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Comparison of Some Numerical methods for the Solution of Fourth Order Integro-Differential Equations

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Abstract

The numerical methods for solving fourth order Integro-differential equations are presented. The methods are based on replacement of the unknown function by Power series and Legendre polynomials of appropriate degree. The proposed methods convert the resulting equation by some examples considered show that the Standard Collocation Method proved superior to the Perturbed Collocation Method. Two examples are considered to illustrate the efficiency and accuracy of the methods.

Keywords: Collocation Method, Legendre polynomials, Integro-Differential Equations

1.0 Introduction

Mathematical formulations of many physical, chemical and biological phenomena have been found to often result in integro-differential equations. Analytical solutions of integro-differential equations are extremely difficult to be obtained even for the few ones whose closed form solutions are available. The procedures for obtaining such closed form solutions are not readily available. Therefore, numerical methods are often applied to solve integro-differential equations. In 1992, Liao[1] employed the basic ideas of the homotopy in topology to propose a general analytical method for BVPs in integro-differential equations, namely the Homotopy Analysis Method [2].
Several Numerical Methods have been used to solve high order integro-differential equations. These include An algorithm for solving higher-order nonlinear Volterra-Fredholm integro-differential equations with constant coefficients[4]. Fourth order integro-differntial equations using Variational Iteration Method[5], Pseudo-Spectral Method[6] and Direct Method for solving integro-differential equations using hybrid fourier and block-pulse functions[7]. Other methods are Petrov-Galerkin Method with a memory term[8], Wavelet-Galerkin method[9] and Spline functions expansion method[10].
In order to discuss the methods, we consider the general fourth order linear integro-differential equation.

$$\rho_0 y(x) + \rho_1 y'(x) + \rho_2 y''(x) + \rho_3 y'''(x) + \rho_4 y^{iv}(x) + \int_0^x k(x,t)y(t)dt = f(x) \qquad (1)$$

With the boundary conditions;

$$y(a) = a_0 \qquad (2)$$
$$y'(a) = a_1 (3)$$
$$y(b) = b_0 \qquad (4)$$
$$y''(b) = b_1 \qquad (5)$$

Where $f(x), k(x,t)$ are known functions, a_0, a_1, b_0 and b_1 are constants and $y(x)$ is the unknown function to be determined.

2.0 Methodology and Techniques

In this section, we discussed the numerical methods based on Power series and Legendre polynomials as the basis functions for the solution of equations (1)-(5).
Remarks:

1. For Perturbed Collocation Methods by the two basic functions, we have chosen T_1, T_2, T_3 and T_4 as four free tau parameters to be determined along with the constants $a_r (r \geq 0)$ and $T_N(x)$ is the Chebyshev polynomial of degree N of the first kind valid in the interval $[a, b]$ and is defined by;

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$T_N(x) = cos[Ncos^{-1}x],\ \ N \geq 0$
Where,
$T_0(x) = 1$
$T_1(x) = x$
And this is satisfied by the recurrence relation

$$T_{N+1}(x) = 2\left(\frac{2x-a-b}{b-a}\right)T_N(x) - T_{N-1}(x), N \geq 1,\quad a \leq x \leq b$$

2. The Legendre polynomials used in this work are easily obtained using the recurrence relation
$(N+1)P_{N+1}(x) - (2N+1)xP_N(x) + NP_{N-1}(x) = 0, N \geq 1$
Where,
$P_0(x) = 1$
And
$P_1(x) = x$

3. **Standard Collocation Method by Power Series (SCMPS):** We used the method to solve equations (1)-(5) by assuming power series approximation of the form

$$yN(x) = \sum_{r=0}^{N} a_r x^r \tag{6}$$

Where *x* represents the independent variables in the problem, $a_r (r \geq 0)$ are the unknown constants to be determined. Thus, equation (6) is substituted into equations (1)-(5), to obtained;

$$\rho_0 \sum_{r=0}^{N} a_r x^r + \rho_1 \sum_{r=0}^{N} r a_r\, x^{r-1} + \rho_2 \sum_{r=0}^{N} r(r-1)\, a_r x^{r-2} + \rho_3 \sum_{r=0}^{N} r(r-1)(r-2)\, a_r x^{r-3}$$
$$+\rho_4 \sum_{r=0}^{N} r(r-1)(r-2)(r-3) a_r x^{r-4} + \int_0^x k(x,t) \sum_{r=0}^{N} a_r t^r dt = f(x) \tag{7}$$

together with the boundary conditions

$$\sum_{r=0}^{N} a_r a^r = a_0 (8)$$
$$\sum_{r=0}^{N} r(r-1) a_r a^{r-2} = a_1 \tag{9}$$
$$\sum_{r=0}^{N} a_r b^r = b_0 \tag{10}$$
$$\sum_{r=0}^{N} r(r-1) a_r b^r = b_1 \tag{11}$$

