Journal of the Nigerian Association of Mathematical Physics
Volume28, No. 1, (November, 2014), **pp 469 – 474**
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Existence of Optimal Parameters for a Non –Linear Black-Scholes Option Pricing Model with Transaction Cost and Portfolio Risk Measures

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Abstract

This paper studied the parameter associated with the risk adjusted non-Linear Black-Scholes option pricing model which incorporates the transaction cost and the risk of the portfolio measures.The existence, uniqueness and continuous dependence of the weak solution of the risk adjusted Black–Scholes model are established.The existence of the optimal parameters is established.

Keywords: Risk adjusted Black-Scholes model, Weaksolution, Optimal parameters, Transaction cost, Portfolio risk Measure

1.0 Introduction

Option price model for incomplete market proposed by [1] looked at the case were the volatility σ of the underlying stock process is uncertain but bounded from bellow and above by given constants $\sigma_1 < \sigma_2$.The risk from the unprotected volatile portfolio is described by the variance of the synthetized portfolio.Transaction costs as well as the volatile portfolio risk depend on the time –lag between two consecutive transactions [2].Minimizing their sum yields the optimal length of the hedge interval –time lag.This leads to a fully nonlinear parabolic PDE.If transaction costs are taken into account perfect replication of the contingent claim is no longer possible.Modeling the short rate $r = r(t)$ by a solution to a one factor stochastic differential equation[3].

$$dS = \mu(s,t)dt + \sigma(s,t)dw \qquad (1.1)$$

Where $\mu(S,t)dt$ represent a trend or drift of the process and $\sigma(S,t)$ represents volatility part of the process. The risk adjusted Black-Scholes equation can be viewed as an equation with a variable volatility coefficient

$$\partial_t V + \frac{\sigma^2(s,t)}{2} S^2 \left(1 - \mu(S\partial_S V)^{\frac{1}{3}}\right) \partial_S^2 V + rs\partial_S V - rV = 0, \qquad (1.2)$$

Where $\sigma^2(s,t)$ depends on a solution $V = V(s,t)$and $\mu = 3\left(\frac{C^2 R}{2\pi}\right)^{\frac{1}{3}}$, c is the transaction cost and R the portfolio risk measure. If $\mu = 0$ we recover the equation discussed in [4].

Taking $\hat{\sigma}^2(s,t) = \sigma^2(1 - \mu(S\partial_S^2 V(S,t))^{\frac{1}{3}}$, equation(1.2) becomes

$$\partial_t V + \frac{\hat{\sigma}^2}{2} S^2 \partial_S^2 V + rS\partial_S V - rV = 0. \qquad (1.3)$$

By setting $S = e^x, u(x,t) = V(e^x,t) and\, h(e^x) = g(x)$, we obtain the following parabolic PDE .

$$\frac{\partial u(x,t)}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} - (\Lambda - \alpha)\frac{\partial u(x,t)}{\partial x} + \Lambda u(x,t) = 0 \,,(x,t) \in \mathbb{Q} \quad ,u(x,0) = g(x) \in \mathbb{R}, \qquad (1.4)$$

where$g(x)$ is the pay-off function. For $T > 0, \mathbb{Q} = \mathcal{R} \times (0,T), \alpha = \frac{\sigma^2(1-\mu(S\partial_S^2 v(s,t))^{\frac{1}{3}}}{2}$ $and\ \Lambda = r.$

In this paper we discuss the parameters that are leading the Risk adjusted Black-Scholes option pricing model such that equation (1.4) exhibits the desired behavior. More precisely, let

$$\mathcal{P}_{ad} = \{q = (\alpha,\Lambda) \in [\alpha_{min},\alpha_{max}] \times [\Lambda_{min},\Lambda_{max}]\} ,$$

where

$$\alpha_{min} > 0\ and \Lambda_{min} > 0.$$

Defined a functional $J(q)$ by

$$J(q) = \|u(q,t) - z_d\|^2_{L^2(0,T;H)}, \qquad (1.5)$$

where the data z_d can be thought of as the desired value of $u(q;t)$. The parameter identification problem for (1.4) with the objective function (1.5) is to find

$$q^* = (\alpha^*,\Lambda^*) \in \mathcal{P}_{ad}$$

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satisfying

$$J(q^*) = inf_{q\in\mathcal{P}_{ad}} J(q). \quad (1.6)$$

Let

$$q \to u(q)$$

from,$\mathcal{P}$ in to $C([0,T]; H$ be the solution map . In what follows, the existence and uniqueness of the weak solution of (1.4) is established in the next section. Continuity of the solution with respect to data is established in section 3.

