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Hermite Series Expansion for Solving Perturbed Quantum Harmonic Oscillator Problems

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Abstract

It is obvious that all harmonic oscillators in the universe are affected by perturbation such as friction or defects in the elastic potential. In this paper, we determine the eigenenergies of the perturbed quantum harmonic oscillator by method of Hermite series expansion. The first notable result of the work in this paper is the discovery of the indefinitely fine revisions of thewell known sequence of Schrödinger's quantum mechanical eigenenergies due to a quartic perturbation potential..

Keywords: Hermite series expansion, eigenenergies, perturbation, oscillators

1.0 Introduction

Method of Hermite Series Expansion had been employed to derive the eigenenergies and their corresponding Eigenfunctions of the relativistic quantum linear simple harmonic oscillator [1] with kinetic energy operator $\hat{T}$ given by [2-4]

$$\hat{T} = m_0c^2 - \frac{\hbar^2}{2m_0}\frac{\partial^2}{\partial_x^2} - \frac{\hbar^4}{8m_0^3c^2}\frac{\partial^4}{\partial x^4} - \cdots \quad (1)$$

and potential energy V given by

$$V(x) = \frac{1}{2}m_0\omega_0^2x^2 \quad (2)$$

In this paper we show how to apply this method to a perturbed quantum harmonic oscillator.

2.0 Theoretical Analysis

Consider a given entity of non-zero rest mass m_0, in the one dimensional Hooke's field under the influence of friction having natural frequency,ω_0, under a quartic perturbation potential energy V given by:

$$V(x) = \frac{1}{2}m_0\omega_0^2x^2 \pm \epsilon x^4 \quad (3)$$

where$\boldsymbol{\epsilon}$is a small constant. The perturbed quantum mechanical energy wave equation is given by:

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \left\{-\frac{\hbar^2}{2m_0}\frac{\partial^2}{\partial x^2} + \left(\boldsymbol{\frac{1}{2}m_0\omega_0^2x^2} \pm \boldsymbol{\epsilon x^4}\right)\right\}\Psi(x,t) \quad (4)$$

where$\Psi(x,t)$ is the quantum mechanical wave function, which is subject to the condition of uniqueness and regularity everywhere and continuity across all boundaries and normalization.

Now the variables may be separated as:

$$\Psi(x,t) = U(x)exp\left[\frac{-iEt}{\hbar}\right] \quad (5)$$

where E is quantum mechanical energy and U is the quantum mechanical energy wavefunction which satisfies the equation:

$$0 = \frac{\hbar^2}{2m_0}U''(x) + \left[E - \boldsymbol{\frac{1}{2}m_0\omega_0^2x^2} \pm \boldsymbol{\epsilon x^4}\right]U(x) \quad (6)$$

Let ξ be a new independent variable defined by:

$$\xi = \left(\frac{m_0\omega_0}{\hbar}\right)^{\frac{1}{2}}x \quad (7)$$

Then quantum mechanical energy wave equation for the perturbed quantum harmonic oscillator (6) transforms as:

$$0 = U''(\xi) + \left[\lambda - \xi^2 \mp \frac{2\epsilon\hbar}{m_0^2\omega_0^2}\xi^4\right]U(x) \quad (8)$$

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Where

$$\lambda = \frac{2E}{\hbar\omega_0} \tag{9}$$

Next, let us seek the solution of (8) in the form:

$$U(\xi) = exp\left[-\frac{1}{2}\xi^2\right]F(\xi) \tag{10}$$

Then the function F satisfies the equation

$$0 = F''(\xi) - 2\xi F'(\xi) + \left[(\lambda - 1) \mp \frac{2\epsilon\hbar}{m_0^2\omega_0^3}\xi^4\right]F(\xi) \tag{11}$$

Toward the solution of (11), let us note that the state of perturbed harmonic oscillator is contained within the space $L_2(-\infty,\infty)$ of all square integral functions over the interval $(-\infty,\infty)$. Also, the Hermite Polynomials constitute a sequence of complete orthogonal function in the space. It therefore follows that there exist constants A_n such that

$$F(\xi) = \sum_{n=0}^{\infty} A_n H_n(\xi) \tag{12}$$

Hence simplifying by using the well known recurrence properties of the Hermite Polynomials, we obtain the following:

$$F'(\xi) = \sum_{n=0}^{\infty} 2(n+1)A_{n+1}H_n(\xi) \tag{13}$$

$$F''(\xi) = \sum_{n=0}^{\infty} 2^2(n+1)(n+2)A_{n+2}H_n(\xi) \tag{14}$$

$$\xi F'(\xi) = \sum_{n=1}^{\infty} nA_nH_n(\xi) + \sum_{n=0}^{\infty} 2(n+1)(n+2)A_{n+2}H_n(\xi) \tag{15}$$

