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Construction of Orthogonal Basis Function and Formulation of Continuous Hybrid Schemes for the Solution of Third Order ODEs

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Abstract

The paper presents development of a block algorithm for the numerical solution of ordinary differential equations with application to third order initial value problems. Collocation and interpolation techniques were adopted and a set of orthogonal polynomials valid in interval [-1, 1] with respect to weight function w(x) = x + 1 was constructed and employed as basis function for the development of continuous hybrid schemes. To make the continuous implicit schemes self-starting, a block method of discrete hybrid form was derived. Findings from the analysis of the basic properties of the method using appropriate existing theorems show that the developed schemes are consistent, zero-stable and hence convergent. On implementation, the schemes compared favourably well with the existing methods owing to the fact that they are accurate and efficient.

Keywords: Collocation, Interpolation, Orthogonal polynomials, Block method, Implicit, hybrid. **AMS Subject Classification**: 65L05, 65L06.

1.0 Introduction

Initial value problems of the form

$$y^{m} = f(x, y, y^{'}, ..., y^{m-1}) \qquad (1)$$

$$y(x_0) = y_0, y'(x_0) = y_0^{'}, ..., y^{m-1}(x_0) = y_0^{m-1}$$

arise frequently in applications and are useful in modelling of variety of physical phenomena such as oscillating systems, chemical reactor dynamics, predator-prey problem, electrical networks, advanced mass spring systems, biological dynamics and so on. Since most of these equations are difficult to solve, efficient ODEs solvers are much needed to approximate them. The various techniques of solving such problems have been considered in literature. In the recent years, the solutions of (1) for cases m = 1, 2 & 3 have been extremely discussed [1-4]. The approach of reducing the higher order of (1) to a system of first order and the consequent setback has also been discussed in [5].

Predictor-corrector method was later adopted and applied but has its setbacks which were discussed in [5].

To cater for the setback of predictor-corrector methods, the approach of block method came into being.

Milne [6] proposed a method called block method as a means of obtaining starting values which Rosser [7] developed into algorithms for general use. Later, the modified self-starting block method was given in [8] as

$$Y_m = \text{e}y_n + h^{«-ª} df(y_n) + h^{«-ª} bF(y_m) \qquad (2)$$

where **e** is s x s vector, d is r-vector and **b** is r x r vector, s is the interpolation points and r is the collocation points. F is a k-vector whose jth entry is $f_{n+j} = f(t_{n+j}, y_{n+j})$ and « is the order of the differential equation.

Given a predictor equation in the form

$$Y_m^{(0)} = ey_n + h^{«} df(y_n) \qquad (3)$$

Putting (3) in (2) gives

$$Y_m = \text{e}y_n + h^{«} df(y_n) + h^{«} bF(ey_n + h^{«} dfy_n) \qquad (4)$$

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which is called a self-starting block-predictor-corrector method [9, 10].
The block (4) is a simultaneous producing approximations to the solution of (1) at a block of desired points
However, the effectiveness of these ODE solvers depends on the types of trial functions used in developing the schemes. Various trial functions such as, the Chebyshev polynomials which was introduced in [11] as basis function for the solution of linear differential equations in term of finite expansion, the Legendre polynomials, Power series and the Canonical polynomials have been used to derive continuous schemes
The aim of this paper is to adopt the block method approach and develop orthogonal polynomials which are employed as basis function to derive a block method that provides direct solution to (1).

