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A Sixth -Order Implicit Method for the Numerical Integration of Initial Value Problems of Third Order Ordinary Differential Equations

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Abstract

In this paper we derive a linear multistep method with step number $k = 4$ ***using Taylor series as basis function for approximate solution .An order six scheme is produced which is used for the direct solution of third order initial value problems in ordinary differential equation . Taylor's series algorithm of the same order was developed to implement our method. This implementation strategy is found to be efficient and more accurate as the result has shown in the numerical experiments.***

Keywords: Linear multistep methods, third order, Taylor series, approximate solutions, direct solution basis function

1.0 Introduction

In this article, we considered the direct method for solving a third order initial value problem in ordinary differential equations of the forms

$$y''' = f(x, y) \qquad (1.1a)$$

$$y(a) = \partial_0, y'(a) = \partial_1, y''(a) = \partial_2 ;$$

$$y^{(iv)} = f(x, y, y', y''),$$

$$y(a) = y_0, y'(a) = \eta_1, y''(a) = \eta_2 \qquad (1.1b)$$

where $a, y, f \in R$

This class of problem has a lot of applications in sciences and engineering especially in biological sciences and control theory. The conventional method of solving (1.1) is to first reduce it to an equivalent system of first order differential equation and then appropriate methods of solution for first order equation are applied to solve the resulting equations. This approach has been discussed extensively[1-3]. Conventional Linear Multistep Method (LMM) for solving first order ordinary differential equation is of form:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} \quad , \alpha_k = 1 \qquad (1.3)$$

where α_j and β_j are uniquely determined.

According to Awoyemi [4] ; the approach of reducing to a system of first order system lead to waste of human and computer time.Variants of the method in (1.2) have been developed to improve the accuracy of results. These includes the hybrid LMM[5], the second derivative method and the general multi-derivative LMM [6-7] .

Eminent scholars have made efforts to solve higher order initial value problems especially the second and third order ordinary differential equations.

For instance, LMM based on Numerov method has been considered by[9-12] ; to solve second order initial value problems of the form: $y'' = f(x, y)$ and $y'' = f(x, y, y')$ directly without reducing to a system of first order.

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Also LMM for the direct solution third order IVPS have been proposed several authors[4],[13-17]. While the method [4] was implemented in predictor-corrector mode with starting values from Taylor series , those of[13-17] were combined with additional methods obtained from continuous k-step LMM to solve third order ODES directly. Therefore we are motivated by the results obtained by these authors to develop and implement LMM for the solution of (1.1) using Taylor series

2.0 Methodology

We consider the LMM for higher order initial value problem. The general multistep method or$k-$ step method for the solution of (1.1) may be written as

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h^r\sum_{j=0}^{k}\beta_j f_{n+j} \quad , \tag{2.1a}$$

where $f_{n+j} = f(x_{n+j}, y_{n+j},..., y_{n+j}^{(n-1)})$ (2.1b)

and r is the order of the differential equation, α_j and β_j are the coefficients to be determined.
The equation (2.1) can be written as

$$\sum_{j=0}^{k}\alpha_j y(x+jh) = h^r\sum_{j=0}^{k}\beta_j y^r(x+jh) \tag{2.2}$$

The above equation can only be used when the values $y(x)$ and $y^r(x)$ at k successive previous points are known. These kvalues are assumed to be given.
In order to determine the coefficients α_j 's and β_j 's , we write the local truncation error of (2.1) as

$$T_{n+k} = \sum_{j=0}^{k}\alpha_j y(x+jh) - h^r\sum_{j=0}^{k}\beta_j y^r(x+jh) \tag{2.3}$$

Where the function $y(x)$ is assumed to have continuous derivatives of sufficiently high order.

By expanding $y(x+jh)$ and $y^r(x+jh)$ in Taylor series about x_n and collecting terms, we obtain

$$T_{n+k} = C_0 y(x_n) + C_1 hy'(x_n) + ... + C_{p+1}h^{p+1}y^{p+1}(x_n) + C_{p+2}h^{p+2}y^{p+2}(\xi)$$

where $x_{n-k+1} < \xi < x_{n+1}$ and

$$C_q = \frac{1}{q!}\left[\sum_{j=1}^{k} j^q\alpha_j - q(q-1)...(q-(n-1)\sum_{j=1}^{k} j^{q-n}\beta_j\right], q = 0,1,2,..., p+2 \tag{2.4}.$$

The matrix equation arising from the above is of the form

$$AX = B \tag{2.5}$$

where $A = [c_0, c_1, ..., c_p]^T$, $X = [\alpha_0,...,\alpha_{k-1}, \beta_0,...,\beta_k]^T$, $B = [0,0,...,0]^T$ and T means Transpose. The matrix equation arising from (2.4) is solved to obtain the unknown parameters..
Note that the local truncation error T_{n+k} vanishes identically when $y(x)$ is a polynomial of degree less than or equal to $p+1$. The constants C_i and $p+1$ are therefore independent of $y(x)$. Thus the constants can be determined by imposing an accuracy of order p on the C_q, s
The method above can be used to derive linear multistep method of any order.