Case 1: We considered the fourth order integro-differential equation of the form

$$y^{iv}(x) + y(x) + \int_0^x k\,(x,t) y(t) dx = f(x), 0 \leq x, t \leq 1 \tag{12}$$

$by\ setting\ \rho_1 = \rho_2 = \rho_3 = 0\ and\ \rho_0 = 1$ in equation (1)
Thus, equation (6) is substituted into equation (12), we obtained

$$y_N^{iv}(x) + y_N(x) + \int_0^x k(x,t) y_N(t) dx = f(x) \tag{13}$$

Equation (13) is re-written as

$$\sum_{r=0}^{N} r(r-1)(r-2)(r-3) a_r x^{r-4} + \sum_{r=0}^{N} a_r x^r + \int_0^x k(x,t) \sum_{r=0}^{N} a_r t^r dt = f(x) \tag{14}$$

$$\sum_{r+4=0}^{N}(r+4)(r+3)(r+2)(r+1)a_{r+4}x^r+\sum_{r=0}^{N}a_rx^r+\int_0^x k(x,t)\sum_{r=0}^{N}a_rt^rdt=f(x)\quad(15)$$

$$\sum_{r=0}^{N}(r+4)(r+3)(r+2)(r+1)a_{r+4}x^r+\sum_{r=0}^{N}a_rx^r+\int_0^x k(x,t)\sum_{r=0}^{N}a_rt^rdt=f(x)\qquad(16)$$

$$\sum_{r=0}^{N}[(r+4)(r+3)(r+2)(r+1)a_{r+4}+a_r]x^r+\int_0^x k(x,t)\sum_{r=0}^{N}a_rt^rdt=f(x)\qquad(17)$$

After evaluating the integral part of equation (17), the left-over is then collocated at the point $x=x_k$, to obtain.

$$\sum_{r=0}^{N}[(r+4)(r+3)(r+2)(r+1)a_{r+4}+a_r]x_k^r+\int_0^x k(x_k,t)\sum_{r=0}^{N}a_rt^rdt=f(x_k)\qquad(18)$$

Where,

$$x_k=a+\frac{(b-a)k}{N-2},k=1,2,3,\dots,N-3.\qquad(19)$$

Thus, equation (18) gives rise to (N-3) algebraic linear equations in (N+1) unknown constants. Four extra equations are obtained using equations (8)-(11). Altogether, we have (N+1) algebraic linear equations in (N+1) unknown constants. These (N+1) algebraic linear equations are then solved by Gaussian elimination method to obtain the (N+1) unknown constants which are then substituted back into equation (6) to obtain the approximate solution.

Case 2: We considered the fourth order integro-differential equation of the form

$$y^{iv}(x)+\rho_2y''(x)+\int_0^x k(x,t)y(t)dx=f(x),0\le x,t\le 1\qquad(20)$$

By setting $\rho_0=\rho_1=\rho_3=0$ and $\rho_4=1,\rho_2\neq 0$ in equation (1)

Thus, equation (6) is substituted into equation (20), we obtained

$$y_N^{iv}(x)+\rho_2y_N''(x)+\int_0^x k(x,t)y_N(t)dx=f(x)\qquad(21)$$

Equation (21) is re-written as

$$\sum_{r=0}^{N}r(r-1)(r-2)(r-3)a_rx^{r-4}+\rho_2\sum_{r=0}^{N}r(r-1)a_rx^{r-2}+\int_0^x k(x,t)\sum_{r=0}^{N}a_rt^rdt=f(x)\quad(22)$$

$$\sum_{r+4=0}^{N}(r+4)(r+3)(r+2)(r+1)a_{r+4}x^r+\rho_2\sum_{r+2=0}^{N}(r+1)(r+1)a_{r+2}x^r+\int_0^x k(x,t)\sum_{r=0}^{N}a_rt^rdt=f(x)\qquad(23)$$

$$\sum_{r=0}^{N}(r+4)(r+3)(r+2)(r+1)a_{r+4}x^r+\rho_2\sum_{r=0}^{N}(r+2)(r+1)a_{r+2}x^r+\int_0^x k(x,t)\sum_{r=0}^{N}a_rt^rdt=f(x)(24)$$

$$\sum_{r=0}^{N}[(r+4)(r+3)(r+2)(r+1)a_{r+4}+\rho_2(r+2)(r+1)a_{r+2}]x^r+\int_0^x k(x,t)\sum_{r=0}^{N}a_rt^rdt=f(x)\qquad(25)$$

