Journal of the Nigerian Association of Mathematical Physics
Volume 29, (March, 2015), pp 11 – 20

Further Results on Some Properties of Special Metrics in Location Optimization Models

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Abstract

The planar location problem has been studied for many years. Yet, a number of important real world issues and variants have not been investigated and merit further attention. This paper describes new statements of the problem with new and different metrics, their properties and relationship existing between them.

These enumerated metrics are special cases of the class of metrics that not only give a good characterization of distance but also provide better interpretation of travel distances especially for real life location problems. Using their properties, we introduce new modifications into the classification scheme of planar models. Furthermore, it is shown that these metrics can provide an insight into better classification of planar models due to its numerous applications. An extension of the modifications is suggested to other location models such as discrete and network models.

Keywords: distance function, locational search problem, special metrics, angular metrics, optimization.
MSC: 90C80, 90C85.

1.0 Introduction

Facility location problem (also known as location search problem) are concerned with finding optimal location for one or more facilities with respect to existing facilities whose locations are known. In this case, decisions concerning the number of new facilities, the choice of a distance function, and the solution procedure are critical to the determination of the optimal location or locations[1]. A distance function is used in location models to estimate the travel distance between a new and existing facilities. Distance measures are essential in the solution of many pattern recognition problems such as classification, clustering and information retrieval [2-4]. There are several numbers of distance measures encountered in many different fields such as anthropology, biology, chemistry, computer science, ecology, mathematics, physics, psychology, statistics etc. From mathematical point of view, distance is defined as a quantitative degree of how far apart two objects. Those distance measures satisfying the metric properties are simply called metric. How far apart two objects or approximation of their separation is valuable for the strategic and planning analysis of transportation and logistics problems. Locational search models have been very well studied over the years [5-12]. Even though the contexts in which the models are situated may differ, their main features are always the same: a space including a metric, customers whose locations in the given space are known, and facilities whose locations have to be determined according to some objective functions [11].

Many significant results have been published by a host of researchers in recent years. Some of the works have been reported in forms of books, articles and surveys [13-19]. Nonetheless, a number of meaningful and practical issues in location models remain unattended. This article surveys the modelling issues associated with these outstanding problems and suggests realistic problem statements that have, to the best of author's knowledge, apparently have been insufficiently attempted in the literature.

The objective of this paper is to extend some of the recent results on special metrics in planar location models. The extensions will be illustrated on the general classification scheme proposed in [20]. In particular, we obtain new modifications of the scheme, which are important to study of planar models, and new modifications of the discrete and network models. The most recent materials related to this paper can be found in [21].

2.0 Preliminaries

There are two general types of mathematical distance functions for measuring the distance between objects in space. It is assumed throughout the work that a distance function measures the shortest possible distance between objects. The first distance function type provides the following mapping:

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DEFINITION 2.1.For the points $x, y, z \in \Re^n$,

$r : \Re^n \times \Re^n \to \Re$ (2.1)

where $\Re^n \times \Re^n$ is the set of ordered pairs of points and $\Re$ is the real number. A function r that has the following properties is called a metric.

M1: $r(x,x) = 0,\ r(x,x) > 0, if\ \ x \neq y,$ (2.2)

M2: $r(x,y) = r(y,x),$ (2.3)

M3: $r(x,y) \leq r(x,z) + r(z,y)$ (2.4)

Thus, r determines the non-negative real number, which represents the shortest distance from x to y.

The second type of distance functions is called norms.

DEFINITION 2.2.For $x, y \in \Re^n$, a norm is a function whose mapping is

$k : \Re^n \to \Re$ (2.5)

and which has the properties

N1: $k(x) > 0$ if $x \neq 0$ (2.6)

N2 $k(x) = 0$ if $x = 0$ (2.7)

N3: $k(ax) = |a| k(x)$ for all $a \in \Re$, and (2.8)

N4: $k(x+y) = k(x) + k(y)$ (2.9)

where the function $k(x)$ denotes the nonnegative distance from 0 to x .Several of these norms and its variants such as block, mixed and round norms have had widespread use in facilities location applications [1, 5, 22-24].

A norm can be used to define a metric by setting

$r(x,y) = k(x-y)$ (2.10)

and is called the metric induced by the norm.

