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Modified Kutta's Algorithm

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Abstract

In this paper a modified Kutta's Algorithm is derived through a tactful application of a geometric progression and binomial expansion for rational powers. The new method is constructed from the traditional Kutta's formula, a one-step explicit method for the solution of initial value problems (IVPs) in ordinary differential equations.(ODEs)

The performance of the new formula is test by numerical computation of some selected IVPs and the results compares favourably with those from three other existing Runge-Kutta Methods It has also been proved that the method is absolutely stable, convergence, consistence anda very fast computing time.

Keywords: modified Kutta algorithm, Initial value problem, ordinary Differential Equations and stability.
2010 Mathematical Subject Classification: Primary 65L20; Secondary 65L06

1.0 Introduction

Here we present a modified Kutta algorithm

$$y_{n+1} - y_n = \frac{h}{4}\left(\sqrt{k_1k_2} + k_2 + k_3 + \sqrt{k_3k_4}\right). \quad (1.1)$$

through a geometric root mean and a binomial processfor the numerical solution of initial value problems

$$y' = f(x, y),\ y(a) = \eta,\ a \le x \le b\,, \quad (1.2)$$

whose solution function $y' \in [a, b \to R]$, where a and b are finite. The literature presented here shows various one-step schemes in existence in the area, see [1],[2], [3]and [4] respectively,where it has been stated that a numerical method becomes useful only when it has properties like consistency, convergence and stability inherent in it. Also,a one-step method is said to be consistent, if the difference equation of the computation formula exactly approximates the differential equation it intends to solve [5].We are encouraged by the work of [6] and[7, 8] to investigate the efficiency of our method because of their various contributions in error analysis. It has been noted that bounds for the local truncationerror do not form a suitable basis for monitoring local truncation error with a view to constructing a step – control policy similar to that developed for predictor- corrector methods [7, 9]. He said what we need, in place of a bound, is a readily computable estimate of the local truncation error, similar to that obtained by Milne's device for predictor- corrector pairs. One of such is the Richardson extrapolation [10]. Under the usual localizing assumptionthat no previous errors have occurred,several such estimates exist for the general one-step method defined by;

$$y_{n+1} - y_n = h\phi(x_n, y_n, h)$$
$$h\phi(x_n, y_n, h) = \sum_{i}^{R} c_i k_i \quad (1.3)$$

$$k_i = hf\left(x_n + c_i h, y_n + \sum_{j=1}^{i=1} a_{ij}k_j\right), c_i = 0, i = 1, 2, 3, ...r$$
$$a_i = \sum_{j=1}^{i-1} b_{ij}, \qquad \forall,\ i = 2, 3, ...R \quad (1.4)$$

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$y_{n+1} = y_n + \sum_{t+1}^{r} w_i k_j$. (1.5)Where the parameters c_i , a_{ij} and w_i are arbitrary. For the purpose of linearity, we need to modify the above parameters as followsb$_{21}$=a$_1$, b$_{31}$=a$_2$, b$_{32}$=a$_3$, b$_{41}$= a$_4$,b$_{42}$=a$_5$, and b$_{43}$=a$_6$.We shall highlight two of the well known fourth order Runge- Kutta Formulae (RKF) for the purpose of clarity as follows:

a) Classical RKF,

$$y_{n+1} - y_n = \frac{h}{6}(k_1 + 2k_2 + 2k_2 + k_3) \qquad (1.6)$$

and

(b) Kutta's formula given as:

$$y_{n+1} - y_n = \frac{h}{8}(k_1 + 3k_2 + 3k_3 + k_4) \qquad (1.7)$$

Equation (1.6) has widely been research into without due attention paid to (1.7) hence our interest is in class (b) which represents the original Kutta formula. An approach developed first and tested in [11], where a new 4th order Runge- Kutta method was carved out of the existing forth order classical Runge-Kutta method of (1.6). The classical Runge-Kutta method is based on Arithmetic mean for $k_i \; \forall \; i = 1,2,3 and 4$ also called a one-Sixth Runge-Kutta method because it averages out to six components. On the other hand, [12] use the Geometric root mean for $k_i \; \forall \; i = 1,2,3 and 4$ to develop a One-third Kutta formula which averages to three components. For the definition of these properties see [7, 8] and[13], Where it was said that the local truncation error at x_{n+1} of the general explicit one –step method given by $y_{n+1} - y_n = h\phi(x_n, y_n, h)$ (1.8) Is defined by

$T_{n+1} = y(x_{n+1}) - y(x_n) - h\phi(x_n, y(x_n), h)$ (1.9)and $y(x)$ is the theoretical solution of the initial value problemwhere, the order p and the error constant C_{p+1} of (1.1) is obtain from the given local truncation error. Applying this definition to our discussion, the rounding off error will be ignored because we shall adopt the Richardson's extrapolation process of estimating the discretization error. This method is useful in calculating global (not local) truncation error.