2.0 Existence and Uniqueness of weak solution

Since the type of equation in (1.4) do not belong to $L^2(\mathcal{R})$ we introduce weighted lebesgue and sobolev spaces $L^2_\beta(\mathcal{R})$and$H^1_\beta(\mathcal{R})$ for $\beta > 0$ as follows.

$$L^2_\beta(\mathcal{R}) = \{u \in L^1_{loc}(\mathcal{R}): ue^{-\beta|x|} \in L^2(\mathcal{R})\} \quad (2.1)$$

$$\mathrm{H}^1_\beta(\mathcal{R}) = \{u \in L^1_{loc}(\mathcal{R}): ue^{-\beta|x|} \in L^2(\mathcal{R}), u'e^{-\beta|x|} \in L^2(\mathcal{R})\}. \quad (2.2)$$

The respective inner products and norms are defined by

$$(u,v)_{L^2_\beta(\mathcal{R})} = \int_\mathcal{R} uve^{-2\beta|x|}dx, \quad (2.3)$$

$$(u,v)_{H^1_\beta(\mathcal{R})} = \int_\mathcal{R} uve^{-2\beta|x|}dx + \int_\mathcal{R} u'v'e^{-2\beta|x|}dx , (2.4)$$

$$\|u\|_{L^2_\beta(\mathcal{R})} = \left(\int_\mathcal{R} |u|^2 e^{-2\beta|x|}dx\right)^{\frac{1}{2}}, (2.5)$$

$$\|u\|_{H^1_\beta(\mathcal{R})} = \left(\int_\mathcal{R} |u|^2 e^{-2\beta|x|}dx + \int_\mathcal{R} |u'|^2 e^{-2\beta|x|}dx\right)^{\frac{1}{2}}. \quad (2.6)$$

We define the dual space of $\mathrm{H}^1_\beta(\mathcal{R})$ as

$$\left(H^1_\beta(\mathcal{R})\right)^* = \{u\backslash u: H^1_\beta(\mathcal{R}) \to \mathcal{R} \text{ is linear and continuous}\}. \quad (2.7)$$

The duality pairing between $H^1_\beta(\mathcal{R})$ and$\left(H^1_\beta(\mathcal{R})\right)^*$ is given by

$$\langle u,v\rangle = \int_\mathcal{R} |u|^2 e^{-2\beta|x|}dx. \quad (2.8)$$

In what follows, we state,

Lemma1:let$f = L^2_\beta(\mathcal{R})$.For $\emptyset \in C_0^\infty, supp\emptyset = (-1,1), \int_\mathcal{R} \emptyset d\,(x)dx = 1, and\ \emptyset_\epsilon = \frac{1}{\epsilon}\emptyset\left(\frac{x}{\epsilon}\right)$, then

$$\emptyset_\epsilon * f \to f \ in\ L^2_\beta(\mathcal{R}). \quad (2.9)$$

Proof: Suppose $q = e^{-2\beta|x|}$, then we have

$$(\emptyset_\epsilon * f).q = \left(\emptyset_\epsilon * (f.q) + (\emptyset_\epsilon * f).q - \emptyset_\epsilon * (f.q)\right). \quad (2.10)$$

Since $f.g \in L^2\ and\ \emptyset_\epsilon * (f.q)\ in\ L^2$, it suffices to show that

$$\|g_\epsilon\|_{L^2} = (\|(\emptyset_\epsilon * f).q - \emptyset_\epsilon * (f.q)\|) \to 0 \quad for\ \epsilon \to 0. \quad (2.11)$$

The fundamental theory of calculus for q gives

$$g_\epsilon(x) = \int_\mathcal{R} \emptyset_\epsilon(x-y)f(y)\left(q(x) - q(y)\right)dy. \quad (2.12)$$

Using$supp\emptyset_\epsilon = (-\epsilon,\epsilon)$, we get$|g_\epsilon(x)| \le \int_\mathcal{R} |\emptyset_\epsilon(x-y)||f(y)|(2\epsilon\ sup|q'(t)|)dy$

$= \int_\mathcal{R} |\emptyset_\epsilon(x-y)||f(y)|(2\epsilon\ sup|q'(y+s)|)dy = \overline{g_\epsilon}(x)$.(2.13)

Since $\overline{g_\epsilon}(x) = L^2$uniformly, and$|g_\epsilon(x)| \le 2\epsilon|\overline{g_\epsilon}(x)|$, thus $\|g_\epsilon(x)\|_{L^2} \to 0$as $\epsilon \to 0$.