Metric or distance function is necessary for the estimation of actual distance in location optimization. The Minkowski distance of order p or l_p metrics is the most widely used measure of distance between two points X and Y in $k-$ dimensional space, defined as [25]

$$l_p\left((x_1, x_2, ..., x_k), (y_1, y_2, ..., y_k)\right) = \left(\sum_{i=1}^{k} |x_i - y_i|^p\right)^{\frac{1}{p}},\ p \geq 1 \qquad (2.11)$$

For $p = 1,$ we get the city block distance (also called the Manhattan distance); for $p = 2,$ we get the well-known Euclidean distance; if we let $p = \infty,$ we get the Chebyshev distance or l_∞ metric.

Other variants of l_1 metric applicable in data mining, bio-informatics and criminal behavior clustering include [1, 4, 26]:

- Canberra $\sum_{i=1}^{k} \frac{|x_i - y_i|}{x_i + y_i}$ (2.12)
- Soresen (or Bray-Curtis often used in botany, ecology and environmental science)

$$\frac{\sum_{i=1}^{k} |x_i - y_i|}{\sum_{i=1}^{k} (x_i + y_i)} \qquad (2.13)$$

- Gower $\frac{1}{k} \sum_{i=1}^{k} |x_i - y_i|$ (2.14)

- Soergel $$\frac{\sum_{i=1}^{k}|x_i - y_i|}{\sum_{i=1}^{k}\max(x_i, y_{i)}}$$ (2.15)

- Jaccard $$\frac{\sum_{i=1}^{k}|x_i - y_i|}{\left(n - \sum(1-x_i)(1-y_i)\right)}$$ (2.16)

- Mahalanobis (generalized form of Euclidean distance, [26, 27])

$$\|x-y\|_A = \sqrt{(\det A)^{\frac{1}{n}}(x-y)A^{-1}(x-y)^T} \tag{2.17}$$

where A is a positive-definite matrix.

3.0 Previous Work on Weber location Problem with Different and Special Metrics

In this section, we look at the mathematical formulation of the Weber location problem. Let k existing facilities $P_1, P_2, ..., P_k$ be given in the plane $\Re^2$, with corresponding positive weights $w_1, w_2, ..., w_k$. If $X(x_1, x_2) \in \Re^2$ denote the location of the new facility, then the problem of minimizing a sum of weighted Euclidean distances between the new and existing facilities is the problem of minimizing the function

$$f(X) = \sum_{i=1}^{k} w_i r(X, P_i) \tag{3.1}$$

Planar location problems with forbidden regions have been extensively studied and relatively well solved [28,29]. It is common to have forbidden regions in the plane where it is not possible to locate facilities. In this case, the problem (3.1) is reformulated as the minimization of the following objective function

$$f(X) = \sum_{i=1}^{k} w_i r_F(X, P_i) \tag{3.2}$$

The works of [5, 29], introduced non-convex barrier problem and shown that optimal solution can be found by solving a finite (and in case of line barriers a polynomial) number of related unconstrained subproblem.

$$f(X) = \sum_{i=1}^{k} w_i r_B(X, P_i) \tag{3.3}$$

On the other hand, there is at times the need to locate many facilities in the plane. In this case, the decision is to determine the location of new facilities including interactions between them and existing facilities, or association of a new facility for each existing facility following the mathematical model. The multi-facility Weber problem is defined by the following model [30-32]:

$$F(X_j) = \sum_{i=1}^{m}\sum_{j=1}^{n} w_{i,j} r_{MF}(X_j, P_i) + \sum_{j=1}^{n-1}\sum_{k=j+1}^{n} v_{j,k} r_{MF}(X_j, X_k) \tag{3.4}$$

The choice of distance functions is an important factor in location model representation. The l_p metrics have received the most attention from location analysts. But many other types of distances have been exploited, for special purposes in the facility location problems. A review of metrics exploited in many variations of location problems include distance functions based on mixed distances [7], Tchebychev norms [33], weighted one-infinity norm [34, 35], minisum location problem under A-distance [36], continuous and network distances [37],location problems based on angular distances [38], location problems on a sphere and arbitrary surface [40, 47].