2.0 Derivation of the method:

This paper discusses thederivation of the new Kutta scheme by means of binomial processes in line with[11], where he derived a one-third Runge-Kutta using geometric mean as against the traditional fourth order Runge-Kutta which is an arithmetic progression in nature. Generally, an R-Stage Kutta method is defined by;

$$y_{n+1} - y_n = h\phi(x_n, y_n, h)$$

$$\phi(x_n, y_n, h) = \sum_{i}^{R} c_i k_i$$

$$k_1 = f(x, y) \qquad (2.1)\; y_{n+1} = y_n + \sum_{t+1}^{r} w_i k_j .$$

$$k_i = hf\left(x_n + c_i h, y_n + \sum_{j=1}^{i=1} a_{ij} k_j \right), c_i = 0, i = 1,2,3,....r$$

$$a_i = \sum_{j=1}^{i-1} b_{ij} \qquad i = 2,3,...R$$

(2.2)

where the parameters, c_i , a_{ij} and w_i are arbitrary. For the purpose of linearity, we need to modify the above parameters as followsb$_{21}$=a$_1$, b$_{31}$=a$_2$, b$_{32}$=a$_3$, b$_{41}$= a$_4$,b$_{42}$=a$_5$, and b$_{43}$=a$_6$

$$y_{n+1} - y_n = \frac{h}{8}(k_1 + 3k_2 + 3k_3 + k_4)$$

$$y_{n+1} - y_n = \frac{h}{4}\left(\sqrt{k_1 k_2} + k_2 + k_3 + \sqrt{k_3 k_4}\right). \quad (2.3)$$

$$k_1 = f(x_n, y_n) \quad (2.4)$$

$$k_2 = f(x_n + ha_1, y_n + ha_1 k_1) \quad (2.5)$$

$$k_3 = f(x_n + ha_3, y_n + h(a_2 k_1 + a_3 k_2)) \quad (2.6)$$

$$k_4 = f(x_n + ha_4, y + h(a_4 k_1 + a_5 k_2 + a_6 k_3)) \quad (2.7)$$

In order to find the values of the parameters in the right hand side of the equation (2.4) to (2.7) we adopt the general principles of Taylor series to derive the functional values of k_i's using equations (2.1)-(2.2), starting with

$$k_1 = f(x_n, y_n) = y_n \quad (2.8a)$$

$$k_2 = 1 + ha_1 f_y + \frac{h^2}{2} a_1^2 k_1 f_{yy} + \frac{h^3}{6} a_1^2 k_1^2 f_{yyy} + \ldots \quad (2.8b)$$

$$k_3 = k_1 + h\left(a_2 + a_3\right) k_1 f_y + h^2 a_1 a_3 k_1 f_y{}^2 + \frac{h^2}{2}\left(a_2 + a_3\right)^2 k_1^2 f_{yy}$$
$$+ \frac{h^3}{2} a_1 a_3 \left(a_1 + 2\left(a_2 + a_3\right)\right) k_1^2 f_y f_{yy} + \frac{h^3}{6}\left(a_2 + a_3\right)^3 k_1^3 f_{yyy} \quad (2.8c)$$

$$k_4 = k_1 + h\left(a_4 + a_5 + a_6\right) k_1 f_y + h^2\left(a_1 a_5 + a_6\left(a_2 + a_3\right)\right) k_1 f_y{}^2$$
$$+ \frac{h^3}{2}\left(a_1^2 a_5 + a_6\left(a_2^2 + a_3^2 + 2a_2 a_3\right)\right)\left(a_1 a_5 + a_6\left(a_2 + a_3\right)\right) k_1^2 f_{yy} \quad (2.8d)$$
$$+ h^3 a_1 a_3 a_6 k_1 f_y{}^3 + \frac{h^3}{6}\left(a_4 + a_5 + a_6\right)^3 k_1^3 f_{yyy}$$

We introduce the binomial expansion of $(1+x)^{1/2} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \ldots$ (2.9a)

By setting $\sqrt{k_1 k_2} = f(1+x)^{\frac{1}{2}}$, (2.9b) *where* $k_1 k_2 = f^2(1+x)$,

$$x = \left(\frac{k_1 k_2}{f^2} - 1\right) \quad (2.9c)$$

Now substituting for x in the binomial expansion, we have

$$\sqrt{k_1 k_2} = 1 + \frac{1}{2}\left(\frac{k_1 k_2}{f^2} - 1\right) - \frac{1}{8}\left(\frac{k_1 k_2}{f^2} - 1\right)^2 + \frac{1}{16}\left(\frac{k_1 k_2}{f^2} - 1\right)^3 \ldots \quad (2.10)$$

Which Simplifies to

$$\sqrt{k_1 k_2} = 1 + \frac{h}{2} a_1 f_y + \frac{h^2}{4} a_1^2 k_1 f_{yy} + \frac{h^3}{4} a_1^3 k_1^2 f_{yyy} - \frac{h^2}{8} a_1^2 f_y^2 - \frac{h^2}{8} a_1^3 k_1 f_y f_{yy} + \frac{h^3}{16} a_1^3 f_y^3 \ldots \quad (2.11)$$

Similarly,

$$k_2 = 1 + ha_1 f_y + \frac{h^2}{2} a_1^2 k_1 f_{yy} + \frac{h^3}{6} a_1^2 f_{yyy} ... \qquad (2.12)$$

$$k_3 = k_1 + h(a_2 + a_3) k_1 f_y + h^2 a_1 a_3 k_1 f_y^2 + \frac{h^3}{2}\left\{a_1^2 a_3 + 2\left(a_1 a_2^3 + a_1 a_2 a_3\right)\right\} k_1^2 f_y f_{yy}$$
$$+ \frac{h^2}{2}(a_2 + a_3)^2 k_1^2 f_{yy} + \frac{h^3}{6}(a_2 + a_3)^3 k_1^3 f_{yyy} + ... \qquad (2.13)$$