2.0 Construction of Orthogonal Basis Function

Let the function $q_n(x)$ be defined as

$$q_r(x) = \sum_{r=0}^{n} C_r^{(n)} x^r \qquad (5)$$

where $q_r(x)$ satisfies

$$< q_m(x), q_n(x) > = 0, \quad m \neq n, [-1,1] \qquad (6)$$

For the purpose of constructing the basis function, we use additional property that

$$q_n(1) = 1 \qquad (7)$$

For r = 0 in (5),

$$q_0(x) = C_0^{(0)}$$

From (7),

$$q_0(1) = C_0^{(0)} = 1$$

Hence,

$$q_0(x) = 1$$

For r = 1 in (5),

$$q_1(x) = C_0^{(1)} + C_1^{(1)} x \qquad (8)$$

By definition (7), (8) gives

$$C_0^{(1)} + C_1^{(1)} = 1 \qquad (9)$$

and

$$< q_0, q_1 > = \int_{-1}^{1} (x+1) q_0(x) q_1(x) dx$$

which implies

$$C_1^{(1)} + 3C_0^{(1)} = 0 \qquad (10)$$

Solving (9) and (10) and substituting the outcomes into (8), we have

$$q_1(x) = \frac{1}{2}(3x - 1) \qquad (11)$$

When r = 2 in (5),

$$q_2(x) = C_0^{(2)} + C_1^{(2)} x + C_2^{(2)} x^2 \qquad (12)$$

By definition (7), (12) gives

$$C_0^{(2)} + C_1^{(2)} + C_2^{(2)} = 1 \qquad (13)$$

$$< q_0, q_2 > = \int_{-1}^{1} (x+1) q_0(x) q_2(x) dx$$
$$= 0$$

which implies

$$2C_0^{(2)} + \frac{2}{3}C_1^{(2)} + \frac{2}{3}C_2^{(2)} = 0 \qquad (14)$$

$$< q_1, q_2 > = \int_{-1}^{1} (x+1)\, q_1(x) q_2(x) dx$$
$$= 0$$

which gives

$$\frac{2}{3}C_1^{(2)} + \frac{4}{15}C_2^{(2)} = 0 \qquad (15)$$

Solving (13), (14), (15) and substituting the resulting values into (12), we have

$$q_2(x) = \frac{1}{2}(5x^2 - 2x - 1) \qquad (16)$$

When n = 3 in (5),

$$q_3(x) = C_0^{(3)} + C_1^{(3)}x + C_2^{(3)}x^2 + C_3^{(3)}x^3 \qquad (17)$$

By definition (7), (17) gives

$$C_0^{(3)} + C_1^{(3)} + C_2^{(3)} + C_3^{(3)} = 1 \qquad (18)$$

$$< q_0, q_3 > = \int_{-1}^{1} (x+1)\, q_0(x) q_3(x) dx$$
$$= 0$$

which implies

$$2C_0^{(3)} + \frac{2}{3}C_1^{(3)} + \frac{2}{3}C_2^{(3)} + \frac{2}{5}C_3^{(3)} = 0 \qquad (19)$$

$$< q_1, q_3 > = \int_{-1}^{1} (x+1)\, q_1(x) q_3(x) dx$$
$$= 0$$

This leads to

$$\frac{2}{3}C_1^{(3)} + \frac{4}{15}C_2^{(3)} + \frac{2}{5}C_3^{(3)} = 0 \qquad (20)$$

$$< q_2, q_3 > = \int_{-1}^{1} (x+1)\, q_2(x) q_3(x) dx$$
$$= 0$$

Solving (18) – (20) and substituting the resulting values into (17), we obtain

$$q_3(x) = \frac{1}{8}(35x^3 - 15x^2 - 15x + 3) \qquad (21)$$

In the same vein, $q_n(x), n \geq 4$ are developed.

The next four polynomials which are used in this paper are listed hereunder.

$$q_4(x)=\frac{1}{8}(63x^4-28x^3-42x^2+12x+3)$$

$$q_5(x)=\frac{1}{16}(231x^5-105x^4-210x^3+70x^2+35x-5)$$

$$q_6(x)=\frac{1}{16}(429x^6-198x^5-495x^4+180x^3+135x^2-30x-5)$$

$$q_7(x)=\frac{1}{128}(6435x^7-3003x^6-9009x^5+3465x^4+3465x^3-945x^2-315x+35)$$

3.0 Literature Review

To investigate the applicability of the derived orthogonal polynomials, we briefly review here the work of Adam-Moulton on derivation of three-step implicit method whose discrete scheme is

$$y_{n+3}=y_{n+2}+\frac{h}{24}(f_n-5f_{n+1}+19f_{n+2}+9f_{n+3})$$

For this purpose, we shall seek an approximation of the form

$$y(x)=\sum_{r=0}^{s+k-1}a_r q_r(x) \qquad (22)$$

where $q_r(x)$ is the orthogonal polynomials derived.

