0012
4	0.2	0.0028

Table 5 Results from Analytical, Picard, Euler and Modified Euler Methods for h = 0.05

	$y(x)$			
x	Analytical Method(AM)	Picard Method(PM)	Euler Method(EM)	Modified Euler Method(MEM)
0.1	0.0003	0.0003	0.0005	0.0004
0.2	0.0028	0.0028	0.0037	0.0029

Table 6 Some examples of FODE using PEMEM

S/N	Problem	PM				Problem	EM			
	y' = x-y	x	y_P	y_E	Error	y' =1+y	x	y_E	y_E	Error
1	y(0) = 1	0.05	0.9513	0.9525	0.0012	y(0) = 1	0	1.0000	1.0000	0.0000
2		0.1	0.9098	0.9097	0.0001		0.1	1.2000	1.2103	0.0103
3		0.15	0.8714	0.8714	0.0000		0.2	1.4200	1.4428	0.0228
4		0.2	0.8374	0.8378	0.0000		0.3	1.6620	1.6997	0.0377
5		0.25	0.8075	0.8078	0.0003		0.4	1.9282	1.9836	0.0554

Problem	MEM		
y' = 1 + y	x	y_M	Error
y(0) = 1	0	1.0000	0.0003
	0.1	1.2103	0.0008
	0.2	1.4428	0.0013
	0.3	1.6997	0.0018
	0.4	1.9836	0.0025

Table 5 Displays the summary of numerical solution obtained from Picard, Euler, Modified Euler and Analytical Method (PEMEM & AM) for the specified values of (x).
Furthermore, Figure 1 shows the nature of the numerical solution of equation (1)

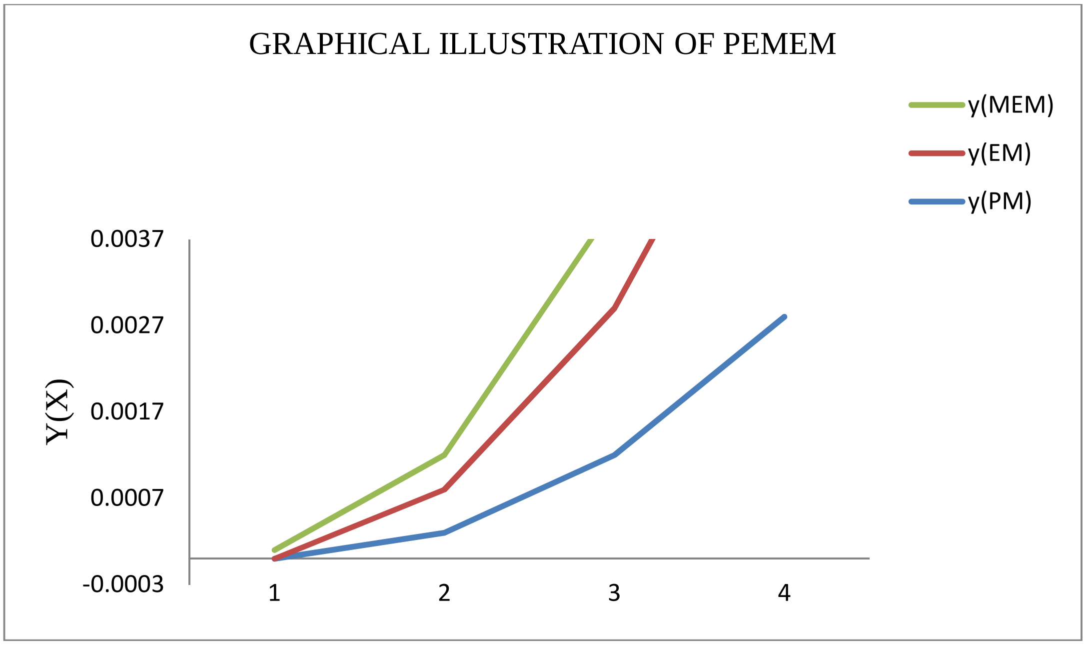

Figure 1 Graphical Illustration of Solutions Obtained for problem (1,2 and 3) Using the Three Methods (PEMEM).