After evaluating the integral part of equation (25), the left-over is then collocated at the point $x=x_k$, to obtained

$$\sum_{r=0}^{N}[(r+4)(r+3)(r+2)(r+1)a_{r+4}+\rho_2(r+2)(r+1)a_{r+2}]x_k^r+\int_0^x k(x_k,t)\sum_{r=0}^{N}a_rt^rdt=f(x_k)\qquad(26)$$

Where,

$$x_k=a+\frac{(b-a)k}{N-2},k=1,2,3,\dots,N-3.\qquad(27)$$

Thus, equation (26) gives rise to (N-3) algebraic linear equations in (N+1) unknown constants. Four extra equations are obtained using equations (8)-(11). Altogether, we have (N+1) algebraic linear equations in (N+1) unknown constants. These (N+1) algebraic linear equations are then solved by Gaussian elimination method to obtain the (N+1) unknown constants which are then substituted back into equation (6) to obtain the approximate solution.

4. Perturbed Collocation Method by Power Series (PCMPS):

Case1: We used the method to solve equation (12) by substituting equation (6) into a slightly perturbed equation (17), we obtained;

$$\sum_{r=0}^{N}[(r+4)(r+3)(r+2)(r+1)a_{r+4}+\rho_2(r+2)(r+1)a_{r+2}]x^r+\int_0^x k(x,t)\sum_{r=0}^{N}a_r t^r dt = f(x) \qquad (28)$$

$$+T_1T_N(x)+T_2T_{N-1}(x)+T_3T_{N-2}(x)+T_4T_{N-3}(x)$$

After evaluating the integral part of equation (28), the left-over is then collocated at the point $x = x_k$, to obtained;

$$\sum_{r=0}^{N}[(r+4)(r+3)(r+2)(r+1)a_{r+4}+\rho_2(r+2)(r+1)a_{r+2}]x_k^r+\int_0^x k(x_k,t)\sum_{r=0}^{N}a_r t^r dt = f(x) \qquad (28)$$

$$+T_1T_N(x)+T_2T_{N-1}(x)+T_3T_{N-2}(x_k)+T_4T_{N-3}(x_k)$$

Where,

$$x_k = a+\frac{(b-a)k}{N+2}, k = 1,2,3,\dots,N+3. \qquad (30)$$

Thus, equation (29) gives rise to (N+1) algebraic linear equations in (N+5) unknown constants. Four extra equations are obtained using equations (8)-(11). Altogether, we have (N+5) algebraic linear equations in (N+5) unknown constants. These (N+5) unknown constants which are then substituted back into equation (6) to obtain the approximate solution.

5. Standard Collocation Method by Lengendre Polynomials (SCMLP)

We used the method to solve equations (1)-(5) by assuming Legendre Polynomial approximation of the form

$$y_N(x) = \sum_{r=0}^{N}a_rP_r(x)(31)$$

Where, x represents the independent variables in the problem, $a_r(r \geq 0)$ are unknown constants to be determined and $P_r(x)(r \geq 0)$ are Legendre polynomials of appropriate degree.
Thus, equation (31) is substituted into equations (1)-(5), we obtained

$$\rho_0\sum_{r=0}^{N}a_rP_r(x)+\rho_1\sum_{r=0}^{N}a_rP_r'(x)+\rho_2\sum_{r=0}^{N}a_rP_r''(x)+\rho_3\sum_{r=0}^{N}a_rP_r'''(x)+\rho_4\sum_{r=0}^{N}a_rP_r^{iv}(x)$$
$$+\int_0^x k(x,t)y(t)dt = f(x) \qquad (32)$$