Lemma 2: $D(\mathcal{R})$ the space of test function in $\mathcal{R}$, is dense in $\mathrm{H}^1_\beta(\mathcal{R})$.

Proof .Let $f \in H^1_\beta(\mathcal{R})\ and\ \Phi \in C^\infty$such that

$$\Phi(x) = \begin{cases} 1, & if|x| \le 1 \\ 0, & if|x| \ge 2 \end{cases}.$$

Now we show that

$$f_\epsilon = \left(f.\Phi(\epsilon(.))\right) * \Phi_\epsilon \in C_0^\infty,$$

where

$\Phi_\epsilon = \frac{1}{\epsilon}\Phi\left(\frac{x}{\epsilon}\right), f_\epsilon \to finH^2_\beta(\mathcal{R})$. ie

$$f_\epsilon \to f\ and\ \nabla f_\epsilon \to \nabla f\ in\ L^2_\beta(\mathcal{R}) \quad (2.14)$$

$$\nabla f_\epsilon = \left(f.\Phi(\epsilon(.))\right) * \Phi_\epsilon + \epsilon\left(f.\Phi(\epsilon(.))\right) * \Phi_\epsilon. (2.15)$$

It suffices to show

$$\left(f.\Phi(\epsilon(.))\right) * \Phi_\epsilon \to f \quad in\ L^2_\beta(\mathcal{R}). \tag{2.16}$$

By the Lebesgue Dominated convergence theorem [5],we get

$$f.\Phi(\epsilon(.)) \to f\ in\ L^2_\beta(\mathcal{R}). \tag{2.17}$$

Hence Lemma 1 concludes the proof.

Since $D(\mathcal{R})$ is dense in $H^1_\beta(\mathcal{R})\ and\ L^2_\beta(\mathcal{R})$, the lemma follows immediately.

Lemma 3:$H^1_\beta(\mathcal{R}) \subset L^2_\beta(\mathcal{R}) \subset \left(H^1_\beta(\mathcal{R})\right)^*$, from Gelfand triple.

Note. Since $D(\mathcal{R})$ is dense in $H^1_\beta(\mathcal{R})$,the definition of $\langle .,. \rangle$ allows us to interprete the operator $\mathcal{A}$ as a mapping from $H^1_\beta \to \left(H^1_\beta\right)^*$.

For our simplicity, we use

$V = H^1_\beta(\mathcal{R}), V^* = \left(H^1_\beta(\mathcal{R})\right)^*$ and $H = L^2_\beta(\mathcal{R})$

To use the variational formulation let us defined the following bilinear form on $V \times V$

$$\mathfrak{a}_{(\alpha,\Lambda)}(u,v) = \alpha\int_{\mathcal{R}} u'v'e^{-2\beta|x|} + \Lambda\int_{\mathcal{R}} uve^{-2\beta|x|}dx - (\Lambda-\alpha)\int_{\mathcal{R}} u'v e^{-2\beta|x|}dx \tag{2.18}$$

For $\alpha > 0\ and\ \Lambda > 0$.

One can show $\mathfrak{a}_{(\alpha,\Lambda)}(u,v)$ is bounded and coercive in V .Define linear operator $A_{(\alpha\equiv,\Lambda)}: D\left(A_{(\alpha,\Lambda)}\right) = \{u: u \in V, A_{(\alpha,\Lambda)}u \in V^*\}$ into V^* by $\mathfrak{a}_{(\alpha,\Lambda)}(u,v) = \left(A_{(\alpha,\Lambda)}, u, v\right)$ for all $u \in D\left(A_{(\alpha,\Lambda)}\right)$ for all $v \in V$.

