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Effect of Varying Step-Size in Approximation of Differential Equations

Abdullahi Salisu

Department of Computer Science, Niger State Polytechnic, Zungeru, Niger State, Nigeria.

Abstract

In this paper, we use the Conventional Finite Difference Approximation schemes of the first and second order derivatives of a function to examine and analyze the effect of varying step-size in finite approximation. We then use these finite difference quotients to solve some differential equation problems using different values of step-size to help establish the effect of varying step-size on the approximated solution. The technique is illustrated using an Excel (Spreadsheet) package.

1.0 Introduction

The use of simple operations to find approximate solutions to complex problems constitutes the main goal of numerical analysis. Solutions to differential equations are obtainable by analytical or numerical methods, however, where differential equations defy solution analytically, approximate solutions are often obtainable by the application of numerical methods [1]. Most Differential equations are not too complicated to be solved by an explicit analytical formulae, thus the development of accurate numerical approximation scheme is essential for both extracting qualitative information as well as achieving understanding the behavior of the solutions [2-4].Numerical Methods for solving ordinary differential equations depend on a step-size "*h*", since the truncation error goes to zero as *h*goes to zero, atleast for nice problems. Step-size would be limited only the number of steps we have time to take, however, as step–size decreases and the number of steps increase, arithmetic error also increases. In this paper, we solve some differential equations using small and smaller step-sizes comparing the solutions to see if they are converging. A number of other works considering the effect of step-size on numerical solution exits (see [5-7]).

2.0 Methodology

2.1 Description of Step-Size

Suppose we are solving $u' = f(t,u) on\ [0,T]$ and we are using a method of order p;i.e the error in our approximation for $u(T)$ is bounded by some constant times h^p. Solve using 2^N steps, so $h=2^{-N}T$ and let U_N denote the solution.
Assume that only is the error bounded by a multiple of h^p but that for small enough h the error is approximated equal to a constant times h^p. that is assume

$$U(T) - U_N(T) \approx ch^p \tag{1.1}$$

Let

$$D_N = U_N(T) - U_{N-1}(T) \tag{1.2}$$

$$\approx (U(T) - c\,(2^{-N}\,T)^p) - \left(U(T) - c\left(2^{-(N-1)}T\right)^p \right) \tag{1.3}$$

$$= 2^{-pN}\,(2^p - 1)cT^p \tag{1.4}$$

$$\frac{D_{N-1}}{D_N} \approx \frac{2^{-pN+P}\,(2^p - 1)cT^p}{2^{-pN}(2^p - 1)cT^p} = 2^p \tag{1.5}$$

And so

$$R_N \equiv \frac{\ln\ |D_{N-1}/D_N|}{\ln 2} \approx p \tag{1.6}$$

The derivation between R_N and p gives a measure of how the method is converging. Typically, R_N gets closer to p while the method is becoming more accurate and then deviate from p as the error starts increasing due to accumulated arithmetic error.

Corresponding author: ***Abdullahi Salisu*** E-mail: saddullah2k3@yahoo.com, Tel.: +2348035925783

2.2 Description of the Conventional Numerical Methods

Let consider

$$\frac{u(x+h)-u(x)}{h} \approx u'^{(x)} \tag{1.7}$$

Used to approximate the first derivative of the function *U*(x). Indeed, if it is differentiable at x, then u′(x) is by definition the limit as $h \to 0$ of the finite difference quotients, where h is the step-size which may be either positive or negative but is assume to be small $|h| \ll 1$. when$h > 0$ eqn (1.7) is refer as forward difference scheme while $h < 0$ eqn (1.7) gives backward difference scheme [2,3]

Assume that u(x) is atleast twice continuously differentiable, and examine the first order Taylor expansion

$$u(x+h) = u(x) + u'(x)h + \frac{1}{2}u''(\xi)h^2 \tag{1.8}$$

We have used Cauchy form for the reminder term, in which ξ represents some points lying between *x* and *x*+*h*. the Error and the derivative being approximated is given by

$$\frac{u(x+h)-u(x)}{h} - u'(x) = \frac{1}{2}u''(\xi)h \tag{1.9}$$

Since the Error is proportional to *h*, we can re-write the equation as

$$u'(x) = \frac{u(x+h)-u\ \ (x)}{h} + 0(h) \tag{1.10}$$

This is a first order approximation.

We again approximate $u''(x)$ by sampling *u* at the particular points *x*, *x*+ *h*, *x*-*h*. Which combination of the functions values *u*(x-*h*), *u*(x), *u*(x + *h*) are used.