Finally,

$$\sqrt{k_3 k_4} = 1 + \frac{h}{2}\left\{(a_2 + a_3) + (a_4 + a_5 + a_6)\right\} f_y$$
$$+ \frac{h^2}{8}\left\{4(a_1 a_3 + a_1 a_5) + 4a_6(a_2 + a_3) + 4(a_4 + a_5 + a_6)\left((a_2 + a_3) - (a_4 + a_5 + a_6)\right) - (a_2 + a_3)^2\right\} f_y^2$$
$$+ \frac{h^2}{4}\left\{(a_2 + a_3)^2 + (a_4 + a_5 + a_6)^2\right\} k_1 f_{yy} + \frac{h^3}{16}\left\{8a_1 a_3 a_6 + 4a_6(a_2 + a_3)^2 + 4a_1 a_5(a_2 + a_3) + \right.$$
$$4a_1 a_3(a_4 + a_5 + a_6) + (a_2 + a_3)^3 + (a_4 + a_5 + a_6)^3 \quad + 4a_1 a_5(a_4 + a_5 + a_6) - 4a_1 a_6(a_2 + a_3)$$
$$\left. -4a_6(a_2 + a_3)(a_4 + a_5 + a_6) - (a_2 + a_3)(a_4 + a_5 a_6)^2 - (a_2 + a_3)^2(a_4 + a_5 + a_6)\right\} f_y^3$$
$$+ \frac{h^3}{12}\left[(a_4 + a_5 + a_6)^3 + (a_2 + a_3)^3\right] k_1^2 f_{yyy} +$$
$$+ \frac{h^3}{8}\left\{\begin{array}{l} 2a_1^2 a_3 + 2a_1^2 a_5 + 4a_1 a_3(a_2 + a_3) + 4a_1 a_5(a_4 + a_5 + a_6) + 2a_6(a_2 + a_3)^2 \\ +4a_6(a_2 + a_3)(a_4 + a_5 + a_6) + (a_2 + a_3)^2(a_4 + a_5 + a_6) - (a_2 + a_3)^3 - (a_4 + a_5 + a_6)^3 \end{array}\right\} \qquad (2.14)$$

Substituting (2.11) up to (2.14) into (2.3) and setting $A = a_2 + a_3$ and $B = a_4 + a_5 + a_6$ (2.15)

we obtain values for the parameters a_i's. Hence $y_{n+1} - y_n = h + \frac{1}{8}h^2(3a_1 + 3A + B) f_y + \frac{h^3}{16}(3a_1^2 + 3A^2 + B^2) k_1 f_{yy}$

$$+ \frac{1}{32}h^3(-a_1^2 + 4a_1 a_5 + 12a_1 a_3 4a_6 A + 2AB - A^2 - B^2) f_{y^2} + \frac{1}{48}h^4(3a_1^3 + 3A^3 + B^3) k_1^2 f_{yyy}$$
$$+ \frac{1}{128}h^4[-4a_1^3 + 11a_1^2 a_5 + 27a_1^2 a_3 + 22a_1 a_5 B + 54a_1 a_5 A + 22a_6 AB + 11a_6 A^2 +$$
$$13AB(A + B) + 2(A^3 + B^3)] k_1 f_y f_{yy} + \frac{1}{64}h^4[a_1^3 + 8a_1 a_3 a_6 + 4a_1 a_5 A + 4a_1 a_3 B + 4a_6 A^2$$
$$-4a_1 a_5 B - 4a_1 a_3 A - 4a_6 AB - AB(A + B) + A^3 + B^3] f_{y^2} \qquad (2.16)$$

For the purpose of obtaining values for the parameters a_i we obtain the Taylor Series expansion in one variable y, such that,

$$y_x - y_{(x)} = hy'(x_n) + \frac{h^2}{2!}y''(x_n) + \frac{h^3}{3!}y'''(x_n) + \frac{h^4}{4!}y^{iv}(x_n) + o(h^5)... \qquad (2.17)$$

Where $y' = k_1,\ y'' = k_1 f_y,\ \ y''' k_1^2 f_{yy} + k_1 f_{y^2}\ \ and\ \ y^{iv} = k_1^3 f_{yyy} + 4k_1^2 f_y f_{yy} + k_1 f_{y^3}$ (2.18)

so that, (1.20) becomes:

$$y_{x+1} - y_x = hk_1 + \frac{h}{2}h^2 k_1 f_y + \frac{h^3}{6}(k_1^2 f_{yy} + k_1 f_{y^2}) + \frac{h^4}{24}(k_1^3 f_{yyy} 4k_1^2 f_y f_{yy} + k_1 f_{y^3}) \qquad (2.19)$$

Now comparing equation (1.19) with equation (1.22) above, we have the following equations for $k^n{}_{i-n} \quad \forall i = 2,3,4,5,\ n = 1,2,3,4,5$

$$k_1 = 1, \qquad (2.20a)$$
$$3a_1 + 3A + B = 4, \qquad (2.20b)$$
$$3a_1^2 + 3A^2 + B^2 = \frac{8}{3} \qquad (2.20c)$$
$$d(-a_1^2 + 4a_1a_5 + 12a_1a_3 + 4a_6A + 2AB - A^2 - B^2) = \frac{16}{3} \qquad (2.20d)$$
$$3a_1^3 + 3A^3 + B^3 = 2 \qquad (2.20e)$$
$$-4a_1^3 + 11a_1^2a_5 + 27a_1^2a_3 + 22a_1a_5B + 54a_1a_3A + 22a_6AB + 11a_6A +$$
$$13AB(A+B) + 2(A^3 + B^3) = \frac{64}{3} \qquad (2.20f)$$
$$a_1^3 + 8a_1a_3a_6 + 4a_1a_5A + 4a_1a_3B + 4a_6A^2 - 4a_1a_5B - 4a_1a_3A -$$
$$4a_6AB - AB(A+B)A^3 + B^3 = \frac{8}{3} \qquad (2.20g)$$

For easy computation, we set

$$A = a_2 + a_3 = \frac{1}{3} \qquad (2.20h)$$
$$B = a_4 + a_5 + a_6 = 1 \qquad (2.20i)$$