Definition 4. Let X be a Banach space and $a, b \in \beta$ with $a < b, 1 \le p < \infty$.Then $L^2(0,T;X)$ and $L^\infty(0,T\,;X)$denote the space of measurable functions u defined on (a,b) with values in V such that the function $t \to \|u(.,t)\|_X$ is square integrable and essentially bounded. The respective norms are defined by

$$\|u\|_{L^2(0,T;X)} = \left(\int_a^b \|u(.,t)\|^2_X dt\right)^{\frac{1}{2}} \tag{2.19}$$

$$\|u\|_{L^\infty(0,T;X)} = ess.sup_{a\le t\le b}\|u(.,t)\|_X. \tag{2.20}$$

Definition 5.A function $u: [0,T] \to V$ is a weak solution of (1.4) if

(i) $u \in L^2(0,T;V)$and $u_t \in L^2(0,T;V^*)$;

(ii) For every $\in V, \langle u_t(t), v\rangle + \mathfrak{a}_{(\alpha,\Lambda)}(u(t),v) = 0$,for t pointwise almost every (a.e.) in $[0,T]$; $u(0) = u_0$.

Note .The time derivative u_t understood in the distributionsense.The following two lemmas are of critical importance for the existence and uniqueness of the weak solutions.

Lemma 6. Let $\hookrightarrow H \hookrightarrow V^*\ If\ u \in L^2(0,T;V)$,$u' \in L^2(0,T;V^*)$,*thenu* $\in C([0,T];H)$. Moreover, for any $v \in V$,the real –valued function $t \to \|u(t)\|_H{}^2$ is weakly differentiable in $(0,T)$ and satisfies

$$\frac{1}{2}\frac{d}{dt}\{\|u\|^2\} = \langle u', u\rangle \tag{2.21}$$

For proof, see [6]

Lemma 7.(Gronwall's Lemma) Let $\xi(t)$ be a nonnegative, summable function on $[0,T]$ which satisfies the integral inequality

$$\xi(t) \le C_1\int_0^t \xi(s)ds + C_2, \tag{2.22}$$

for constant $C_1C_2 \ge 0$,almost everywhere $t \in [0.T]$.Then

$$\xi(t) \le C_2(1 + C_1te^{C_1t})\text{a.e on } 0 \le t \le T. \tag{2.23}$$

In particular, if

$$\xi(t) \le C_1\int_0^t \xi(s)ds \text{ a.e on } 0 \le t \le T,\text{then } \xi(s) = 0\ a.e \text{ on } [0,T]. \tag{2.24}$$

For proof, see [7].

Lemma 8. The weak solution of (1.4) is unique if it exists.

Proof. Let u_1 and u_2 be two weak solution of (1.4). Let $u = u_1 - u_2$.To prove Lemma 8 it suffices to show that $u = 0$ pointwise a.e.on $[0,T]$.since $\langle u_t(t), v\rangle + \mathfrak{a}_{(\alpha,\Lambda)}(u(t),v) = 0$ for any $v \in V$,we take $v = u \in V$ to get

$$\langle u_t(t), u\rangle + \mathfrak{a}_{(\alpha,\Lambda)}(u(t),u) = 0 \tag{2.25}$$

(2.25) is true point wise a.s .on $[0,T]$.Using (2.1) and the coercivity estimate, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_H \le \gamma\|u\|^2_H, u(0) = 0$$

For some $\gamma > 0$.By Lemma 7, $\|u\|_H = 0$ for all $t \in [0,T]$.Thus $u = 0$ pointwise a.e in $[0,T]$.

To show existence of the weak solution of (1.4) .we first show existence and uniqueness of approximation solution. Now we define the approximate solution of (1.4)

Definition9. A function$u_m: [0,T] \to V_M$ is an approximate solutions of (1.4) if

(i) $u_m \in L^2(0,T,V_M)$and)$u_{m_t} \in L^2(0,T,V_M)$;

(ii) for every $v \in V_m$ and $\langle u_{M_t}(t), v\rangle H + \mathfrak{a}_{(\alpha,\Lambda)}(u_M(t),v) = 0$ pointwise a.e in $[o,T]$