We collocate and interpolate (22) at $x=x_{n+i}, i=0(1)3$ and $x=x_{n+2}$ respectively to obtain a system of equations which are solved and the resulting values of a_r are substituted back into (22) to have a continuous schemes. Evaluating the continuous scheme at the grid point $x=x_{n+3}$ yields the Adams-Moulton explicit three-step method.

We shall now employ the set of polynomials to formulate a continuous scheme through which numerical solutions of initial value problems in ordinary differential equations are obtained.

4.0 Methods and Materials

We consider here equation (22) to obtain the solution of (1) in the sub-interval $[x_n, x_{n+p}]$ of [a, b] taking our basis function to be an orthogonal function where $x=\frac{2x-x_n-ph}{ph}$ and p varies as the method to be derived.

Here, we are formulating a two-step method i.e. p = 2 and, s and k are points of interpolation and collocation respectively. The procedure involves interpolating (22) at s = 0, $\frac{2}{3}$, 1 and collocating the third order derivative of (22) at k = 0, $\frac{2}{3}$, 1, 2. The a_r, r = 0(1)6 from the resulting system of equations are obtained as

$$a_0=\frac{2}{3}y_n-3y_{n+2/3}+\frac{10}{3}y_{n+1}-3y_{2/3}+\frac{737h^3}{136080}f_n+\frac{2447h^3}{34020}f_{n+1}+\frac{757h^3}{272160}f_{n+2}+\frac{277h^3}{6048}f_{n+2/3}$$

$$a_1=\frac{11}{15}y_n+\frac{52}{15}y_{n+1}-\frac{21}{5}y_{n+2/3}+\frac{2377h^3}{340200}f_n+\frac{8767h^3}{85050}f_{n+1}+\frac{419h^3}{97200}f_{n+2}+\frac{3673h^3}{75600}f_{n+2/3}$$

$$a_2=\frac{3}{5}y_n+\frac{6}{5}y_{n+1}-\frac{9}{5}y_{n+2/3}+\frac{1367h^3}{226800}f_n+\frac{671h^3}{8100}f_{n+1}+\frac{1963h^3}{453600}f_{n+2}+\frac{1223h^3}{50400}f_{n+2/3}$$

$$a_3=\frac{8h^3}{3465}f_n+\frac{64h^3}{1485}f_{n+1}+\frac{32h^3}{10395}f_{n+2}-\frac{4h^3}{385}f_{n+2/3}$$

$$a_4=\frac{h^3}{1386}f_n+\frac{2h^3}{189}f_{n+1}+\frac{h^3}{594}f_{n+2}-\frac{h^3}{77}f_{n+2/3}$$

$$a_5=\frac{17h^3}{45045}f_n-\frac{8h^3}{4095}f_{n+1}+\frac{61h^3}{90090}f_{n+2}+\frac{9h^3}{10010}f_{n+2/3}$$

$$a_6=\frac{h^3}{8580}f_{n+2}-\frac{2h^3}{2145}f_{n+1}-\frac{h^3}{4290}f_n+\frac{3h^3}{2860}f_{n+2/3}$$

and substituted into (22) to have the continuous implicit method

$$y(x) = \sim_0(t) y_n + \sim_{\frac{2}{3}}(t) y_{n+\frac{2}{3}} + \sim_1(t) y_{n+1} + h^3 (\ _0(t) y_n + \ _{\frac{2}{3}}(t) y_{n+\frac{2}{3}} + \ _1(t) y_{n+1} + \ _2(t) y_{n+2}) \qquad (23)$$