The use of block norms (a special case of norm distances) for characterizing actual travel distances in city-block travel was suggested by Witzgall [39]. Ward and Wendell [23, 24] also studied block norms for the estimation of actual distances and developed facility location model that can be solved using linear programming. Love and Morris [34, 35] have investigated six distance models based on Cartesian and spherical coordinates (also called Cartesian and spherical distances). The spherical distances were proposed for modeling very long (trans-continental) actual distances on maps [40]. Perreur and Thisse [41] proposed the use of special distance functions named " central metrics" in location models, applicable when the transportation system follow a particular pattern such as star-shaped, circum-radial and circumferential. Spath [42] introduced Jaccard metric in minisum problems and proposed its usage in multiple location-allocation from cluster analysis. An algorithm was proposed in Watson [43] for single-facility location problem with Jaccard distance. Klein [48] suggested the use of a distance model, which is called the "Moscow metric", useful in the construction of Voronoi diagrams for cities such as Moscow, Amsterdam and Karlsruhe. In cities mentioned, once a Voronoi diagram has been determined for a geographical area, the nearest neighbor to a query point can be found efficiently. This has been shown to be useful in emergency response and criminal control [1]. Recent efforts in this direction can be found in [21-22, 32, 38, 44-46].

Literature in location analysis has witnessed a wide array of distance functions or metrics to represent travel distances between new and existing facilities. The location models considered here assume "special metrics" applicable in shortened distance only or where special transportation means are employed. These metrics are useful in clustering, business and planning, transportation, criminal analysis, and optimal location of facilities involving rotating telescopic boom, manipulator etc [22, 32, 38, 47]. Despite their practical relevance, location problems with this kind of distance measures have received little attention in the location literature.

Herein, we give definitions and properties of some special class of metrics recently considered in location models.

DEFINITION 4.1. ([26, 46]) For any $x, y \in \Re^2$, we define the *lift metric* between x and y, $r_L(x, y)$, as

$$r_L(x, y) = \begin{cases} |x_1 - y_1|, x_2 = y_2, \\ |x_1| + |x_2 - y_2| + |y_1|, x_2 \neq y_2. \end{cases} \quad (3.5)$$

We call r_L the lift metric.

This metric suggests a distance model for road networks with only one main street and the other crossing it at right angles. In [46], authors considered the use of lift distances in 2-D (in cities like Zankitos). Discussed also are single-facility location problems using lift distances and an algorithm proposed for its solution. The 3-D is also possible, for instance in buildings with 1 elevator and 100 floors and many offices, i.e where to locate restaurant in such building (assuming nobody uses stairs). Another application that translate into location problem with the lift metric is presented in the use of loading machine (forklift) moving along a rack storage [32, 46]. Problems are drawn from recognition that certain features of existing models are not realistic and from encounters with practical problems in industry. Indeed, the lift metric can be and has been successfully considered in solving discrete problems [46].

THEOREM4.1:*Let M be a metric space with a metric r_L and let $x, y \in M$. Then r_L defined by*

$$r_L(x, y) = \begin{cases} |x_1 - y_1|, x_2 = y_2 \\ |x_1| + |x_2 - y_2| + |y_1|, x_2 \neq y_2 \end{cases} \quad (3.6)$$

is a metric.

Proof: Out of the properties of a metric, only the triangle inequality is worth showing. Exploiting the validity of the triangle inequality for r_L two times

For $x_2 = y_2$, given

$$r_L(x, y) = |x_1 - y_1|$$

consider $r_L(x, y) = |x_1 - y_1|$

$$\leq |x_1 - z_1 + z_1 - y_1| \leq |x_1 - z_1| + |z_1 - y_1|$$

$$\leq r_L(x, z) + r_L(z, y)$$

For $x_2 \neq y_2$, we have

$$r_L(x, y) = |x_1| + |x_2 - y_2| + |y_1|$$, then

$$r_L(x, y) \leq |x_1| + |x_2 - z_2| + |z_1| + |z_1| + |y_2 - z_2| + |y_1|$$

$\le r_L(x,z)+r_L(z,y)$

Next, we consider a single-facility Weber location problem under the lift metric. The 2-dimensional continuous Weber problem can be restated as follows. Let k demand points $P_1,\dots,P_k$ be given where $P_i\left(p_1^i, p_2^i\right), i=1,\dots,k.$ It is necessary to find a new point $X(x_1,x_2)\in\Re^2$ which has minimal sum of weighted distance with respect to given points. Thus, we need to solve the unconstrained optimization problem