Using the value of $a_1 = \frac{2}{3},\ A = \frac{1}{3}\ and\ B = 1$, we obtain values for the remaining parameters a_2, a_3, a_4, a_5 and a_6 by solving equations (iv), (vi) and (vii) by using MATLAB program we obtain values which represent the coefficients of the parameters $a_{i's}$

$$a_1 = \frac{2}{3},\ a_2 = -\frac{165}{293},\ a_3 = \frac{788}{879},\ a_4 = \frac{369}{263},\ a_5 = -\frac{545}{1763},\ a_6 = -\frac{908}{9669} \qquad (2.20)$$

Hence we have the new algorithm:

$$y_{n+1} - y_n = \frac{h}{4}(\sqrt{k_1k_2} + k_2 + k_3 + \sqrt{k_3k_4}) \qquad (2.21)$$

$$k_1 = f(y_n),\ k_2 = f(y_n + \frac{2h}{3}k_1),$$

$$k_3 = f(y_n + h(\frac{-165}{293}k_1 + \frac{788}{1763}k_2))$$

and

$$k_4 = f(y_n + h(\frac{369}{263}k_1 - \frac{545}{1763}k_2 - \frac{908}{9669}k_3)) \qquad (2.22)$$

3.0 Implementation of the Method:

In this section, we prove that the method is convergent and implement it using two singular initial value problems

3.1 Theorem:

We assert that our method (2.21) to (2.22) is consistent and converges to a known function if

$$y' = f(x, y),\ y(a) = \eta,\ a \le x \le b \qquad (3.1)$$

Proof: In order to establish the convergence of the method, we show that (2.21) is consistent with the initial value problem (1.1); that is

$$\phi(x, y, 0) = f(x, y) \qquad (3.2)$$

Note here that a necessary condition for a method to converge is that it has to be consistent. And if the method is stable, it is sufficient to prove convergence. In a similar manner, we apply the above rules to show that our method;

$$y_{n+1}-y_n=\frac{h}{4}\left(\sqrt{k_1k_2}+k_2+k_3+\sqrt{k_3k_4}\right) \quad (3.3)$$

with $K_1 = f(y_n), K_2 = f(y_n + a_1hk_1),\ k_3 = f(y_n + h(a_2k_1 + a_3k_2))$

and $k_4 = f(y_n + h(a_4k_1 + a_5k_2 + a_6k_3))$ (3.4)

is consistent with equation (3.6). Using the exact solution $y(x_n)$ of the initial value problem $y' = f(y) = f(x,y),\quad y(x_0) = y_0$.(3.5)by substituting the set of equations in (3.10) into (3.9) such that

$$T_n(h^5) = y_{n+1} - y_n - \frac{h}{4}\Big\{\{(f(y_n) * f(y_n + ha_1f(y_n)))^{\frac{1}{2}} + f(y_n + ha_1f(y_n)) +$$

$$f(y_n + ha_2f(y_n) + ha_3f(y_n) + a_1h(y_n)) + f(y_n + ha_2f(y_n) + ha_3f(y_n) + a_1h(y_n)) * f$$

$$\{y_n + ha_4f(y_n) + ha_5f(y_n) + a_1h(y_n) + ha_6f(y_n + h(a_2f(y_n) + a_3f(y_n) + a_1h(y_n)))\}^{\frac{1}{2}} \quad (3.6)$$

Dividing all through by h andtaking the limit of both sides as h⟶0, we have:

$$T_n(h) = \frac{y_{n+1}-y_n}{h} - \frac{1}{4}(f(y_n) * f(y_n))^{\frac{1}{2}} + f(y_n) + f(y_n) + (f(y_n) * f(y_n))^{\frac{1}{2}}$$

$$\lim_{h\to 0} T_n(h) = \lim_{h\to 0}\left(\frac{y_{n+1}-y_n}{h}\right) = \frac{1}{4}\left[f(y_n) + f(y_n) + f(y_n) + f(y_n)\right] \quad (3.7)$$

$$y' = f(y) = f(x,y),\quad y(x_0) = y_0. \quad (3.8)$$

Hence the method is consistent and convergent.

3.2 Comparison of results:

We now apply the new formula to solve two different initial value problems and compare the result with two other different methods for accuracy and error examination.The methods considered are as follows

MODIFIED KUTTA'S ALGORITHM (MKA):

$$y_{n+1}-y_n=\frac{h}{4}\left(\sqrt{k_1k_2}+k_2+k_3+\sqrt{k_3k_4}\right) \quad (3.9)$$

$$k_1 = f(y_n),\ k_2 = f(y_n + \frac{2}{3}hk_1),$$

$$k_3 = f(y_n + h(-\frac{165}{293}k_1 + \frac{788}{879}k_2), \quad (3.10a)$$

$$k_4 = f(y_n + h(\frac{369}{263}k_1 - \frac{545}{1763}k_2 - \frac{908}{9669}k_3)) \quad (3.10b)$$

ONE-THIRD RUNGE-KUTTA METHOD ($\frac{1}{3}RKM$):

$$y_{n+1}-y_n=\frac{h}{3}\left(\sqrt{k_1k_2}+\sqrt{k_2k_3}+\sqrt{k_3k_4}\right) \quad (3.11)$$

$$k_1 = f(y_n),\ k_2 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right), k_3 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{16}(-k_1 + 9k_2)\right)$$

$$k_4 = f\left(x_n + h, y_n + \frac{h}{24}(-3k_1 + 5k_2 + 22k_3)\right) \quad (3.12)$$