Together with the boundary conditions

$$\sum_{r=0}^{N}a_rP_r(a) = a_0(33)$$
$$\sum_{r=0}^{N}a_rP_r''(a) = a_1(34)$$
$$\sum_{r=0}^{N}a_rP_r(b) = b_0(35)$$
$$\sum_{r=0}^{N}a_rP_r''(b) = b_1(36)$$

Case 1: We considered the fourth order integro-differential equation is described in equation (12).
Thus, equation (12) leads to

$$\sum_{r=0}^{N} a_r P_r^{iv}(x) + \rho_1 \sum_{r=0}^{N} a_r P_r(x) \int_0^x k(x,t) \sum_{r=0}^{N} a_r P_r(t)\, dt = f(x) \tag{37}$$

Equation (37) is expanded to get

$$a_0 P_0^{iv}(x) + a_1 P_1^{iv}(x) + \cdots + a_N P_N^{iv}(x) + a_0 P_0(x) + a_1 P_1(x) + \cdots + a_N P_N(x) + \int_0^x k(x,t) \sum_{r=0}^{N} a_r P_r(t) dt = f(x) \tag{38}$$

After further simplification, we obtained

$$\left[P_0^{iv}(x) + P_0(x) a_0\right] a_0 + \left[P_1^{iv}(x) + P_1(x) a_1\right] a_1 + \cdots + \left[P_1^{iv}(x) + P_N(x)\right] a_N + \int_0^x k(x,t) \sum_{r=0}^{N} a_r P_r(t) dt = f(x) \tag{39}$$

After evaluating the integral part in equation (39), the left-over is then collocated at the point $x = x_k$, we obtained

$$\left[P_0^{iv}(x_k) + P_0(x_k) a_0\right] a_0 + \left[P_1^{iv}(x_k) + P_1(x_k) a_1\right] a_1 + \cdots + \left[P_1^{iv}(x_k) + P_N(x_k)\right] a_N + \int_0^x k(x,t) \sum_{r=0}^{N} a_r P_r(t) dt = f(x_k) \tag{40}$$

Where,

$$x_k = a + \frac{(b-a)k}{N+2}, k = 1, 2, 3, \ldots, N-3. \tag{41}$$

Thus, equation (40) gives rise to (N-3) algebraic linear equations in (N+1) unknown constants. Four extra equations are obtained using equations (33)-(36). Altogether, we have (N+1) algebraic linear equations in (N+1) unknown constants. These (N+1) algebraic linear equations are then solved by Gaussian elimination method to obtain the (N+1) unknown constants which are then substituted back into equation (31) to obtain the approximate solution.

Case 2: We considered the fourth order integro-differential equation as described in equation (20).
Thus, we obtained from equation (32).

$$\sum_{r=0}^{N} a_r P_r^{iv}(x) + \rho_2 \sum_{r=0}^{N} a_r P_r''(x) + \int_0^x k(x,t) \sum_{r=0}^{N} a_r P_r(t)\, dt = f(x) \tag{42}$$

Equation (42) is expanded to get

$$a_0 P_0^{iv}(x) + a_1 P_1^{iv}(x) + \cdots + a_N P_N^{iv}(x) + \rho_2 a_0 P_0(x) + \rho_2 a_1 P_1(x) + \cdots + \rho_2 a_N P_N(x) + \int_0^x k(x,t) \sum_{r=0}^{N} a_r P_r(t) dt = f(x) \; (43)$$

Further simplification of equation (43), we obtained

$$\left[P_0^{iv}(x) + \rho_2 P_0''(x)\right] a_0 + \left[P_1^{iv}(x) + \rho_2 P_1''(x)\right] a_1 + \cdots + \left[P_N^{iv}(x) + \rho_2 P_1''(x)\right] a_N + \int_0^x k(x,t) \sum_{r=0}^{N} a_r P_r(t) dt = f(x) \tag{44}$$

After evaluating the integral part in equation (44), the left-over is then collocated at the point $x = x_k$ we obtained

$$\left[P_0^{iv}(x_k) + \rho_2 P_0''(x_k)\right] a_0 + \left[P_1^{iv}(x_k) + \rho_2 P_1''(x_k) a_1\right] a_1 + \cdots + \left[P_1^{iv}(x_k) + \rho_2 P_N''(x_k)\right] a_N + \int_0^x k(x,t) \sum_{r=0}^{N} a_r P_r(t) dt = f(x_k) \tag{45}$$

Where

$$x_k = a + \frac{(b-a)k}{N+2}, k = 1, 2, 3, \ldots, N-3.$$

Thus, equation (45) gives rise to (N-3) algebraic linear equations in (N+1) unknown constants. Four extra equations are obtained using equations (33)-(36). Altogether, we have (N+1) algebraic linear equations are then solved by Gaussian elimination method to obtain the (N+1) unknown constants which are then substituted back into equation (31) to obtain the approximation solution.