$$f_L(X)=\sum_{i=1}^{m} w_i r_L(X,P_i) \quad (3.7)$$

where w_i is the positive weight for each p_i. The Weber problem (3.7) can be written explicitly using lift metric as

$$f_L(X)=\left(\sum_{i=1,i\in Q_L}^{m} w_i\left(|x_1|+\left|x_2-p_2^i\right|+p_1^i\right)+\sum_{i\notin Q_L} w_i\left|x_1-p_1^i\right|\right) \quad (3.8)$$

where $Q_L=\left\{i: x_2=p_2^i, 1\le i\le m\right\}$. In most of the continuous location problems the convex hull or rectangular hull of the existing facilities is contain the optimal solution. Herein, we show that the optimal solution of problem (3.8) lies in the smaller rectangle containing all points.

DEFINITION 4.2. Let $p_1, p_2,\dots,p_n$ be n points in the plane. We define the new points as

$R_1=\left(x_{1\min}, y_{1\min}\right), R_2=\left(x_{1\min}, y_{2\max}\right)$

$R_3=\left(x_{2\max}, y_{1\min}\right), R_4=\left(x_{2\max}, y_{2\max}\right)$

THEOREM 4.2. An optimal solution exists in the smaller rectangular hull of the existing facilities.

Proof. Let B be the rectangular hull of the points R_1,R_2,R_3,R_4 and $x=(x_1,x_2)$ be a point out of B. We show that x is not an optimal solution. We present the proof for the case $x_1>x_{2\max}$, the proof for the other cases will be the same. Let $x'=\left(x_{2\max}, x_2\right)$. For each $p_i=(a_i,b_i)$, $i=1,\dots,n,$ we have

$r_L(x,P_i)>r_L(x',P_i)\ge 0$

Therefore since $w_i\ge 0$

$$F(x)=\sum_{i=1}^{n} w_i r_L(x,P_i)>\sum_{i=1}^{n} w_i r_L(x',P_i)=F(x')$$

So x is not an optimal solution.

REMARK 4.1. Theorem 4.2 can be formulated using convex hull when an Euclidean metric is employed in the minisum location models.

DEFINITION 4.3.([26, 44]) Given a norm $\|\bullet\|$ on $\Re^2$, the *French Metro metric* is a metric on $\Re^2$, defined by

$$r_{FM}(x,y)=\begin{cases}\|x-y\|, x=cy,\\ \|x\|+\|y\|, x\ne cy.\end{cases} \quad (3.9)$$

We call r_{FM} the French metro metric. This kind of metric is applicable where the cities have road networks with streets either straight line emanating from a fixed centre or are straight lines through the central point. Authors [41] proposed among other distance functions, a star-shaped transportation networks applicable in bus system with a central station. The work of authors [44]presented the two possible cases. In case 1, both points A, B belong to the same line (ray) then $A=cB$, c is a constant and in case 2, $A\ne cB, \forall c\in\Re$. In French metro metric, if both points lay on the same ray, the distance between them is calculated as the Euclidean distance. Otherwise, the distance is calculated as the sum of the distances from both points to zero point. The French metro distance is important in practice. If we have a manipulator with a telescopic rotating boom [38] as the transportation means and this machine is allowed to rotate in shortened position only (zero length of the boom), this problem is a problem with French metro metric. Other real- life applications in the contexts of business and service planning, transportation, etc., can be found in [44].

Next, we also use similar approach above to formulate the Weber problem with French Metro metric. According to [44], the distance function between k given points $P_1, P_2, ..., P_k$ and the optimal point X is given by

$$r_{FM}(X, P_i) = \begin{cases} \|X, P_i\|, P_i = cX \\ \|X\| + \|P_i\|, P_i \neq cX \end{cases} \tag{3.10}$$

Transform using polar coordinates, formula (3.10) becomes

$$r_{FM}(X, P_i) = \begin{cases} |p^i_{R1} - x_{R1}|, p^i_{R2} = x_{R2} \\ p^i_{R1} + x_{R1}, p^i_{R2} \neq x_{R2} \end{cases} \tag{3.11}$$

The corresponding unconstrained optimization problem is given by

$$f_{FM}(X) = \sum_{i=1}^{k} w_i r_{FM}(X, P_i) \tag{3.12}$$

REMARK 4.2.The French metro metric is similar with the lift metric in the following sense: instead of orthogonal projections of two given points [46], we use projections of both pointson the zero point (0,0). Therefore, the length of the vectors from the lift metric is identical with zero in the French metro metric.

