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Effect of Small Perturbation on Propagation of Gravity Waves

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Abstract

The propagation of gravity waves along the common surface of two superposed liquids is studied. Using a selection of Cartesian axes, we analyze the effect of a small perturbation of steady flow.

1.0 Introduction

Gravity waves (gravitational waves) are discrete forces radiated by gravity and are essentially waves such as on the surface of water where gravity provides the restoring force [1]. It is the main restoring force in a system, with gravitational energy making up more than 95% of the energy in long-period waves in the ocean, the rest being compressional energy in the slightly compressible water and compressional and shear energy in the underlying rock.

This model considers the propagation of gravity waves along the common surface of two superposed liquids where we select coordinate axes with the origin in the plane equilibrium surface of the liquids, the y-axis vertical and the xz-plane horizontal. We choose liquids of densities ρ_1 and ρ_2 occupying the regions $y\in(0, h_1)$ and $y\in(-h_2, 0)$ respectively with $y=h_1$ and $y=-h_2$ being the rigid boundaries for the undisturbed flow. We further suppose that the liquids are flowing uniformly with speeds u_1 and u_2 in the direction of the x-axis.

For our typical fluid, the kinematical state is defined by the density field $\rho(\mathbf{r}, t)$ and the velocity field $\mathbf{q}(\mathbf{r}, t)$, where $\mathbf{r}$ is the position vector of a particle, say P, moving with the fluid [2]. So, in this consideration, we assume the liquids to be incompressible fluids. In conservation of mass, the condition is that, the rate of increase of the mass of the fluid in any region, say D, bounded by a closed surface, say S, must equal the rate at which fluid is flowing inwards through S, i.e.

$$\frac{d}{dt}\int_D \rho \, dV = -\int_S \rho \, \mathbf{q} . d\mathbf{S} \tag{1}$$

where $d\mathbf{S}$ has the direction of the outward normal. Assuming that (1) holds for every region and applying the divergence theorem, we obtain the continuity equation:

$$\frac{\partial \rho}{\partial t} + \mathbf{q} . \nabla \rho + \rho \, div \, \mathbf{q} = 0 \tag{2}$$

From the definition of the kinematical state of the fluid, as P moves from the position $\mathbf{r}$ at time t to the position $\mathbf{r}+\delta\mathbf{r}$ at time $t+\delta t$, the velocity increases by $\mathbf{q}(\mathbf{r}+\delta\mathbf{r}, t+\delta t) - \mathbf{q}(\mathbf{r}, t)$ to yield the acceleration of P as

$$\mathbf{f} = \frac{\partial \mathbf{q}}{\partial t} + \frac{dr}{dt}.\nabla \mathbf{q} = \frac{\partial \mathbf{q}}{\partial t} + \mathbf{q}.\nabla \mathbf{q} \tag{3}$$

Since the equation of motion holds for every region, we can re-write the Euler equation of motion as

$$\frac{\partial \mathbf{q}}{\partial t} + \mathbf{q}.\nabla \mathbf{q} + \frac{1}{\rho}\nabla p + \nabla w = 0 \tag{4a}$$

This is ordinarily based on the following assumptions about the system of forces acting on the fluid:

i. Dissipation of energy as a result of viscosity, heat transfer, etc. may be neglected so that there are no shear stresses and so the internal stress can be derived from a single scalar field, the pressure $p(\mathbf{r},t)$;

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ii. We derive the external forces from the potential energy per unit mass which is a scalar field [3]. Denote this field by $W(\mathbf{r})$ and the equation of motion of the fluid in the region D is therefore

$$\int_D \rho \mathbf{f}\, dV = -\int_D \rho \nabla W dV - \int_S p d\mathbf{S} \tag{4b}$$

The right side of (4b) can be transformed to $\int_D \nabla\, pdV$.

3.0. Governing equations of liquid motions under gravity and their specifications

Normally, in considering the propagation of waves in a liquid, it is assumed that the liquid is an incompressible fluid [4]. Following this assumption and with a few approximations for small motions of the fluids, we can re-write (2) as

$$\frac{\partial \rho}{\partial t} + \rho\, div\ \mathbf{q} = 0 \tag{5}$$

and (4a) as

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{1}{\rho}\nabla p + \nabla w = 0 \tag{6}$$

In (5) and (6), the velocity is small compared with the instantaneous local speed of propagation of changes in any property of the fluid e.g., $|\mathbf{q}| << |\partial\rho/\partial t| / |\nabla\rho|$ except near points where $\partial\rho/\partial t = 0$ and $\nabla\rho \neq 0$ even though such regions can be assumed to be very small to produce appreciable effects on the solution of the differential equations.