(iii) $u_M(0) = P_{Mg}$

To prove the existence of approximate solution, we take $v = u_m$ in

$\langle u_t(t), u\rangle + \mathfrak{a}_{(\alpha,\Lambda)}(u(t), u) = 0$

to get following system of ODEs

$C^j_{M_t} + \sum_{k=1}^{M} \mathfrak{a}^{jk} C^k_M = 0\ , C^j_M(0) = g^j$ (2.26)

where

$C^k_M \in H, C^k_{M_t} \in H$,for$0 \le t \le T$,$\mathfrak{a}^{jk}(t) = a(w_j, w_k)$,and$g^j = (g, w_j)_HforC: [0,T] \to \mathcal{R}^N$,

equation(2.26)can be written as

$\vec{C}_{M_t} + A(t)\vec{C}_M = 0\ , \vec{C}_M(0) = \vec{g},$ (2.27)

since

$A \in L^\infty(0,T; \mathcal{R}^{M\times M}$,for$\vec{C}_M = \psi(\vec{C}_M)$.

Equation (2.27) can be written as

$\psi(\vec{C}_M(t)) = \vec{g} - \int_0^t A(s)\vec{C}_M(s)\, ds.$ (2.28)

The following lemma is immediate from contraction mapping theorem and (2.28).

Lemma 10: For any $M \in N$,there a unique approximate solution $u_m: [0,T] \to V_m$ of (2.28).

The following theorem provides the energy estimate for approximate solutions.

Theorem11.There exist a constant Cdepending only on $T\ and\ \Omega$ such that the approximate solution u_m satisfies

$\|u_m\|_{L^2(0,T;H)} + \|u_m\|_{L^\infty(0,T;V)} + \|u_{m_t}\|_{L^2(0,T;H)} \le C\|g\|_H$ (2.29)

Proof: For every $v \in u_m$ we have$\langle u_{M_t}(t), v\rangle H + \mathfrak{a}_{(\alpha,\Lambda)}(u_M(t), v) = 0$. Take$v \in u_m(t)$,then we have

$\langle u_{M_t}(t), v\rangle H + \mathfrak{a}_{(\alpha,\Lambda)}(u_M(t), v) = 0$,point wise a.e in $(0,T)$. (2.30)

Using (2.30) and the coercivity estimate, we find that there exists constants $\rho > 0, \gamma > 0$ such that

$\frac{1}{2}\frac{d}{dt}(e^{-2\gamma t}\|u_M\|^2_H) + \rho e^{-2\gamma t \|u_M\|^2_V} \le 0.$ (2.31)

Integrating (2.31) with respect to t, using the initial condition$u_M(0) = P_m(g)$, and$\|P_m(g)\|_H \le \|g\|_H$,

we get

$\frac{1}{2}\frac{d}{dt}(e^{-2\gamma t}\|u_M\|^2_H) + \rho e^{-2\gamma t \|u_M\|^2_V}$. (2.32)

Taking the supremum over $[0,T]$,we get

$\|u_m\|_{L^2(0,T;H)} + \|u_m\|_{L^2(0,T;V)} \le C\|g\|^2_{H.}$ (2.33)

Since $u_{M_t}(t) \in V^*_M$, we have

$\|u_{M_t}(t)\|_{V^*} = \sup_{v\in V^*_M} \frac{(u_{M_t}(t),v)H}{\|v\|_V}, v \ne 0\ .$ (2.34)

Using the notion of approximate solution and boundedness of A we have

$\|u_m\|_{L^\infty(0,T;H)} + \|u_m\|_{L^2(0,T;V)} + \|u_{m_t}\|_{L^2(0,T;H)} \le C\|g\|_H$ (2.35)

To complete the proof of weak solution, we now show the convergence of the approximate solutions by using weak compactness argument.