3. CLASSICAL RUNGE-KUTTA METHOD (CRKM):

$$y_{n+1}-y_n=\frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \quad (3.13)$$

$$k_1 = f(y_n),\ k_2 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right),$$

$$k_3 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_2\right), k_4 = f\left(x_n + h, y_n + hk_3\right), \quad (3.14)$$

4. ORIGINAL KUTTA'S METHOD:

$$y_{n+1} - y_n = \frac{h}{8}(k_1 + 3k_2 + 3k_3 + k_4) \quad (3.15)$$

$$k_1 = f(y_n), k_2 = f\left(y_n + \frac{1}{3}hk_1\right), \quad (3.16a)$$

$$k_3 = f\left(y_n - \frac{1}{3}hk_1 + hk_2\right), \quad k_4 = f\left(y_n + hk_1 - hk_2 + hk_3\right) \quad (3.16b)$$

Using the formulae in equations (3.9) to (3.16b) we solve two singular ivps given below.

Table I:Numerical Solution Of The Innocent Looking Ivp: $y' = 1 + y^2, y_0 = 1 \quad 0 \le x \le 1,$

whose theoretical solution is $y(x) = \tan\left(x + \frac{\pi}{4}\right)$ **and has a single pole at** $x = \frac{\pi}{4},$

XN	TSOL	MKA	ERROR	CRKM	ERROR	1/3RKM	ERROR	KUTTA	ERROR
.1E+00	0.1223E+01	0.1223E+01	0.1528E-03	0.1223E+01	0.3339E-07	0.1223E+01	0.6639E-05	0.1223E+01	0.9052E-06
.2E+00	0.1508E+01	0.1508E+01	0.5077E-03	0.1508E+01	0.1480E-05	0.1508E+01	0.2775E-04	0.1508E+01	0.1129E-05
.3E+00	0.1896E+01	0.1894E+01	0.1414E-02	0.1896E+01	0.1096E-04	0.1896E+01	0.9968E-04	0.1895E+01	0.4644E-05
.4E+00	0.2465E+01	0.2461E+01	0.4103E-02	0.2465E+01	0.6307E-04	0.2465E+01	0.3829E-03	0.2464E+01	0.4819E-04
.5E+00	0.3408E+01	0.3394E+01	0.1422E-01	0.3408E+01	0.4030E-03	0.3408E+01	0.1825E-02	0.3407E+01	0.3663E-03
.6E+00	0.5332E+01	0.5260E+01	0.7200E-01	0.5328E+01	0.3958E-02	0.5318E+01	0.1364E-01	0.5328E+01	0.3833E-02
.7E+00	0.1168E+02	0.1079E+02	0.8905E+00	0.1155E+02	0.1274E+00	0.1138E+02	0.2982E+00	0.1155E+02	0.1219E+00
.8E+00	-.6848E+02	0.6011E+02	0.1286E+03	0.1922E+03	0.2606E+03	0.1233E+03	0.2982E+00	0.2085E+03	0.2769E+03
.9E+00	-.8688E+01	0.1215E+09	0.1215E+09	0.3120E+18	0.3120E+18	0.2582E+12	0.2582E+12	0.7001E+18	0.7001E+18
.1E+01	-.4588E+01	0.1538E+84	0.1538E+84	0.3278+261	0.3278+261	0.7454+123	0.7454+123	0.6355+266	0.6355+266

Table 2:Solution OF $y' = y, \quad y(0) = 1 \quad 0 \le x \le 1,$ **whose theoretical solution is** e^x

XN	TSOL	CRKM	ERROR	MKA	ERROR	1/3RKM	ERROR	KUTTA	ERROR
.1E+00	0.1105E+01	0.1105E+01	0.8474E-07	0.1105E+01	0.2943E-06	0.1105E+01	0.1908E-06	0.1105E+01	0.8474E-07
.2E+00	0.1221E+01	0.1221E+01	0.1873E-06	0.1221E+01	0.6506E-06	0.1221E+01	0.4218E-06	0.1221E+01	0.1873E-06
.3E+00	0.1350E+01	0.1350E+01	0.3105E-06	0.1350E+01	0.1079E-05	0.1350E+01	0.6993E-06	0.1349E+01	0.3105E-06
.4E+00	0.1492E+01	0.1492E+01	0.4576E-06	0.1492E+01	0.1589E-05	0.1492E+01	0.1030E-05	0.1491E+01	0.4575E-06
.5E+00	0.1649E+01	0.1649E+01	0.6321E-06	0.1649E+01	0.2196E-05	0.1649E+01	0.1423E-05	0.1648E+01	0.6321E-06
.6E+00	0.1822E+01	0.1822E+01	0.8383E-06	0.1822E+01	0.2912E-05	0.1822E+01	0.1888E-05	0.1822E+01	0.8382E-06
.7E+00	0.2014E+01	0.2014E+01	0.1081E-05	0.2014E+01	0.3755E-05	0.2014E+01	0.2434E-05	0.2013E+01	0.1080E-05
.8E+00	0.2226E+01	0.2226E+01	0.1365E-05	0.2226E+01	0.4742E-05	0.2226E+01	0.3074E-05	0.2225E+01	0.1365E-05
.9E+00	0.2460E+01	0.2460E+01	0.1697E-05	0.2460E+01	0.5896E-05	0.2460E+01	0.3822E-05	0.2459E+01	0.1697E-05
.1E+01	0.2718E+01	0.2718E+01	0.2084E-05	0.2718E+01	0.7241E-05	0.2718E+01	0.4694E-05	0.2718E+01	0.2084E-05

Table I and Table 2 shows the performance of the new method as compared with the numerical solution of other existing RKF. The results indicate that the new algorithm performances well in the solution ivps in ordinary differential equations

4.0 Stability analysis of method

Our duty here is to investigate and establish the stability of the method by following [13], where it was revealed that "in all computational methods, the use of a scheme for numerical solution of initial value problem (1.1) will generate errors at some stages of the computation due to inaccuracy inherent in the formula and the arithmetic operations adopted during computer implementation. The magnitude of the error determines the degree of accuracy and stability of the method". Thus, it is important that the numerical solution approximates the exact solution and that the numerical solution tends to the exact solution as the step size tends to zero.