9.0 Error Analysis

$E^{g}_{m\ \ ho\ (i)} = |y(x_i) - Y(x_i)|, i = 1,2,\ldots,$ (55)

$r\ -w\ \ a : E^{F}_{m\ \ ho\ (i)} = |y(x_i) - Y(x_i)|, i = 1,2,\ldots,$ (56)

$a\ \ the\ l\ \ e\ \ a : E^{L}_{m\ \ ho\ (i+1)} = |y(x_{i+1}) - y(x_i)|, i = 1,2,\ldots,$ (57)

whe $y(x_i) = S$ b D V M ho (D) a
$Y(x_i) = E$ S (S io b A M ho (A))

Table 7 Tabular Representation of The Global Errors in AM & PEMEM

E^{g}_{A}	E^{g}_{P}	E^{g}_{E}	E^{g}_{M}
0.0000	0.0000	0.0000	0.0001
0.0000	0.0000	0.0002	0.0001
0.0000	0.0000	0.0005	0.0000
0.0000	0.0000	0.0009	0.0001

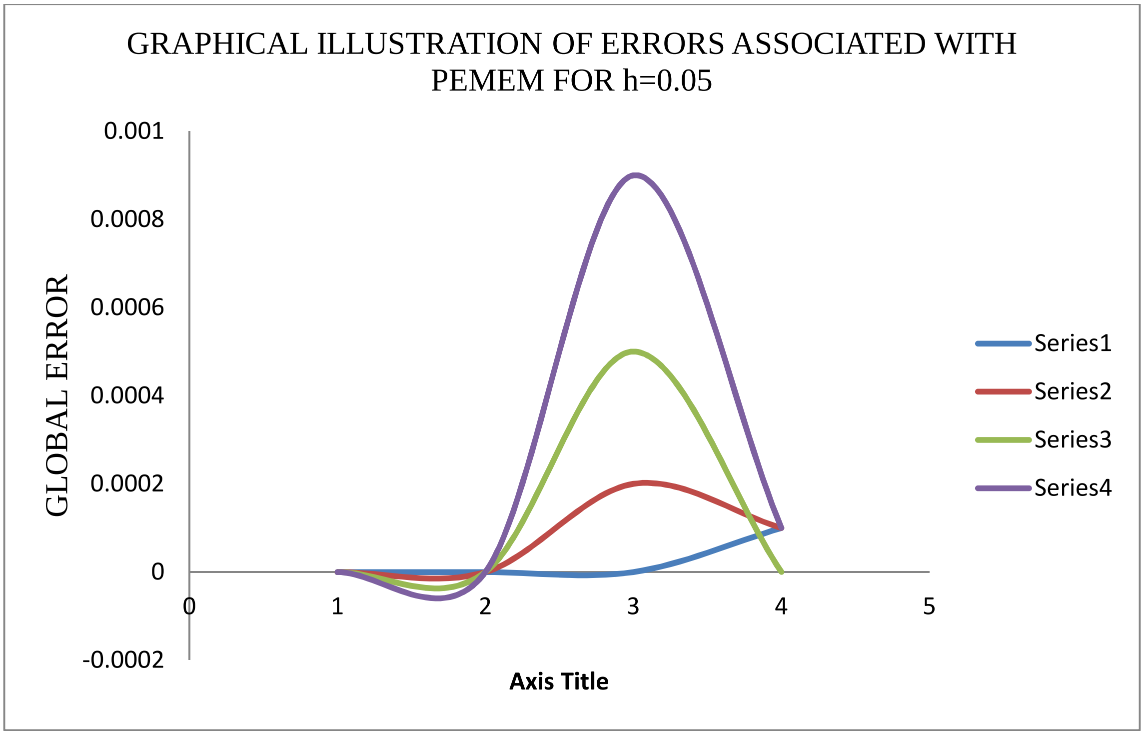

Figure 2 Graphical illustration of Associated Error in the Solution of Equation (1) Using AM, PM, EM and MEM