6. Perturbed Collocation Method by Legendre Polynomials (PCMLP)

Case 1: We used the method to solve equations (33)-(37) by substituting equation (31) into a slightly perturbed equation (39), we obtained;

After evaluating the integral part in equation (46), the left-over is then collocated at the point $x = x_k$, we obtained.

$$\left[P_0^{iv}(x) + P_0(x)\right]a_0 + \left[P_1^{iv}(x) + P_1(x)\right]a_1 + \cdots + \left[P_N^{iv}(x) + P_N(x)\right]a_N + \int_0^x k(x,t)\sum_{r=0}^{N} a_r P_r(t)dt = f(x) \quad (46)$$
$$+T_1T_N(x) + T_2T_{N-1}(x) + T_3T_{N-2}(x) + T_4T_{N-3}(x)$$

After evaluating the integral part in equation (46), the left-over is then collocated at the point $x = x_k$, we obtained

$$\left[P_0^{iv}(x_k) + P_0(x_k)\right]a_0 + \left[P_1^{iv}(x_k) + P_1(x_k)\right]a_1 + \cdots + \left[P_N^{iv}(x_k) + P_N(x_k)\right]a_N + \int_0^x k(x,t)\sum_{r=0}^{N} a_r P_r(t)dt = f(x_k) \quad (47)$$
$$+T_1T_N(x_k) + T_2T_{N-1}(x_k) + T_3T_{N-2}(x_k) + T_4T_{N-3}(x_k)$$

Where,

$$x_k = a + \frac{(b-a)k}{N+2}, k = 1, 2, 3, \ldots, N+1.$$

Thus, equation (47) gives rise to (N+1) algebraic linear equations in (N+5) unknown constants. Four extra equations are obtained using equations (33)-(36). Altogether, we have (N+5) algebraic linear equations in (N+5) unknown constants. These (N+5) algebraic linear equations are then solved by Gaussian elimination method to obtain the (N+5) unknown constants which are then substituted back into equation (31) to obtain the approximate solution.

Case 2: We used the method to solve equations (33)-(37) by substituting equation (31) in to a slightly perturbed equation (44), we obtained

$$\left[P_0^{iv}(x) + \rho_0 P_0''(x)\right]a_0 + \left[P_1^{iv}(x) + \rho_2 P_1''(x)\right]a_1 + \cdots + \left[P_N^{iv}(x) + \rho_2 P_N''(x)\right]a_N$$
$$+\int_0^x k(x,t)\sum_{r=0}^{N} a_r P_r(t)dt = f(x) + T_1T_N(x) + T_2T_{N-1}(x) + T_3T_{N-2}(x) + T_4T_{N-3}(x) \quad (48)$$

After evaluating the integral part in equation (48), the left-over is then collocated at the point $x = x_k$, we obtained

$$\left[P_0^{iv}(x_k) + \rho_2 P_0''(x_k)\right]a_0 + \left[P_1^{iv}(x_k) + \rho_2 P_1''(x_k)\right]a_1 + \cdots + \left[P_N^{iv}(x_k) + \rho_2 P_N''(x_k)\right]a_N$$
$$+\int_0^x k(x_k,t)\sum_{r=0}^{N} a_r P_r(t)dt = f(x_k) + T_1T_N(x_k) + T_2T_{N-1}(x_k) + T_3T_{N-2}(x_k) + T_4T_{N-3}(x_k) \quad (49)$$

Where,

$$x_k = a + \tfrac{(b-a)k}{N+2}, k = 1, 2, 3, \ldots, N-1. \quad (50)$$

Thus, equation (49) gives rise to (N+1) algebraic linear equations in (N+5) unknown constants. Four extra equations are obtained using equations (33)-(36). Altogether, we have (N+5) algebraic unknown constants. These (N+5) algebraic linear equations are then solved by Gaussian elimination method to obtain the (N+5) unknown constants which are then substituted back into equation (31) to obtain the approximate solution.