Butcher[7], observed that if the step length used is too small, excessive computation time and round-off error will result. We should also consider the opposite case, and ask whether there is any upper bound on step length. Often there is such a bound and it is reached when the method becomes numerically unstable, that is the numerical solution produced, no longer corresponds qualitatively with the exact solution.According to Lambert [1], the traditional criterion for ensuring that a numerical method is stable is called “Absolute Stability”, and this analysis will therefore, be carried out to establish the absolute stability of our method by subjecting it to the linear test equation;

$$y' = \lambda y; \qquad \lambda \in C; \ \mathrm{Re}(\lambda) < 0 \tag{4.1}$$

where λ is complex.

Butcher [7] emphasized that all Runge-Kutta methods including the implicit ones, when applied to the test equation, reduce to an equation of the form;

$$y_{n+1} = R(\lambda h) \tag{4.2}$$

where $R(\lambda h)$ is called the stability polynomial function. Bearing this in mind we write$\mu = \lambda h$, so that it produces a linear system for the computation of $k_i \ \forall \ i = 1,2,3 and 4$ which will be solved for, and then inserted into our method to produce

$$\frac{y_{n+1}}{y_n} = R(\mu) \tag{4.3}$$

[10], says, the key issue for understanding the long term dynamics of Runge-Kutta methods near some fixed points, concerns the region where R (μ) ≤ 1; that is, the Stability region of the numerical method. The polynomial, for which R(μ) ≤ 1 is known as the Stability polynomial of the method, and this method is absolutely stable for a given$\mu = \lambda h$, if all the roots of the polynomial function lie within the unit circle. The region containing all these points in the complex plane is said to be a region of absolute stability, if the method is stable for all

$$\mu = \lambda h \in R(\mu). \tag{4.4}$$

It is also possible according to Lambert [1], that applying a method to the test equation (1.1) (where y_n is a scalar) yields

$$\left\{ \begin{aligned} Y_i &= y_n + \mu \sum_{j=1}^{s} a_{ij} Y_j \\ y_{n+1} &= y_n + \mu \sum_{i=1}^{s} b_i Y_i \end{aligned} \right\} \tag{4.5a}$$

Now defining

$$Y, e \in R^s \ by \ Y := [Y_1, Y_2, ..., Y_s]^T \ and \qquad e := [1,1,....,1]^T; \tag{4.5b}$$

we may then write (4.5a) in the form

$$Y = y_n e + \mu A Y, \qquad and \quad y_{n+1} = y_n + \mu b^T Y. \tag{4.6}$$

Where A is the matrix of coefficients.

Solving the first of these for Y and substituting in the second gives

$$y_{n+1} = y_n \left[1 + \mu b^T (I - \mu A)^{-1} e\right], \tag{4.7}$$

Where I the s x s unit matrix, the stability function is therefore given by

$$R(\mu) = 1 + \mu b^T (I - \mu A)^{-1} e \tag{4.8}$$

However, in another approach, Dekker and Verwer [9] gives an alternative form of $R(\mu)$, where they observed that the solution for y_{n+1} by Cramer’s rule is $y_{n+1} = \frac{N}{D}$ (4.9) where; N= $y_n \det\left[I - \mu A + \mu e b^T\right]$;

$$D = \det\left[I - \mu A\right] \tag{4.10}$$

$$\text{Hence} \quad y_{n+1} = R(\mu) y_n = \frac{N}{D} \tag{4.11}$$

where

$$\frac{y_{n+1}}{y_n} = R(\mu) = \frac{\det\left[I - \mu A + \mu e b^T\right]}{\det\left[I - \mu A\right]} \qquad (4.12)$$

We observed here that, irrespective of the values given to the parameters in matrix A after satisfying the order requirements, for a given P = 1, 2, 3, 4, all p-stage Runge-Kutta methods of order p have the same interval of absolute stability. These intervals are given in Table 3, where R_p denote any p-stage Runge-Kutta method of order p.

Table 3 (Interval Of Absolute Stability For Order P, For P $\leq$ 4) $\mu = \lambda h$

Method	s	**Interval of absolute stability**
R_1	$1+\mu$	(-2, 0)
R_2	$1+\mu+\frac{1}{2}\mu^2$	(-2, 0)
R_3	$1+\mu+\frac{1}{2}\mu^2+\frac{1}{6}\mu^3$	(-2.51, 0)
R_4	$1+\mu+\frac{1}{2}\mu^2+\frac{1}{6}\mu^3+\frac{1}{24}\mu^4$	(-2.78, 0)

All Runge-Kutta methods of order four have the stability polynomialR_4 shown in the table above.Below are the curves showing the different regions of absolute stability for the various orders as indicated in Table3, the curves are put together to visualize their shape as the orders grow.