10.0 Result and Discussion

In Equations [(4), (15), (21) and (50)] shows the derived general form of the three Methods namely (Picard, Euler and Modified Euler) and the Analytical Method (AM) respectively. Similarly, Equations [(55),(56) and (57)] gives Expression for the Local, Global and Final global Errors respectively. Also; equations [(9) – (14)], [(17) – (20)] and [(25) – (36)] gives the approximate numerical solution to four decimal place of problem (1,2 and 3) using the proved equations in [(4), (15), (21) and (50)] for the three methods (Picard, Euler and Modified Euler) and the numerical results in chapter three. Again equations [(51) – (54)] and numerical solution for the analytical method of the general solution of problem 4 in equation (50). In addition, graphical illustrations for the general solutions and associated error were shown and displayed in figures 1 and 2 for the three methods (Picard, Euler and Modified Euler) and Analytical Method (AM) respectively. Tables [1 – 5] shows the numerical results together with their associated errors where necessary of the solutions obtained from solutions for the problems (1,2,3 and 4) using the three methods (Picard, Euler and Modified Euler) as well as the analytical method respectively. Table 1show the numerical solution obtained from Picard Method for the successive iterations. Similarly, numerical solution from the Analytical Method and the associated error were also displayed. Table 2 shows the numerical solution obtained from Euler Method (EM) for the successive iterations. In addition, numerical solution from the Analytical Method and the associated error were also displayed. Table 3 shows the numerical solution obtained from Modified Euler Method (MEM) for the successive iterations. More so, numerical solution from the Analytical Method and the associated error were also displayed.

Table 4 shows the numerical solution obtained from Analytical Method (AM). In addition, numerical solution from the Analytical Method and the associated error were also displayed. Table 5 displays the summary of numerical solution obtained from Picard, Euler, Modified Euler and Analytical Method (PEMEM & AM) for the specified values of (x) and table 6 thus; as seen it is important to note that PM gives an average error of 10.8%, EM 25% and MEM gives 20%. Hence it is still evident to say that PM is more accurate than EM and MEM followed by MEM to EM by average percentage error. Furthermore, in figure 1 shows the nature of the numerical solution obtained from Picard Euler, Modified Euler and Analytical Method (PEMEM). In Figure 2. Similarly, figure 2 shows the nature of the associated error in the numerical solution of problem 1, 2, 3 and 4 relative to the solution from Analytical Method (AM)for the three methods namely; Picard, Eulerand Modified Euler Method (PEMEM).

More so, the graphical illustration in figure 1 and 2 also displays the distinction and uniqueness by associated errors in the numerical solutions of the problem 1,2,3 and 4 using the three methods (PEMEM)and Analytical Method. It is clear to say that; Picard Method is the most accurate method for the solution of the problem which gives closely or approximately the same solution of the problem as the solution obtained from the analytical solution. Hence considered as most efficient, accurate and fastest by convergence.

Mores so, Modified Euler Method (MEM) is considered second position by accuracy but involves much iteration before converging to the required approximate solution (converges slowly). Conversely, Euler Method (EM) is the last position by error associated in the iterations involved in the solution of the problem 1,2,3 and 4, the graphical solutions helps to show how unique by either convergence or associated error a given method is, as such comparing the line curve for the solution of the problem in figure 1; the graphical solutions displayed for each method as well as the results in Table 1-3 with that of the Analytical Method (AM) in Table 1, it was observed that Picard Method (PM) Emerged the Winner by Accuracy, less iteration process, faster rate of convergence, negligible associated error and gives the same approximate solution as the Analytical Method (AM).

11.0 Conclusion

Explicitly and analytically solution has been obtained for the problem considered for each method and justified; as such it is very important to conclude that Picard Method (PM) is a more Stable Numerical Scheme (SNS) for the solution of First Order Differential Equation (FODE) with Initial Value Problem (IVP) with negligibly zero Globally Associated Error (GAE). However, on the other hand Euler Method (EM) and Modified Euler Method (MEM) are considered to be Conditionally Stable Numerical Scheme (CSNS). Even though, Modified Euler Method (MEM) is more accurate than Euler Method (EM) with less Globally Associated Error (GAE) but requires a lot of multiple iterations before converging to the approximate solution at the given point of evaluation than Euler Method (EM).Thus; by the aim and objective of this paper, successful conclusion is said that Picard Method is the Best, Accurate, most Convergent, Stable and reliable for the solution of First order Differential Equations (FODE) with Initial Value Problem (IVP) over the two other Methods (Euler Method (EM) and Modified Euler Method (MEM)) with respect to the Analytical Method (AM).

12.0 References

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