Remarks:

i. Errors; For the purpose of this work, we have defined maximum error used as

$$Maximum\ error = a \leq \frac{max}{x} \leq b|y(x) - y_N(x)|$$

7. Numerical Examples

Example 1: Consider the fourth order linear integro-differential equation

$$y^{iv}(x) - y(x) + \int_0^x y(t)dt = x + (x+3)e^x, 0 \leq x \leq 1 \quad (51)$$

subject to the boundary conditions

$$y(0) = 1, y(1) = 1 + e, y''(0) = 2, y''(1) = 3e$$

The exact solution of this problem is

$$y(x) = 1 + xe^x$$

And the absolute maximum errors obtained are as shown in table 1.

Table 1: Absolute Maximum Errors for example 1.

N	SCMPS	SCMLP	PCMPS	PCMLP
6	3.1000E-5	5.647E-4	1.63E-3	4.547E-3
7	3.1000E-5	5.546E-4	4.33E-3	4.268E-3
8	3.1000E-5	5.178E-4	8.63E-3	3.926E-3

Example 2: Consider the fourth order integro-differential equation

$$y^{iv}(x) - 2y''(x) + \int_0^1 xy(t)dt = 3x, \qquad 0 \leq x \leq 1 \quad (52)$$

Subject to the boundary conditions

$$y(0) = 0, \qquad y(1) = 1, \qquad y''(0) = 0, y''(1) = -\frac{84}{47}$$

The exact solution of this problem is

$$y(x) = -\frac{14}{47}x^3 + \frac{61}{47}x$$

And the absolute maximum errors obtained are as shown in Table 2.

Table 2: Absolute Maximum Errors for example 2.

N	SCMPS	SCMLP	PCMPS	PCMLP
6	2.000E-6	1.000E-6	5.000E-5	2.105E-5
7	2.000E-6	1.000E-6	5.000E-5	2.105E-5
8	2.000E-6	1.000E-6	5.000E-5	2.105E-5

8. Discussion of Results and Conclusion

In this paper, we have shown that both the Power Series and Legendre Polynomial Collocation methods can efficiently solve a very general case of Fourth-order integro-differential equations with high order accuracy. Moreover, the results obtained by Legendre polynomials proved superior over the results obtained by Power series. Also, the Standard Collocation Method produced results which are as good as the ones produced by Perturbed Collocation method as this is evident in the tables of results presented.

References:

[1] S.J. Liao. 1992. The proposed homotopy analysis technique for the solution of non-linear problems. Ph.D. Thesis, Shanghai Jiao Tong University.

[2] M. Saeidy, M. Matinfar and J.Vahidi, 2010. Analytical Solution of BVPs for Fourth-order Integro-differential Equations by using Homotopy Analysis Method. Int. Journal of Nonlinear Science Vol. 9 No. 4, pp. 414-421.

[3] Wang, W., 2006. An algorithm for solving the higher-order nonlinear Volterra-Fredholm integro-differential equation with mechanization. Applied Math. Comput., 172: 1-23.

[4] Kurt, N. and M. Sezer, 2008. Polynomial solution of high-order Fredholm integro-differential equations with constant coefficients. J. Franklin Inst., 345:839-859.

[5] N.H. Sweilam, 2007. Fourth order integro-differential equations using variational iteration method. Comput. Math. Appl., 54:1086-1091.

[6] M.M. Khader and A.S. Hendy, 2012. The approximate and exact solutions of the fractional order delay differential equations using Legendre pseudo-spectral method, Int. Journal of Pure and Appl. Math., 74(3): 287-297.

[7] Asady, B. and M.T. Kajani, 2005. Direct method for solving integro-differential equations using hybrid fourier and block pulse functions. Int. J. Comput. Math., 82:889-895.

[8] Mustapha, K., 2008. A petrov-Galerkin method for integro-differential equations with a memory term. Int. J. Open Problems Comput. Math., Vol. 1(1).

[9] S.M. El-Sayed, M.R. Abdel-Aziz, 2003. Comparison of Adomian's decomposition method and wavelet-Galerkin method for solving integro-differential equations. Appl. Math. Comput. 136:151-159.

[10] S.T. Mohammed and M.M. Kader, 2011. Numerical solutions to the second order Fredholm integro0differential equations using the spline functions expansion, Global Journal of Pure and Applied Mathematics, 34:21-29