3.0 Derivation of four–step third order method (k=4)

By putting $k=4$ and $n=3$ in equation (2.1) , we have the equation

$$\sum_{j=0}^{4}\alpha_j y_{n+j} - h^3\sum_{j=0}^{4}\beta_j y'''_{n+j} = 0$$

which yield

$$\alpha_0 y_n + \alpha_1 y_{n+1} + \alpha_2 y_{n+2} + \alpha_3 y_{n+3} + \alpha_4 y_{n+4} - h^3 [\beta_0 y_n + \beta_1 y_{n+1} + \beta_2 y_{n+2} \beta_{n+3} y_{n+3} + \beta_4 y_{n+4}] = 0 \qquad (3.1)$$

The local truncation error is given by

$$T_{n+4} = \alpha_0 y_n + \alpha_1 y_{n+1} + \alpha_2 y_{n+2} + \alpha_3 y_{n+3} + \alpha_4 y_{n+4} - h^3 [\beta_0 y_n + \beta_1 y_{n+1} + \beta_2 y_{n+2} \beta_{n+3} y_{n+3} + \beta_4 y_{n+4}] \qquad (3.2)$$

Applying Taylor series expansion to (3.2) about x_n and collecting terms in equal powers of h we have the following system of equation

$$T_{n+4} = C_0 y_n + C_1 h y_n' + C_2 h^2 y_n'' + C_3 h^3 y_n''' + \ldots + 0(h^8) \qquad (3.3)$$

By imposing an accuracy of order 6 on (3.3) , i.e

$$C_0 = C_1 = C_2 = C_3 = C_4 = C_5 = C_6 = C_7 = 0, C_8 \neq 0 \text{ , and } T_{n+4} = 0(h^8) \qquad (3.4)$$

We havethe following matrix equation:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 2 & \frac{9}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{6} & \frac{8}{6} & \frac{27}{6} & -1 & -1 & -1 & -1 & -1 \\ 0 & \frac{1}{24} & \frac{16}{24} & \frac{81}{24} & 0 & -1 & -2 & -3 & -4 \\ 0 & \frac{1}{120} & \frac{32}{120} & \frac{243}{120} & 0 & -\frac{1}{2} & -2 & \frac{-9}{2} & -8 \\ 0 & \frac{1}{720} & \frac{64}{720} & \frac{729}{720} & 0 & -\frac{1}{6} & -\frac{8}{6} & -\frac{27}{6} & -\frac{64}{6} \\ 0 & \frac{1}{5040} & \frac{128}{5040} & \frac{2187}{5040} & 0 & -\frac{1}{24} & -\frac{16}{24} & \frac{-81}{24} & \frac{-256}{24} \\ 0 & \frac{1}{40320} & \frac{256}{40320} & \frac{6561}{40320} & 0 & -\frac{1}{120} & -\frac{32}{120} & -\frac{243}{120} & -\frac{1024}{120} \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \\ -8 \\ -\frac{64}{6} \\ -\frac{256}{24} \\ -\frac{1024}{120} \\ \frac{-4096}{720} \\ -\frac{16384}{5040} \\ -\frac{65536}{40320} \end{bmatrix} \qquad (3.5)$$

Solving the set of equation i (3.5) by Gaussian elimination, with $\alpha_4 = 1$, we obtain

$$\alpha_0 = 0, \alpha_1 = -1, \alpha_2 = 3, \alpha_3 = -3, \alpha_4 = 1, \beta_4 = \frac{1}{240}, \beta_3 = \frac{87}{180}, \beta_2 = \frac{63}{120}$$

$$\beta_1 = \frac{-3}{180}, \beta_0 = \frac{1}{240}$$

The method is therefore given as :

$$y_{n+4} - 3y_{n+3} + 3y_{n+2} - y_{n+1} = \frac{h^3}{240} \left(f_{n+4} + 116 f_{n+3} + 126 f_{n+2} - 4 f_{n+1} + f_n \right) \qquad (3.6)$$

which is of order six (p=6)