$$\begin{aligned}
\xi^4 F(\xi) = &\sum_{n=4}^{\infty}\frac{1}{16}A_{n-4}H_n(\xi) + \sum_{n=2}^{\infty}\frac{1}{8}(n+1)A_{n-2}H_n(\xi) + \sum_{n=2}^{\infty}\frac{1}{8}nA_{n-2}H_n(\xi) + \sum_{n=0}^{\infty}\frac{1}{4}(n+1)(n+2)A_nH_n(\xi)\\
&+ \sum_{n=2}^{\infty}\frac{1}{8}(n-1)\,A_{n-2}H_n(\xi) + \sum_{n=0}^{\infty}\frac{1}{4}(n+1)^2A_nH_n(\xi) + \sum_{n=0}^{\infty}\frac{1}{4}n(n+1)A_nH_n(\xi)\\
&+ \sum_{n=0}^{\infty}\frac{1}{2}(n+1)(n+2)(n+3)A_{n+2}H_n(\xi) + \sum_{n=3}^{\infty}\frac{1}{8}(n-2)A_{n-2}H_n(\xi) + \sum_{n=1}^{\infty}\frac{1}{4}n(n+1)A_nH_n(\xi)\\
&+ \sum_{n=1}^{\infty}\frac{1}{4}n^2A_nH_n(\xi) + \sum_{n=0}^{\infty}\frac{1}{2}(n+1)(n+2)^2A_nH_n(\xi) + \sum_{n=2}^{\infty}\frac{1}{4}n(n-1)A_nH_n(\xi)\\
&+ \sum_{n=0}^{\infty}\frac{1}{2}(n+1)^2(n+2)\,A_{n+2}H_n(\xi) + \sum_{n=1}^{\infty}\frac{1}{2}n(n+1)(n+2)A_{n+2}H_n(\xi)\\
&+ \sum_{n=0}^{\infty}(n+1)(n+2)(n+3)(n+4)A_{n+4}H_n(\xi)
\end{aligned} \tag{16}$$

Now substituting (12)-(16) into (11) we obtain:

$$\begin{aligned}
0 = &\sum_{n=0}^{\infty} 2^2(n+1)(n+2)A_{n+2}H_n(\xi) - 2\sum_{n=1}^{\infty} nA_nH_n(\xi) \mp \alpha\sum_{n=4}^{\infty}\frac{1}{16}A_{n-4}H_n(\xi) \mp \alpha\sum_{n=2}^{\infty}\frac{1}{8}(n+1)A_{n-2}H_n(\xi)\\
&\mp \alpha\sum_{n=2}^{\infty}\frac{1}{8}nA_{n-2}H_n(\xi) \mp \alpha\sum_{n=0}^{\infty}\frac{1}{4}(n+1)(n+2)A_nH_n(\xi) \mp \alpha\sum_{n=2}^{\infty}\frac{1}{8}(n-1)\,A_{n-2}H_n(\xi)\\
&\mp \alpha\sum_{n=0}^{\infty}\frac{1}{4}(n+1)^2A_nH_n(\xi) \mp \alpha\sum_{n=0}^{\infty}\frac{1}{4}n(n+1)A_nH_n(\xi) \mp \alpha\sum_{n=0}^{\infty}\frac{1}{2}(n+1)(n+2)(n+3)A_{n+2}H_n(\xi)\\
&\mp \alpha\sum_{n=3}^{\infty}\frac{1}{8}(n-2)A_{n-2}H_n(\xi) \mp \alpha\sum_{n=1}^{\infty}\frac{1}{4}n(n+1)A_nH_n(\xi) \mp \alpha\sum_{n=1}^{\infty}\frac{1}{4}n^2A_nH_n(\xi)\\
&\mp \alpha\sum_{n=0}^{\infty}\frac{1}{2}(n+1)(n+2)^2A_nH_n(\xi) \mp \alpha\sum_{n=2}^{\infty}\frac{1}{4}n(n-1)A_nH_n(\xi) \mp \alpha\sum_{n=0}^{\infty}\frac{1}{2}(n+1)^2(n+2)\,A_{n+2}H_n(\xi)\\
&\mp \alpha\sum_{n=1}^{\infty}\frac{1}{2}n(n+1)(n+2)A_{n+2}H_n(\xi)\\
&\mp \alpha\sum_{n=0}^{\infty}(n+1)(n+2)(n+3)(n+4)A_{n+4}H_n(\xi)
\end{aligned} \tag{17}$$

where

$$\alpha = \frac{2\epsilon\hbar}{m_0^2\omega_0^3} \tag{18}$$

From the coefficients of $H_0(\xi)$ we obtain the recurrence equation:

$$0 = \left[(\lambda - 1) \mp \frac{3}{4}\alpha\right]A_0 \mp 6\alpha A_2 \mp 24\alpha A_4 \tag{19}$$