where

$$\sim_0(t) = \frac{1}{2}(3t^2 + t)$$

$$\sim_{\frac{2}{3}}(t) = -\frac{9}{2}(t^2 + t)$$

$$\sim_1(t) = 3t^2 + 4t + 1$$

$$_0(t) = \frac{h^3}{12960}(-81t^6 + 108t^5 + 135t^4 + 86t^2 + 32t)$$

$$_{\frac{2}{3}}(t) = \frac{h^3}{2880}(81t^6 - 405t^4 + 464t^2 + 140t)$$

$$_1(t) = \frac{h^3}{3240}(-81t^6 - 54t^5 + 405t^4 + 540t^3 + 176t^2 + 14t)$$

$$_2(t) = \frac{h^3}{25920}(81t^6 + 216t^5 + 135t^4 + 4t^2 + 4t)$$

and $t = \dfrac{x - x_n - h}{h}$.

Evaluating equation (23) at x = x_{n+2} yields the discrete equation

$$y_{n+2} = 2y_n + 8y_{n+1} - 9y_{n+\frac{2}{3}} + \frac{7h^3}{324} f_n + \frac{7h^3}{72} f_{n+\frac{2}{3}} + \frac{25h^3}{81} f_{n+1} + \frac{11h^3}{648} f_{n+2} \qquad (24)$$

5.0 Derivation of the Block Method

To develop the block method from the continuous scheme, we adopt the general block formula proposed in [8] in the normalized form given

$$A^{(0)} Y_m = e y_m + h^{«-ª} d f(y_m) + h^{«-ª} b F(y_m) \qquad (25)$$

Evaluating the first and second derivative of (23) at x = x_{n+j}, j = 0, 2/3, 1, 2 and substituting the resulting equations and equation (24) into (25) gives its coefficients as

$$d = \begin{bmatrix} -\frac{17}{540} & \frac{13}{160} & \frac{2}{5} & \frac{139}{1215} & \frac{23}{120} & \frac{7}{15} & \frac{19}{81} & \frac{11}{48} & \frac{1}{3} \end{bmatrix}^T,$$

$$e = \begin{bmatrix} \frac{8}{9} & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{2}{3} & 1 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 2 & \frac{2}{3} & 1 & 2 & 1 & 1 & 1 \end{bmatrix}^T, \quad b = \begin{bmatrix} -\frac{3}{40} & \frac{9}{64} & \frac{9}{10} & \frac{17}{90} & \frac{9}{20} & \frac{9}{10} & \frac{2}{3} & \frac{27}{32} & 0 \\ -\frac{23}{135} & \frac{-7}{120} & 0 & \frac{-104}{1215} & \frac{-3}{20} & \frac{8}{15} & \frac{-20}{81} & \frac{-1}{12} & \frac{4}{3} \\ -\frac{7}{360} & \frac{1}{320} & \frac{1}{30} & \frac{11}{2430} & \frac{1}{120} & \frac{1}{10} & \frac{1}{81} & \frac{1}{96} & \frac{1}{3} \end{bmatrix}^T,$$

where A^0 = 9x9 identity matrix.

6.0 Basic Properties of the Block Method

7.0 Order and Error Constant of the Block

The linear operator L of the block (25) is defined as

$$L\{y(x):h\} = Y_m - e y_n + h^{«-ª} d f(y_m) + h^{«-ª} b F(y_m) \qquad (26)$$

Using Taylor series expansion to expand $y(x_n + ih)$ and $f(x_n + jh)$, (26) becomes

$$L\{y(x):h\} = C_0 y(x) + C_1 h y'(x) + C_2 h^2 y''(x) + ... + C_p h^p y^{(p)}(x) + \qquad (27)$$