4.0 Analysis and Implementation of the Method

The method (3.6) is a specific member of the conventional LMM which can be expressed as

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^3 \sum_{j=0}^{k} \beta_j y_{n+j}''' \qquad (4.1)$$

and can be written symbolically as $\rho(E)y_n - h^n\sigma(E)f_n = 0$, $f_n = f(x_n, y_n)$

E is the shift operator defined $E^j(y_n) = y_{n+j}$ and $\rho(E)$ and $\sigma(E)$ are respectively the first and second characteristics polynomial of the LMM defined as

$$\rho(E) = \sum_{j=0}^{k}\alpha_j E^j,\ \sigma(E) = \sum_{j=0}^{k}\beta_j E^j \quad , \quad \alpha_k \neq 0 \tag{4.2}$$

Following [1] we define the local truncation error associated with (2.2) by the difference operator

$$L\left[y(x), h\right] = \sum_{j=0}^{k}\left[\alpha_j y(x+jh) - h^3\beta_j y'''(x+jh)\right] \tag{4.3}$$

where $y(x)$ is assumed to have continuous derivatives of sufficiently high order. Thereforeexpanding (4.3) in Taylor series about the point x to obtain the expression

$$L\left[y(x), h\right] = C_0 y(x) + C_1 h y'(x) + C_2 h^2 y''(x) + ... + C_{p+2}h^{p+2} \tag{4.4}$$

where they C_q, $q = 0,1,2,...,p+2$ are defined as

$$C_0 = \sum_{j=0}^{k}\alpha_j$$

$$C_1 = \sum_{j=0}^{k} j\alpha_j\ ,$$

$$C_2 = \frac{1}{2!}\sum_{j=0}^{k} j^2\alpha_j$$

$$C_q = \frac{1}{q!}\left[\sum_{j=1}^{k} j^q\alpha_j - q(q-1)(q-2)\sum_{j=1}^{k}\beta_j j^{q-3}\right] \tag{4.5}$$

In the sense of Lambert [1] ,we say that the method (4.1) is of order p and error constant C_{p+2} if

$C_0 = C_1 = C_2 = ... = C_p = C_{p+1} = 0, C_{p+2} \neq 0$

For our method (3.6), p=6 and $C_{p+2} = \frac{84}{40320}$

Zero stability of the 4-step method (3.6)

The method (3.6) is given by

$$y_{n+4} - 3y_{n+3} + 3y_{n+2} - y_{n+1} = \frac{h^3}{240}\left[f_{n+4} + 116f_{n+3} + 126f_{n+2} - 4f_{n+1} + f_n\right]$$

where

$\rho(r) = r^4 - 3r^3 + 3r^2 - r = r(r-1)^3 = 0$, hence $r = 1(\times 3)$

The roots of the equation $\rho(r) = 0$ lie on the unit circle and of multiplicity at most 3

Therefore the method is zero stable

Region of absolute stability of the four step method (3.6)

Applying the boundary locus methodto the method of(3.6), we see that

$$x(\theta) = \frac{240\left(35 + \cos 4\theta - 8\cos 3\theta + 28\cos 2\theta - 56\cos\theta\right)}{29530 + 2\cos 4\theta + 224\cos 3\theta - 424\cos 2\theta + 28448\cos\theta}$$

Evaluating θ between 0° and 180° ,the result is shown in Table 1

Table 1: Interval of Absolute Stability of the method (3.6)

θ	0	30	60	90	120	150	180
$x(\theta)$	0	0.00015	0.00274	0.065	0.625	0.000137	120

Therefore the interval of absolute stability of (3.6) is [0,120]

Implementation

Taylor series method can be used to solve higher order ordinary differential equations directly without the need to first reduce it to an equivalent system of first order.

Consider the initial value problems in (1.1a) and (1.1b). For our method of order p =6, Taylor series expansion is used to calculate

$y_{n+1}, y_{n+2}, y_{n+3}$, and their first, second ,third derivatives up to order p=6 as follows:

$$y_{n+j} \equiv y(x_n + jh) \approx y(x_n) + jhy'(x_n) + \frac{(jh)^2}{2!} y''(x_n) + \frac{(jh)^3}{3!} f_n + \ldots + \frac{(jh)^6}{6!} f_n'''$$

$$y^{(1)}_{n+j} \equiv y'(x_n + jh) \approx y(x_n) + jhy'(x_n) + \frac{(jh)^2}{2!} y''(x_n) + \frac{(jh)^3}{3!} f_n + \ldots + \frac{(jh)^6}{6!} f_n^{iv}$$

$$y^{(2)}_{n+j} \equiv y(x_n + jh) \approx y''(x_n) + jhf_n + \frac{(jh)^2}{2!} f_n' + \frac{(jh)^3}{3!} f_n'' + \ldots + \frac{(jh)^6}{6!} f_n^{v}$$

$$y^{(3)}_{n+j} \equiv y'(x_n + jh) \approx f_n + jhf_n' + \frac{(jh)^2}{2!} f_n'' + \frac{(jh)^3}{3!} f_n''' + \ldots + \frac{(jh)^6}{6!} f_n^{vi}$$