DEFINITION 4.4.In the case of *Moscow metric or Moscow-Karlsruhe (M-K) metric*, the distance between two points $x = (x_1, y_1)$ and $y = (y_1, y_2)$ is defined as:

$$r_M(x, y) = \begin{cases} \|x_1 - y_1\| + \min\{x_1, y_1\}\|x_2 - y_2\|, |x_2 - y_2| < 2, \\ \|x_1\| + \|y_1\|, |x_2 - y_2| \geq 2. \end{cases} \tag{3.13}$$

Applications of Moscow metric have been employed in the construction of Voronoi diagrams [48] for cities like Moscow, Karlsruhe, Amsterdam and finding optimal location in mechanisms involving rotating telescopic boom as a transportation means [38].As stated earlier, Moscow metric was proposed by Klein [48] for the construction of Voronoi diagram on $\Re^2$. The cities like Moscow, Karlsruhe and Amsterdam have streets that are divided into two classes: ''rays'' starting from the central place (Moscow Kremlin or central train station in Amsterdam) or ''rings'' around this central point. The ''rings'' do not cross each other, their form is close to a circle and the centre of these circles is close to the central place. In this case, we can reach point A from another point B using one of the following ways:

1) Directly if A and B belong to the same ''ray'';
2) Moving along the central point ;
3) Moving along the ''ray'' of the point A, then moving along the ''circle'' street.

If the polar coordinates for points $x, y \in \Re^2$ are (r_x, θ_x) and (r_y, θ_y), respectively, then the distance between them is equal to [26]

$$r_M(x, y) = \begin{cases} \min\{r_x, r_y\}\Delta(\theta_x - \theta_y) + |r_x - r_y|, 0 \leq \Delta(\theta_x, \theta_y) < 2, \\ r_x + r_y, 2 \leq \Delta(\theta_x, \theta_y) < \pi \end{cases} \tag{3.14}$$

Where $\Delta(\theta_x, \theta_y) = \min\{|\theta_x - \theta_y|, 2\pi - |\theta_x - \theta_y|\}, \theta_x, \theta_y \in [0, 2\pi]$ is the metric between angles. For more details on location problems based on angular distances, consult [38].

Consider a single-facility minisum location problem under the Moscow metric. Let k demand points $P_1, ..., P_k$ be given where $P_i(p^i_1, p^i_2), i = 1, ..., k.$ It is necessary to find a new point $X(x_1, x_2) \in \Re^2$ which has minimal sum of weighted distance with respect to given points. Thus, we need to solve the unconstrained optimization problem

$$f(X) = \sum_{i=1}^{k} w_i r_M(X, P_i) \tag{3.15}$$

where w_i is the positive weight for each p_i.

REMARK 4.3. In the case of French metro metric, the distance between two points always goes through the central point if these points do not belong to the same ' ray' or directly from the origin if otherwise (cases 1 and 2 in Moscow metric). The Moscow metric extends the French metro metric in a natural way to include movement along the 'ray' of the point A, then moving along the 'circle' street. The similarity exhibited can be exploited to propose an algorithm similar to the one in [44], for the minisum problem under Moscow distance.

DEFINITION 4.5. Given a norm on $\Re^2$ (in general on $\Re^n$), the *British rail distance* is a metric on $\Re^2$, define by [26]

$$\|x\|+\|y\| \tag{3.16}$$

on $x \neq y$ (and it is equal to zero, otherwise). Thus, distance between two points x and a_i is defined by

$$d_{BR}(x,a_i)=\begin{cases} x+a_i, X \neq A_i \\ 0, X = A_i \forall i=1,\ldots,m \end{cases} \tag{3.17}$$

assuming that any path between two points include the central point (origin). In a similar fashion, the unconstrained Weber optimization problem can be formulated as

$$f(X)=\sum_{i=1}^{k} w_i r_{BR}(X,P_i)=\left(\sum_{i=1}^{k} w_i a_i+\sum_{i=1}^{k} w_i x\right) \tag{3.18}$$

REMARK 4.4. This metric is similar to French metro metric as the distance is calculated as the sum of the distances from both points to zero point if the two points do not lay on the same ray (and it is equal to zero, otherwise). Interested readers are referred to [36] to learn more about British rail metric. Since the mathematical structure of the above problems is similar, theorem 4.2 can be formulated for their optimal solutions.