The block (25) and associated linear operator are said to have order p if

$$C_0 = C_1 = ... = C_p = C_{p+1} = 0, \ C_{p+2} \neq 0$$

The term $C_{p+2} \neq 0$ is called the error constant and the local truncation error is given
As

$$t_{n+k} = C_{p+2} h^{(p+2)} y^{(p+2)}(x_n) + 0h^{(p+3)}$$

Thus, the block (26) is of order 5 with error constants C_{p+2} given as

$$C_{p+2} = \left[-\frac{44}{229635} \quad -\frac{37}{60480} \quad -\frac{4}{945} \quad -\frac{1}{135} \quad -\frac{7}{4320} \quad -\frac{1}{135} \quad -\frac{17}{7290} \quad -\frac{1}{480} \quad -\frac{1}{90} \right]^T$$

8.0 Zero stability of the Block

The block (25) is said to be zero stable if the roots z_s = 1, 2, . . ., N of the characteristic polynomial $°(z) = \det(zA - E)$, satisfies $|z| \leq 1$ and the root $|z| = 1$ has multiplicity not exceeding the order of the differential equation. Also, as $h^{«} \to 0, °(z) = z^{r-«}(ª - 1)^{«}$, where « is the order of the differential equation, $r = \dim(A^{(0)})$.

For the block (25), r = 9, « =3 and $°(z) = ª^6(ª - 1)^3$,

Hence the proposed method is zero stable.

9.0 Convergence

The necessary and sufficient condition for a numerical method to be convergent is for it to be Zero stable and consistent. This method is consistent owing to the fact that the order $p \geq 1$, hence it is convergent.

10.0 Numerical example

We implement the scheme on four test problems.

Problem 1

$$y''' = -y,\ y(0) = 1,\ y'(0) = -1,\ y''(0) = 1,\ 0 \leq x \leq 1$$

$$Exact\ \ solution: y(x) = e^{-x}$$

Problem 2

$$y''' = -e^x,\ y(0) = 1,\ y'(0) = -1,\ y''(0) = 3,\ 0 \leq x \leq 1$$

$$Theoretical\ \ solution: y(x) = 2 + 2x^2 - e^x$$

Problem 3

$$y''' = 3\sin x,\ y(0) = 1,\ y'(0) = 0,\ y''(0) = -2,\ 0 \leq x \leq 1$$

$$True\ \ solution: y(x) = 3\cos x + \left(\frac{x^2}{2}\right) - 2$$

Problem 4

$$y''' = e^x,\ y(0) = 3,\ y'(0) = 1,\ y''(0) = 5,\ 0 \leq x \leq 1$$

$$Theoretical\ \ solution: y(x) = 2 + 2x^2 + e^x$$

Table 1: Comparison of the solutions of Exact, the New Method and Existing Method of test problem 1

X	Exact Solution	New Method	Error in New Method	Error in [12] (p = 6)
0.1	0.904837418035960	0.904837417979346	5.661382473931553e-11	1.95961e-11
0.2	0.818730753077982	0.818730752688398	3.895839206791152e-10	2.47756e-09
0.3	0.740818220681718	0.740818219066154	1.615563682832999e-09	4.18183e-08
0.4	0.670320046035639	0.670320041963222	4.072417292277919e-09	1.18007e-07
0.5	0.606530659712633	0.606530651450277	8.262356576693719e-09	2.30843e-07
0.6	0.548811636094027	0.548811621637371	1.445665509525185e-08	3.79421e-07
0.7	0.496585303791410	0.496585280733952	2.305745783193203e-08	5.60411e-07
0.8	0.449328964117222	0.449328929844064	3.427315736770353e-08	7.63714e-07
0.9	0.406569659740599	0.406569611326962	4.841363732133175e-08	9.62881e-07
1.0	0.367879441171442	0.367879375547914	6.562352861116949e-08	1.09640e-06

Table 2: Comparison of the solutions of Exact, the New Method and Existing Method of test problem 2