Then the known values of x_n and y_n are substituted into the differential equations.Next the differential is differentiated to obtain the expression for higher derivative using partial differentiation as follows

$$y_j''' = f(x_j, y_j, y_j', y_j'') = f_j$$

$$y^{iv} = f_x + y' f_y + y'' f_{y'} + ff_{y''}$$

$$= \left(\frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + y'' \frac{\partial}{\partial y'} + f \frac{\partial}{\partial y''} \right) f_j = Df_j$$

$$y^{v} = f_{xx} + (y')^2 f_{yy} + (y'')^2 f_{y'y'} + f^2 f_{y''y''} + 2y' f_{xy} + 2y'' f_{xy'} + 2 ff_{xy''} + 2y' ff_{yy''} + 2y'' ff_{y'y''}$$

$$+ Df_j (f_{y''}) + f_j (y'' + f_{y'})$$

$$= D^2 f_j + (f_{y''}) Df_j + f_j (y'' + f_{y'})_j$$, where

$$D = \left(\frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + y'' \frac{\partial}{\partial y'} + f \frac{\partial}{\partial y''} \right)$$

$\vdots$

$D^p f_j$,

Where p is the order of the method

And $D^2 = D(D)$

Numerical Experiments

Our method of order p=6 was used to solve some initial value problems of both general and special nature using Taylor's series of the same order .Some application problems were also considered.The results were compared with the results of other researchers in this area as can seen in Tables 2-7

The following initial value problems were used as test problems:

Problem 1

$$y''' = -y, y(0) = 1, y'(0) = -1, y''(0) = 1,$$
$$h = 0.1, 0 \le x \le 1$$

Exact solution: $y(x) = e^{-x}$

Problem 2

$$y''' = -e^x, y(0) = 1, y'(0) = -1, y''(0) = -3,$$
$$h = 0.1, 0 \le x \le 1$$

Exact solution: $2 + 2x^2 - e^x$

Problem 3

$$y''' + y'' + 3y' - 5y' = 2 + 6x - 5x^2,$$
$$y(0) = -1, y'(0) = 1, y''(0) - 3, h = 0.1, 0 \le x \le 1$$

Exact solution: $y(x) = x^2 - e^x + e^{-x}\sin(2x)$

Problem 4

$$y''' + 4y' = x, y(0) = 0, y'(0) = 0, y''(0) = 1,$$
$$h = 0.1, 0 \le x \le 1$$

Problem 5:

(Application problem)

$$y''' = y^{-2}, \quad y(0) = y'(0) = y''(0) = 1$$

The above problem has been derived by Tanner[17] to investigate the motion of the contact line for a thin oil drop spreading on a horizontal on a horizontal surface

Problem 6:

(Application problem)

We shall consider the non-linear third order problem :

$$y^n y''' = 1, y(0) = 1, y'(0) = 0, y''(0) = \lambda_{<0}$$

The above third order initial value problemwas derived by Fazal-i-Hag et al [18]to investigate the travelling wave solution of the form : $h(x,t) = y(x), x = \bar{x} - Vt$, where V is the wave velocity and y is the height of a thin film on a solid surface.

Table 2: The result of test problem 1

X	Exact soln	New result	Error in[14] Predictor-corrector Method(P=7)	Error in our new result (P=6)
0.1	0.904837	0.940837	1.36929E-09	1.95961E-11
0.2	0.818731	0.818731	3.12272E-08	2.47756E-09
0.3	0.740818	0.740818	1.27694E-07	4.18183E-08
0.4	0.670320	0.670320	3.25196E-07	1.18007E-07
0.5	0.606531	0.606531	6.54297E-07	2.30843E-07
0.6	0.548812	0.548812	1.14406E-06	3.79421E-07
0.7	0.496585	0.496585	1.81784E-06	5.60411E-07
0.8	0.449329	0.449329	2.69774E-06	7.63714E-07
0.9	0.406570	0.406570	3.80241E-06	9.62881E-07
1.0	0.367879	0.367879	5.14755E-06	1.09640E-06