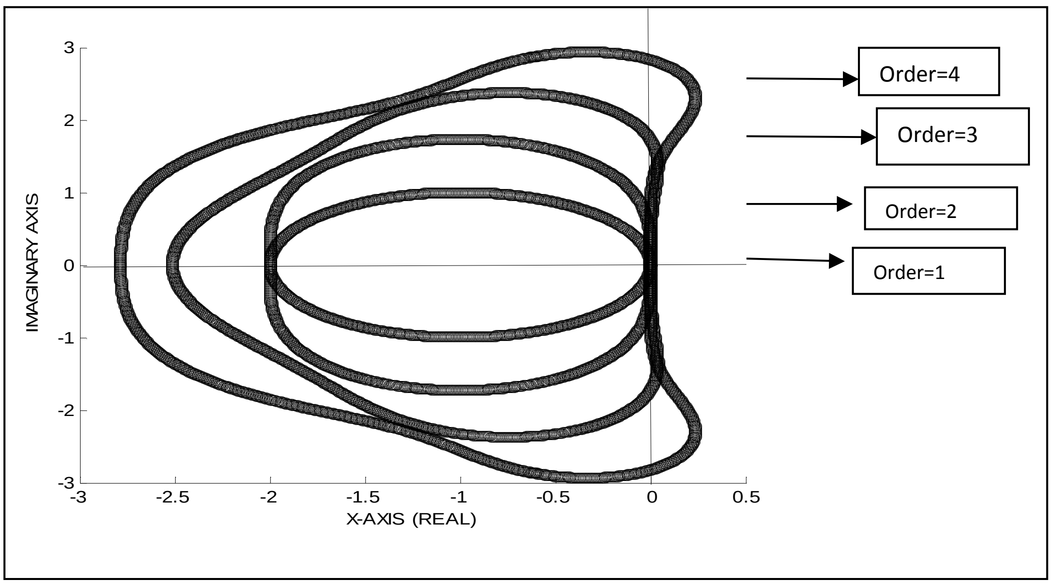

Figure 1:(Jordan Curves Showing the Region of Absolute Stability of Order p, For $1 \leq p \leq 4$)

In figure 1 above, the regions of absolute stability of explicit p-stage, p[th]-order Runge–Kutta methods for $1 \leq p \leq 4$ are plotted in complex $h\lambda$ space. The absolute stability regions are shown in thick black lines, and the ordinate and abscissa are Im ($h\lambda$) and Re($h\lambda$) respectively. Notice that the size of the regions increases with the order of the method.

A close examination of the various approaches of getting the stability polynomial function of a general one-step scheme as revealed in[8], will be more appropriate for our method. It was observed that the other approaches were not feasible; the reason could be attributed to the transformation procedure adopted during the derivation of our method.

We now show that the stability analysis of our method can be established by proving the following theorem.

Theorem II: We assert that the new Kutta algorithm in equations (2.21) and (2.22) is absolutely stable.

Proof: To prove stability of the method, we set the parameters e $a_{i's}$ as follows:

$$a_1 = \frac{2}{3},\ a_2 = -\frac{165}{293},\ a_3 = \frac{788}{879},\ a_4 = \frac{369}{263}, a_5 = -\frac{545}{1763},\ a_6 = -\frac{908}{9669} \qquad (4.13)$$

Such that ,

$$y_{n+1} - y_n = \frac{h}{4}\left(\sqrt{k_1 k_2} + k_2 + k_3 + \sqrt{k_3 k_4}\right)..... \qquad (4.14)$$

$$k_1 = y' = \lambda y, \quad k_2 = \lambda y\left(1 + a_1\lambda h\right), \qquad (4.15a)$$

$$k_3 = \lambda y\left(1 + a_2\lambda h + a_3\lambda^2 h^2\right) \ and \ k_4 = \lambda y\left(1 + a_4\lambda h - a_5\lambda^2 h^2 - a_6\lambda^3 h^3\right) \qquad (4.15b)$$

Therefore substituting (4.13) into (4.14), (4.15a) and (4.15b), we get

$$k_1 = y' = \lambda y,\ k_2 = \lambda y\left(1 + \frac{2}{3}\lambda h\right), \qquad (4.16a)$$

$$k_3 = \lambda y\left(1 + \frac{1}{3}\lambda h + \frac{661}{1106}\lambda^2 h^2\right) \ and \ k_4 = \lambda y\left(1 + \lambda h - \frac{353}{1487}\lambda^2 h^2 - \frac{159}{2833}\lambda^3 h^3\right) \qquad (4.16b)$$

$$y_{n+1} - y_n = \frac{h}{4}\left\{\begin{array}{l}\left(\lambda^2 y^2\left(1 + \frac{2}{3}\lambda h\right)\right)^{\frac{1}{2}} + \lambda y\left(1 + \frac{2}{3}\lambda h\right) + \lambda y\left(2 + \lambda h + \frac{661}{1106}\lambda^2 h^2\right) \\ + \left(\lambda^2 y^2\left(1 + \frac{4}{3}\lambda h + \frac{1288}{1857}\lambda^2 h^2 + \frac{1537}{3324}\lambda^3 h^3\right)\right)^{\frac{1}{2}}\end{array}\right\} \qquad (4.17)$$

By setting $\lambda h = \mu$ and simplifying, we have:

$$\frac{y_{n+1} - y_n}{y_n} = \frac{\mu}{4}\left(\left(1 + \frac{2}{3}\mu\right)^{\frac{1}{2}} + \left(2 + \mu + \frac{661}{1106}\mu^2\right) + \left(1 + \frac{4}{3}\mu + \frac{1288}{1857}\mu^2 + \frac{1537}{3324}\mu^3\right)^{\frac{1}{2}}\right) \qquad (4.18a)$$

Let $\alpha = \left(1 + \frac{2}{3}\mu\right)^{\frac{1}{2}}, \ \beta = \left(2 + \mu + \frac{661}{1106}\mu^2\right), \ \phi = \left(1 + \frac{4}{3}\mu + \frac{1288}{1857}\mu^2 + \frac{1537}{3324}\mu^3\right)^{\frac{1}{2}},$ (4.18b)

Usingbinomial expansion method, we expand these termsin rational power of $\frac{1}{2}$.