Definition 12: Let $L^2(0,T;V^*)\ be$ the dual space of $L^2(0,T;V)$.Let $f \in L^2(0,T;V^*)$ and $u \in L^2(0,T;V)$,then we say $u_M \to u\ in\ L^2(0,T;V)$ weakly if

$\int_0^T \langle f(t), u_M(t)\rangle dt \to \int_0^T \langle f(t), u(t)\rangle\, dt\ \ \forall\ \ f \in L^2(0,T;V^*)$ (2.36)

Lemma 13.A subsequence $\{u_m\}$ of approximate solutions u_m converge weakly in $L^2(0,T;V^*)$ to a weak solution $u \in C([0,T];H) \cap\ L^2(0,T;V)$of (1.4) with $u_t \in L^2(0,T;V^*)$.Moreover,it satisfies

$\|u\|_{L^\infty(0,T;H)} + \|u\|_{L^2(0,T;V)} + \|u_t\|_{L^2(0,T;H)} \le C\|g\|_H$ (2.37)

Proof. Theorem 11 implies that the approximate solutions $\{u_m\}$ are bounded in $L^2(0,T;V)$ and their derivatives $\{u_{m_t}\}$ are bounded in $L^2(0,T;V^*)$. By the Banach-Alaoglu theorem [8], we can extract a subsequence $\{u_m\}$ such that weakly,

$u_m \to u\ \ in\ L^2(0,T;V),\ u_{m_t} \to u_{t\ in}\ L^2(0,T;V^*)$ (2.38)

Let $\emptyset \in C^\infty_0$ (0,T) be a real-valued test function and $w \in V_N$ for some $N = \mathcal{N}$ .Replacing v by $\phi(t)w\ in\ \langle u_{M_t}(t), v\rangle_H + \mathfrak{a}_{(\alpha,\Lambda)}(u_M(t), v) = 0$

and integrating from 0 toT, we get.

$$\int_0^T \langle u_{M_t}(t), \phi(t)w\rangle_H dt + \int_0^T \mathfrak{a}_{(\alpha,\Lambda)}(u_M(t), \phi(t)w) dt = 0\ for\ M \ge N$$

taking the limit as $M \to \infty$,we get

$\int_0^T (u_{M_t}(t), \phi w)_H dt = \int_0^T \langle u_t, \emptyset w\rangle\, dt$ (2.39)

by using boundedness of $\mathfrak{a}_{(\alpha,\Lambda)}$,we get

$\int_0^T \mathfrak{a}_{(\alpha,\Lambda)}(u_M(t), \phi(t)w) dt = \int_0^T \mathfrak{a}_{(\alpha,\Lambda)}(u(t), \emptyset(t)w)\, dt$ (2.40)

using boundedness of $\mathfrak{a}_{(\mathfrak{P},\bar{k})}$,we get

$\langle u_t(t), w\rangle + \mathfrak{a}_{(\alpha,\Lambda)}(u, w) = 0$ (2.41) point wise a.e in $(0,T)$since (2.41) is true for all $w \in V_M$

$U_{M\in\mathcal{N}} V_M$and (2.42)

is dense in V,so (2.42) holds for all $w \in V$. Now it remains to show that $u(0) = u_0$.Using (2.42),integrating by parts and using Galerkin approximation we have

$\langle u(0), w\rangle = \langle u_0, w\rangle \; as \;\; M \to \infty$

for every $w \in V_M$.Thus $u(0) = u_0$.

3.0 Existence of Optimal Parameter

Lemma 14 .Let $v \in V$ .Then the mapping $(\alpha, \Lambda) \to A_{(\alpha,\Lambda)} v$ from

$\mathcal{P}_{ad} = \{q = (\alpha, \Lambda) \in [\alpha_{min}, \alpha_{max}] \times [\Lambda_{min}, \Lambda_{max}]\}$into$V'$ is continuous.

Proof . Suppose that$q_n \to q$ in $\mathcal{R}^2$ as $n \to \infty$.We denote $A = A_{\alpha,\Lambda}$ and $A_n = A_{\alpha_n,\Lambda_n}$.We claim that

$$\|(A_n - A)v\|_{V'} \to 0$$

as $n \to \infty$. Let $w \in V$ with$\|w\| \le 1$.Then

$|\langle (A_n - A)v, w\rangle|^2$

$$\le \int_{\mathcal{R}} (|\alpha_n - \alpha||u'||w'|dx)^2 + \left(\int_{\mathcal{R}} |\bar{\Lambda}_n - \Lambda||u||w|dx\right)^2 + \left(\int_{\mathcal{R}} |\Lambda_n - \Lambda||u'||w|dx\right)^2 + \int_{\mathcal{R}} (|\alpha_n - \alpha||u'||w|dx)$$

$$\le 2|\alpha_n - \alpha|^2 \int_{\mathcal{R}} |u'|(x)^2 dx + |\Lambda_n - \Lambda|^2 \int_{\mathcal{R}} |u'|(x)^2 dx + |\Lambda_n - \Lambda|^2 \int_{\mathcal{R}} |u'|(x)^2 dx \to 0 \quad as\; n \to \infty$$

Lemma 15.Suppose that $\alpha_n, \Lambda_n \to \alpha, \Lambda$in $\mathcal{R}^2$, and $v_n \to v$ weakly in V as $n \to \infty$.Then $A_n v_n \to A_v$ weakly in v'.