Table 3: The result of test problem 2

X-value	Exact Solution	New result (P=6)	Error in[13] (P=7)	Error in our new method
0.1	0.914829	0.914829	7.56477E-10	2.00922E-11
0.2	0.858597	0.858597	3.83983E-09	2.60461E-09
0.3	0.830141	0.830141	9.32410E-09	4.50760E-09
0.4	0.828175	0.828175	1.69587E-08	1.27448E-08
0.5	0.851279	0.851279	2.60999E-08	2.50056E-08
0.6	0.897881	0.897881	3.55095E-08	3.13587E-08
0.7	0.966247	0.966247	4.51295E-08	4.00392E-08
0.8	0.105446	0.105445	5.45075E-08	5.20610E-08
0.9	1.160340	1.160396	6.28431E-08	1.4132E-07
1.0	1.281720	1.281718	6.95410E-08	1.7644E-07

Table 4: The result of test problem 3

X-value	Exact	New result	Error in[13]	Error in our
Solution	(P=6)	(P=7)	method	
0.1	0.915407E+00	0.915407E+00	6.40864E-06	5.4106E-09
0.2	0.862574E+00	0.862575E+00	1.51133E-04	6.8117E-07
0.3	0.841561E+00	0.841573E+00	6.36444E-04	1.1437E-05
0.4	0.850967E+00	0.850999E+00	1.67567E-03	3.2280E-05
0.5	0.888343E+00	0.888407E+00	3.56770E-03	6.3417E-05
0.6	0.950606E+00	0.950711E+00	6.410875E-03	1.0622E-04
0.7	0.103439E+01	0.103456E+01	1.071642E-02	1.6663E-04
0.8	0.113640E+01	0.113667E+01	1.682213E-02	2.6436E-04
0.9	0.125367E+01	0.125412E+01	2.520604E-02	4.5406E-04
1.0	0.138377E+01	0.138464E+01	3.644104E-02	8.6831E-04

Table 5: The result of test problem 4

X -value	Exact Solution	New result	Error in [4] method	Error in our
0.1	4.98752E-003	4.98752E-003	-	1.1899E-11
0.2	1.98011E-002	198011E-002	4.4881E-06	3.0422E-09
0.3	4.39996E-002	4.39997E-002	-	7.7796E-08
0.4	7.68675E-002	7.68677E-001	4.0676E-06	2.2012E-07
0.5	1.174433E-001	1.174437E-001	-	3.9702E-07
0.6	1.645579E-001	1.645583E-001	1.3761E-05	4.2896E-07
0.7	2.16881E-001	2.168801E-001	-	3.5404E-07
0.8	2.72975E-001	2.729709E-001	3.0465E-05	4.0038E-06
0.9	3.31350E-001	3.313340E-001	-	1.5907E-06
1.0	3.905280E-001	3.904788E-001	5.2651E-05	4.8637E-05

Table 6: The result of test problem 5

X-value	Exact solution [17]	New result (p=6)
0.0	1.000000000	1.000000000
0.2	1.221211030	1.221208625
0.4	1.488834893	1.488757726
0.6	1.807361404	1.806566459
0.8	2.179819234	2.175469744
1.0	2.608275822	2.589955860

Table 5: The result of test problem 5 at $x \in [0, 0.2, 0.4, 0.6, 0.8, 1.0], h = 0.1$

Table7 : The result of test problem 6

X-value	Result for n= $5/4, \lambda = -2$ in Fazal-i-Haget et al[18]	New result for p=6	Errors in (p=6)
0.1	0.99017	0.990163	7.0000E-06
0.2	0.96134	0.961271	6.9000E-05
0.3	0.91455	0.914196	3.5400E-04
0.4	0.85089	0.849739	1.1510E-03
0.5	0.77151	0.769127	2.3830E-03
0.6	0.67774	0.674275	3.4650E-03
0.7	0.57106	0.567172	3.8880E-03
0.8	0.45331	0.450101	2.2300E-03
0.9	0.32684	0.325340	1.5000E-03
1.0	0.19505	0.195306	2.5000E-04

5.0 Conclusion

We have developed and implemented ak-step Linear multistep method (LMM) using Taylor series method.A new scheme is obtained which applied tosolve some special and general third order initial value problem in ordinary differential equation.Some application problems have also been solved. Evidence of the better accuracy of method over existing methods as mentioned in [4], [14-15] are given in Tables 2 ,3,4 and 5 respectively.

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