4.0 Improvements in Better Classification of Planar Location Models

In the field of location theory, planar location problems have always played an important role. A large body of literature [4, 5, 15, 28, 30, 38, 47] is witness to the development of location theory within a planar framework and its various successful applications. Historically, planar location models are the oldest location models and deal with geometrical representations of real-world problems, a broad range of different model types exists and must be taken into account when trying to solve location problems.

Since the naming conventions of location models are not unique, to avoid ambiguity, a classification scheme is absolutely necessary to give more transparency in scientific discussions and for overview articles.

Different classification schemes exist based on different criteria such as objective, decision variables and system parameters. On the other hand, schemes are proposed with specific application to network or competitive location models [40]. For a detailed survey on classification schemes, see [20, 49] and references therein. According to Tafazzoli and Mozafari [49], examples from the literature indicated the ability of 5-position classification scheme proposed in [20] to describe location models.

Review of classification scheme based on the use of codes. In this section, we review the general classification scheme developed for location problems in [20]. We have the following five-position classification:

pos1 / pos2 / pos3 / pos4 / pos5,

where the meaning of each position is described in Table 1:

Table 1: Modified five-position classification scheme

Position	Pos1	Pos2	Pos3	Pos4	Pos5
Meaning	Number of new facilities	Type of feasible set	Special restrictions	Type of distance functions	Type of objective function
Examples	$n \in \{1,\ldots,N\}$ Single-facility (n=1), Multi-facility (n>1), with facility regions $(n=\infty)$	**P** -continuous planar/3D **D** discrete **N** network	unconstrained, polyhedral barriers, round barriers, restricted zones, capacity restrictions etc	$l_p, \|\bullet\|$, special metrics, angular distances	$\min\sum$, min max

Several symbols have been proposed for use in each position for planar location models (for details see [20, 28, 49]), for instance, the classical Weber model has the classification $1/P/\bullet/l_2/\sum$ where the symbol $\bullet$ in Position 3 means that standard assumptions hold.

Next, we present the modification of Positions 3 and 4 (pos3 and 4) with the special metrics defined above.

In addition to the criteria in pos 3, we propose capacity restrictions such as the case of movement of load in a shortened distance only. While in pos 4, we propose special and angular metrics as new distance measures.

As an illustration, the classification can be written as $1/P/\bullet/r_L/\sum$ for classical Weber model with lift distance.

By using this scheme we can easily describe models which are not of classical type, multi-facility models, etc.
The metrics listed above describe distances on a plane or *n*-dimensional continuous space. However, they have one common feature: the location problems with these metrics are decomposed into linear programming problems, combinatorial problems or problems involving linear programming and integer programming [7, 20]. Thus, these problems can be formulated for discrete feasible set or for networks. Therefore, special symbol can be introduced for such problems in Position 2 of our classification. We propose notation PD: planar or other continuous problems solved by exact or discrete methods.

5.0 Conclusion and Direction for Future Studies

Mathematical distance functions have had a substantial impact in location analysis. It has
provided a valuable way to represent travel distance for location processes. Such processes such as retail site location, emergency services location and factory location. As seen in the preceding sections, various choices for the metrics have been proposed. Problems with the early choices (Euclidean metrics) concern direct distance between two points on the plane given the location coordinates of each point. However, travel between points may not be possible along a straight line or even a rectangular route.
Recent choices such as lift, French metro, Moscow metrics, etc., provides a better interpretation of travel distances in location theory and in applications. The development of efficient algorithm is an area of active research and important component of location optimization. Very efficient methods have been developed for different models and thus the decision maker needs to decide which model (and method) is the most appropriate for the situation at hand.
However, this research is not exhaustive as more work can be done in the development of algorithms especially for problems involving multi-facility case with angular distances using probabilistic assignment. Moreover, GIS involved location models are worth further investigation, and possibly location of public undesirable facilities, such as dumps, depending on its range in a non-urban environment.

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