Proof.Let $w \in V$,then.

$$\left|\langle A_{n,} v_{n,} w\rangle - \langle A_v, w\rangle\right| = \left|\langle A_n w, v_{n,}\rangle - \langle A_w v\rangle\right| \le |\langle A_n - A\rangle w, v_n| + |\langle Aw, v_n - v\rangle| \qquad (2.43)$$

Since a weakly convergent sequence is bounded, we have

$$|\langle A_n - A\rangle w, v_n| \le \|A_n w - Aw\|V'\|v_n\| \le c\|A_n w - Aw\|V' \to 0$$

as$n \to \infty$ Lemma 14.The second term

$$\left|\langle A_{n,} v_{n,} - v\rangle\right| \to 0$$

since$v_n \to v$ weakly.

Lemma16. Let $q_n \in \mathcal{P}_{ad}$. Then the solution map $q \to u(q)$ from $\mathcal{P}$ into $C([0,T]; H)$ is continuous.

Proof.Let $q_n \to q$ in q_{ad} as $n \to \infty$.Since$U(t;q)$ is the weak solution of (1.4) for any$q \in \mathcal{P}_{ad}$ we have the following estimate.

$\|u_M(t;q_n)\|_{L^\infty(0,T;H)} + \|u_M(t;q_n),\|_{L^2(0,T;V)} + \left\|u_{M_t}(t;q_n)\right\|_{L^2(0,T;H)} \le C\|g\|_H$,(2.44)

where C is constant independent of $q \in \mathcal{P}_{ad}$. Estimate (2.44) shows that $U(t;q)$ is bounded in $W(0,T)$.Since$W(0,T)$ is reflexive.we can choose a sub-sequence $u(t;q_{n_k})$ weakly convergent to a function z in $W(0,T)$.The fact that $u_M(t;q_n)$ is bounded in $W(0,T)$ implies that $u_M(t,q_n)$ is bounded in $L^2(0,T;V)$,so $u(t;q_{n_k})$weakly convergent to a function z in $L^2(0,T;H)$.Since V is compactly imbedded in H,then by the classical compactness theorem[4] $u(t;q_n) \to z$ in $L^{\infty 2}(0,T;H)$,.By (2,44) the derivative $u'(t;q_{n_k})$ and z' are uniformly bounded in $L^\infty(0,T;H)$ .Therefore functions $\{u(t;q_{n_k}), z\}_{k=1}^{\infty}$ are equicontinuous in $C([0,T];H)$..Thus $u(t;q_{n_k}) \to z$ in $C([0,T];H)$ ...In particular $u(t;q_{n_k}) \to z(t)$ in H and $u(t;q_{n_k}) \to z$ weakly in V for any $t \in [0,T]$.By lemma 15,$A_{n_k}]\, u(t;q_n) \to Az(t)$ weakly in V'.Now we see that z satisfies the equation given in definition 5,ie it is the weak solution $u(q)$. The uniqueness of the weak solution implies that $u(q_n) \to u(q) as\; n \to \infty$ in $C([0,T];H)$ for the entire sequence $u(q_n)$ and not for its subsequence. Thus $u(t;q_n) \to u(q)$ in $C([0,T];H)$ as $q_n \to q$ in P as claimed.

4.0 Conclusion

The parameter associated with the risk adjusted Black-Sholes option model was studies where the existence and uniqueness of weak solution of the risk adjusted Black-Scholes option pricing model with variable volatility coefficient given as $\hat{\sigma}^2(s,t) = \sigma^2(1 - \mu(S\partial_S^2 V(S,t))^{\frac{1}{3}}$ was established. The study adjusted the volatility to incorporate both transaction cost and portfolio risk measures and continuity of the weak solution was discussed following the method in [4].

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