X	Exact Solution	New Method	Error in New Method	Error in [12] (p = 6)
0.1	0.914829081924352	0.914829081858189	6.616285297411650e-11	2.00922e-11
0.2	0.858597241839830	0.858597241379556	4.602747072368629e-10	2.60461e-09
0.3	0.830141192423997	0.830141190457356	1.966641183237528e-09	4.50760e-09
0.4	0.828175302358730	0.828175297256612	5.102117950350760e-09	1.27448e-08
0.5	0.851278729299872	0.851278718475214	1.082465772572760e-08	2.50056e-08
0.6	0.897881199609491	0.897881179843941	1.976554986349299e-08	3.13587e-08
0.7	0.966247292529523	0.966247259434682	3.309484120084250e-08	4.00392e-08
0.8	1.054459071507532	1.054459019923943	5.158358939993946e-08	5.20610e-08
0.9	1.160396888843050	1.160396812182158	7.666089274493970e-08	1.4132e-07
1.0	1.281718171540955	1.281718062272431	1.092685231185442e-07	1.7644e-07

Table 3: Comparison of the solutions of Exact, the New Method and Existing Method of test problem 3

X	Exact Solution	New Method	Error in New Method	Error in [13]
0.1	0.990012495834077	0.990012495848359	1.428168694417309e-11	1.65922e-10
0.2	0.960199733523725	0.960199733629373	1.056474907557004e-10	4.76275e-10
0.3	0.911009467376818	0.911009467899794	5.229756627755933e-10	6.23182e-10
0.4	0.843182982008655	0.843182983535743	1.527087678532269e-09	2.91345e-10
0.5	0.757747685671118	0.757747689443271	3.772152812331342e-09	8.71118e-10
0.6	0.656006844729035	0.656006852484582	7.755547204446600e-09	3.92904e-09
0.7	0.539526561853465	0.539526576364118	1.451065279045594e-08	9.55347e-09
0.8	0.410120128041497	0.410120152793067	2.475157034886877e-08	1.80415e-08
0.9	0.269829904811993	0.269829944661690	3.984969709769359e-08	3.03120e-08
1.0	0.120906917604419	0.120906978311808	6.070738874097703e-08	4.73044e-08

Table 4: Comparison of the solutions of Exact, the New Method and Existing Method of test problem 4

X	Exact Solution	New Method	Error in New Method	Error in [13]
0.1	3.125170918075648	3.125170918141810	6.616263092951158e-11	7.56479e-11
0.2	3.301402758160170	3.301402758620444	4.602740411030482e-10	1.83983e-09
0.3	3.529858807576003	3.529858809542644	1.966640628126015e-09	4.42400e-09
0.4	3.811824697641271	3.811824702743388	5.102116951150038e-09	1.03587e-08
0.5	4.148721270700128	4.148721281524786	1.082465761470530e-08	1.12999e-08
0.6	4.542118800390509	4.542118820156059	1.976555008553760e-08	1.46095e-08
0.7	4.993752707470477	4.993752740565318	3.309484153390940e-08	2.05295e-08
0.8	5.505540928492469	5.505540980076057	5.158358806767183e-08	1.95075e-08
0.9	6.079603111156950	6.079603187817842	7.666089185676128e-08	1.08431e-08
1.0	6.718281828459046	6.718281937727569	1.092685231185442e-07	1.54095e-07

11.0 Conclusion

The development of new basis function has been considered and employed to derive a numerical block integrator for the solution of third order initial value problems in ordinary differential equations. The results obtained, as apparent from the foregoing tables, show that the derived orthogonal polynomials can be suitably adopted to construct continuous schemes for numerical solution of ODEs. Numerical results presented in Tables 1-4 have demonstrated an apparent agreement between the exact solutions and the solutions obtained through the methods. On comparison, the solutions obtained through the proposed method compared favourably well with the existing methods.

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