So that $\alpha = 1 + \frac{1}{3}\mu - \frac{1}{18}\mu^2 + \frac{1}{54}\mu^3 + \ldots$ (4.19)

And $(\phi) = \left(1 + \frac{2}{3}\mu + \frac{694}{5571}\mu^2 + \frac{4}{27}\mu^3 + ...\right)$ (4.20)

Adding up (α), β and (ϕ) in equation (4.18b), we have the following:

$$\frac{y_{n+1} - y_n}{y_n} = \left(1 + \frac{\mu}{3} - \frac{\mu^2}{18} + \frac{\mu^3}{54}\right) + \left(2 + \mu + \frac{661\mu^2}{1106}\right) + \left(1 + \frac{2\mu}{3} - \frac{694\mu^2}{5571} + \frac{4\mu^3}{27}\right) \qquad (4.21)$$

$$\frac{y_{n+1}}{y_n} = \left(1 + \mu + \frac{1}{2}\mu^2 + \frac{\mu^3}{6} + \frac{\mu^4}{24}\right) \qquad (4.22)$$

Therefore,

$$R(\mu)=\left(\frac{1}{24}\mu^4+\frac{1}{6}\mu^3+\frac{1}{2}\mu^2+\mu+1\right) \qquad (4.23)$$

Equation (4.23) is the stability polynomial of our method. It is the same with the general 4th order Runge-Kutta polynomial, which is an indication that the method is absolutely stable.
To obtain the roots of the stability polynomial, we solve (4.23) by equating the RHS to zero.
And applying (MATLAB), we have the following roots:

μ_1 = -0.2706+2.50i, μ_2 = -0.2706-2.5048i μ_3 = -1.7294+0.8890i μ_4 = -1.7294-0.8890i
By the same MATHLAB code, weplot the region of Stability for the new method, as shown in figure2 below:

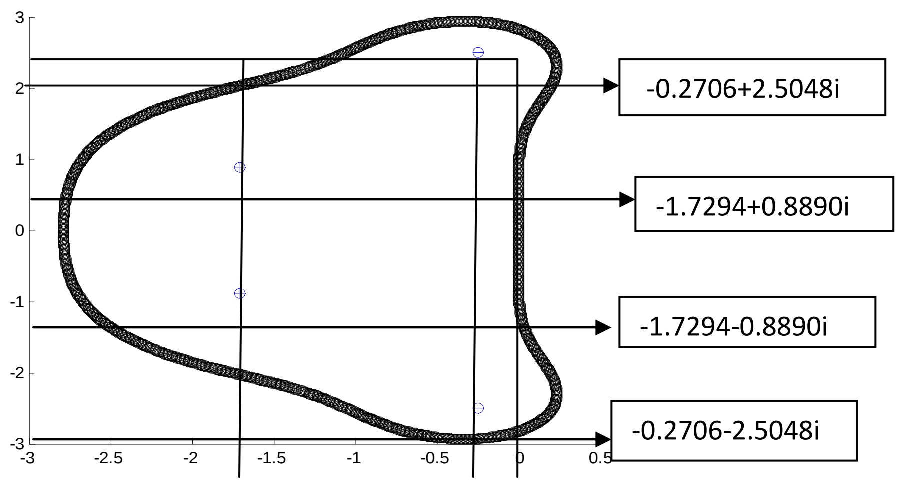

Figure 2: The Region of Absolute Stability of the Modified Kutta's Algorithm

4.2 Conclusion

In deriving the new modified Kutta's algorithm, we applied the Taylor series expansion in combination with the rational binomial theorem to expand the $k_{i's}$ by following the principle of Runge–Kutta. By a careful use of geometric progression, we constructed the method out of the existing Kutta method and obtained values for the parameters $a_{i's}$. After establishing the method, we proved that our algorithm converges and authenticate the validity of our claimby implementing it numerically on two initial value problems in first order ordinary differential equations. Comparing the results fromfour methods we see thatour method improved in its level of performance as the step length increases. We have therefore, showed throughnumerical investigation that our method can solve singular initial value problems in o.d.es, as demonstrated in the tables above and stable as exemplified bythe stability curve obtained in figure 2 above. Hence, the new method is consistent, convergent, absolutely stable and of high accuracy. The method will therefore be suitable for the solution of singular real life problems that can possibly be reduced to first order ordinary differential equation involving initial value conditions.
Observe that solving an initial value problem in ordinary differential equations an error is introduced at each integration step of the formula. The magnitude of this so called local truncation error is a measure of the accuracy of the integration formula. Furthermore, the magnitude of the total error depends on the magnitude of the local truncation errors and their propagation. The local error at each step may be small; though small, the total error may become large due to accumulation and amplification of these local errors. Furthermore,observe in the table of results above that error grows with steplength and that the difference between the theoretical solution and the calculated either reduces or grow with the steplength as can be seen in the methods compared. This growth phenomenon is called numerical instability.
Finally, it is clear that this research will go a long way in reducing the rigor in the solution of initial value problems in ordinary differential equations, and it is worthwhile to encourage further research work in this area of study. The exciting discovery made by us has shown that no area of research is ever exhausted depending on where the interest of a prospective researcher lies.
Since this formula maintains a high degree of accuracy in handling initial value problems in ordinary differential equations, we therefore, recommend it to all numerical analysts and industrial programmers.

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