Next taking the coefficient of $H_1(\xi)$ we obtain the recurrence equation:

$$0 = \left[(\lambda - 1) - 2 \mp \frac{15}{4}\alpha\right]A_1 \mp \frac{51}{2}\alpha A_3 \mp 140\alpha A_5 \tag{20}$$

Next from the coefficient of $H_2(\xi)$ we obtain the recurrence equation:

$$0 = \left[(\lambda - 1) - 4 \mp \frac{39}{4}\alpha\right]A_2 \mp \frac{3}{2}\alpha A_0 \mp 84\alpha A_4 \mp 360\alpha A_6 \tag{21}$$

Generally, from the coefficient of $H_n(\xi)$ we obtain the recurrence equation:

$$\left[(\lambda - 1) - 2n \mp \frac{3}{2}\alpha(n^2 + n + \frac{1}{2})\right] \mp \frac{\alpha}{4}(2n-1)A_{n-2} \mp 2\alpha(n+1)(n^2+3n+2)A_{n+2} \\ \mp \alpha(n+1)(n+2)(n+3)(n+4)A_{n+4} \tag{22}$$

(22) Is the general recurrence relation.

3.0 Results and Discussions

3.1 Ground Energy Level

For the ground level of the perturbed quantum harmonic oscillator, we choose the coefficient of A_0 in the recurrence equation (19) to vanish:

$$(\lambda - 1) \mp \frac{3}{4}\alpha = 0$$

$$\lambda = 1 \pm \frac{3}{4}\alpha \tag{23}$$

or explicitly using (9) and (18)

$$E_0 = \frac{1}{2}\hbar\omega_0 \pm \frac{3}{4}\frac{\epsilon\hbar^2}{m_0^2\omega_0^2} \tag{24}$$

$E_0$is the quantum mechanical energy of the ground level.

3.2 First Energy Level

For the first level of the perturbed quantum harmonic oscillator, we choose the coefficient of A_1 in the recurrence equation (20) to vanish:

$$(\lambda - 1) - 2 \mp \frac{15}{4}\alpha = 0$$

$$\lambda = 3 \pm \frac{15}{4}\alpha \tag{25}$$

Or explicitly using (9) and (18)

$$E_1 = \frac{3}{2}\hbar\omega_0 \pm \frac{15}{4}\frac{\epsilon\hbar^2}{m_0^2\omega_0^2} \tag{26}$$

E_1Is the quantum mechanical energy of the first level.

3.3 Second Energy Level

For the second level of the perturbed quantum harmonic oscillator, we choose the coefficient of A_2 in the recurrence equation (21) to vanish:

$$(\lambda - 1) - 4 \mp \frac{39}{4}\alpha = 0$$

$$\lambda = 5 \pm \frac{39}{4}\alpha \tag{27}$$

or explicitly using (9) and (18)

$$E_2 = \frac{5}{2}\hbar\omega_0 \pm \frac{39}{4}\frac{\epsilon\hbar^2}{m_0^2\omega_0^2} \tag{28}$$

$E_2$is the quantum mechanical energy of the second level.

3.4 General Energy Level

For the general level of the perturbed quantum harmonic oscillator, we choose the coefficient of A_n in the recurrence equation (22) to vanish:

$$(\lambda - 1) - 2n \mp \frac{3}{2}\alpha(n^2 + n + \frac{1}{2}) = 0$$
$$\lambda = 2n + 1 \pm \frac{3}{2}\alpha\left(n^2 + n + \frac{1}{2}\right) \qquad (29)$$
or explicitly using (9) and (18)
$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega_0 \pm 3\left(n^2 + n + \frac{1}{2}\right)\frac{\epsilon\hbar^2}{m_0^2\omega_0^2} \qquad (30)$$
$E_n$is the quantum mechanical energy of the nth level.

Carefully observing each level of the quantum mechanical eigenenergies of the perturbed quantum harmonic oscillator, it may be noted that $E_n$ishitherto unknown to the best of our knowledge. Equation (30) is ageneralisation of the unperturbed quantum mechanical eigenenergies of the linear simple harmonic oscillator, given by;
$$E_n = \frac{1}{2}(2n + 1)\hbar\omega_0 \qquad (31)$$

4.0 Conclusion

This work can now be extended to the derivation of the exact Eigen energies of all perturbed linear harmonic oscillators having potential energy correctionsof the form$\pm\in x^n$; n=3,4,…

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