We expand the functions *u*(x-*h*), *u*(x), *u*(x + *h*) using Taylor expansion as shown

$$u(x+h) = u(x) + u'(x)h + \frac{1}{2}u''(x)h^2 + \frac{1}{6}u'''(x)h^3 + 0(h^4) \tag{1.11}$$

$$u(x-h) = u(x) - u'(x)h + \frac{1}{2}u''(x)h^2 - \frac{1}{6}u'''(x)h^3 + 0(h^4) \tag{1.12}$$

Adding the two formulae (1.11) & (1.12) together to give

$$u(x+h) + u(x-h) = 2u(x) + u''(x)h^2 + 0(h^4) \tag{1.13}$$

Re-arranging terms, we conclude that

$$u''(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} + 0(h^2) \tag{1.14}$$

The result is known as the central difference approximation of the second derivative of a function. Since the Error is proportional to h^2, we conclude that this is a second order approximation.

We also reconsider the first order approximation in equation (1.10) based on the function values at three points x, x+ *h*, & x-*h*, to find the approximate combination of *u*(x- h), *u*(x), *u*(x+ *h*), we return to Taylor Expansion (1.12) & (1.13) , to solve for $u'(x)$, we subtract the two formulae and so

$$u(x+h) - u(x-h) = 2u'(x)h + u''(x)\frac{h^3}{3} + 0(h^4) \tag{1.15}$$

Re-arranging the terms, we are lead to the well central difference formula

$$u'(x) = \frac{u(x+h) - u(x-h)}{2h} + 0(h^2) \tag{1.16}$$

This is a second order approximation to the first derivative

4.0 Numerical Examples

Example 1. Let $u(x) = \sin x$

Analytical Solution: $u'(1) = \cos 1 = 0.5403023$

But by computing with finite different quotients

$$u'(x) \approx \frac{u(x+h)-u\ \ (x)}{h}$$

$$\therefore \cos 1 \approx \frac{sin(1+h) - sin1}{h}$$

Considering different values of "h" the results are presented in table 1, the result are obtained using Microsoft Excel Package.

Example 2: Let $u(x) = e^{x^2}$

Analytical solution: $u''(x) = (4x^2 + 2)\, e^{x^2}$

$$\therefore u''(1) = 6e = 16.30969097$$

Using the finite difference quotient (1.14) above

$$u''(x) \approx \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}$$

$$6e \approx \frac{e^{(1+h)^2} - 2e + e^{(1-h)^2}}{h^2}$$

The results are listed in table 2, using different values of step size "h"

Example 3. Let $u(x) = \sin x$

Analytical solution: $u'(x) = \cos x$

$$u'(1) = \cos 1 = 0.5403023$$

Using finite different quotient (1.15)

$$u'(x) \approx \frac{u(x+h) - u(x-h)}{2h}$$

$$\therefore \cos 1 \approx \frac{sin(1+h) - sin(1-h)}{2h}$$

Considering using different values of "h" the results are listed in the table 3

Table 1: Numerical Results of Example 1 obtained using different values of "*h*"

h	1	0.1	0.01	0.001	0.0001
Approximate solution	0.067826	0.497364	0.536086	0.539881	0.540260
Error	-0.472476	-0.042939	-0.004216	-0.000421	-0.000042

Table 2: Numerical Results of Example 2 obtained using different values of "*h*"

h	1	0.1	0.01	0.001	0.0001
Approximate solution	5.16158638	16.48289823	16.31141265	16.30970819	16.30969115
Error	33.85189541	0.17320726	0.00172168	0.00001722	0.00000018

Table 3: Numerical Results of Example 3 obtained using different values of "*h*"

h	0.1	0.01	0.001	0.0001
Approximate solution	0.539402254217	0.54029330087	0.5403221582	0.54030230497
Error	-0.0090005370	-0.00000900499	-0.00000009005	-0.00000000090

5.0 Analysis and Discussion of Results

We observed in table 1,that reducing the step size "*h*" by a factor of $\frac{1}{10}$ reduces the size of the error by approximately the same factor in example 1. Thus, to obtain 10 digit accuracy we anticipate needing a step size of about $h = 10^{-11}$.In table 2,each reduction in step size "*h*" by a factor of $\frac{1}{10}$ reduces the size of the error by approximately the factor $\frac{1}{100}$, results is a gain of two new decimal digits accuracy, confirming that the finite difference approximation is of second order . In table 3,The results are much more accurate than the one-sided finite difference approximation used in example 1 at the same step-size, since it is second order approximation, each reduction in the step size by a factor of $\frac{1}{10}$ results in two more decimal places of accuracy.

6.0 Conclusion

This study examines the effect of vary step size on the approximated solution, it is interesting to observed that reducing step size also reduces the size of error by approximately the same factor, in case of first order numerical approximation scheme as seen in table1, but in case of second order numerical approximation reducing the step size by a factor $\frac{1}{10}$, will reduces the error by a factor $\frac{1}{100}$ as seen in table 2. Now we can conclude that the central difference approximation is a better approximation scheme than one-sided finite difference scheme as seen in table 1,2, & 3.

Where the errors produced in solving Example 3 using central difference approximation scheme gain two new decimal points, which make it more accurate than one-sided finite difference approximation used in Example 1 at the same step- size.

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