Linearizing (4) with

$$|q.\nabla q| << |q_t| \tag{7a}$$

and taking a constant value of ρ in (2) and (4a), (5) now becomes

$$div\ \mathbf{q} = 0 \tag{7b}$$

and (6) becomes $\frac{\partial \mathbf{q}}{\partial t} + \nabla\left(\frac{p}{\rho} + w\right) = 0$.

Our liquids are in continuous, non-turbulent motion and therefore the boundary surface always consist of the same particles of liquid. Let

$$F(\mathbf{r}, t) = 0 \tag{7c}$$

be such a surface. Then, following the motion of the liquid particles,

$$\frac{\partial F}{\partial t} + \mathbf{v}.\nabla F = 0 \tag{8}$$

Now, at the free surface of the liquid, the surface tension effects produce a discontinuity in the pressure across the surface, i.e.

$$p_1 = p_o - T\left(\frac{R_1 + R_2}{R_1 R_2}\right) \tag{9a}$$

where p_1= value of the pressure inside the surface;
p_o = value of the pressure outside the surface;
T = constant surface tension
R_1, R_2 = principal radii of curvature of the surface.

We now select Cartesian axes with the origin in the plane equilibrium surface of the liquid, the *y*-axis vertical and the *xz*-plane horizontal and denote the elevation of the liquid above the point $(x,0,z)$ by $\eta(x,z,t)$ to produce the equation of the free surface as

$$\eta(x,z,t) - y = 0 \tag{9b}$$

In the same way, the equation of the lower boundary over which the liquid moves is given by

$$h(x,z,t) + y = 0 \tag{9c}$$

where $h(x,z,t)$ is the depth of the liquid in the equilibrium state.

At the free surface, $y = \eta$, so (8) gives

$$\nabla\phi.\nabla\eta - \phi_y = \eta_t \tag{10}$$

while at the lower boundary where *y = -h* we have

$$\nabla\phi.\nabla h + \phi_y = -h_t \tag{11}$$

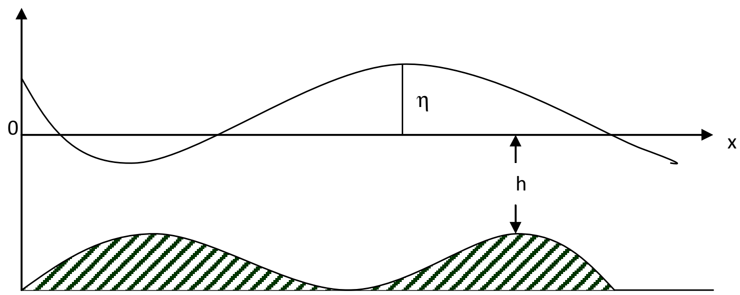

Fig. 1: Two superposed liquids on a common boundary

4.0 Boundary Conditions and their Linearization

The equation at the free surface is

$$\eta(x,z,t) - y = 0 \tag{12a}$$

and at the lower boundary over which the liquid moves is

$$h(x,z,t) + y = 0 \tag{12b}$$

For small motions of the boundaries, if

1. The boundaries suffer only small departures from the horizontal [5, 6], i.e. $|\nabla\eta| << 1$ and $|\nabla h| << 1$, then we may neglect the terms $\nabla\phi.\nabla\eta$ and $\nabla\phi.\nabla h$ in (10) and (11). And in (9a), the factor $1/R_1 + 1/R_2$ may then be replaced by $\frac{\partial^2\eta}{\partial x^2} + \frac{\partial^2\eta}{\partial z^2}$;

2. $\eta << \left(\frac{\phi_y}{\phi_{yy}}\right)_{y=0}$ and as $\phi_y(x,\eta,z,t) \cong \phi_y(x,0,z,t) + \eta\phi_{yy}(x,\eta,z,t)$, the kinematical condition (10) reduces to $\phi_y = \eta_t$ on $y = 0$, i.e. the elevation η is small enough to neglect changes in the vertical component of the velocity **v** between the free surface and the equilibrium plane. In the same way, (11) reduces to $\phi_y = -h_t$ on $y = -h_0$ if $|h - h_0| << \left(\frac{\phi_y}{\phi_{yy}}\right)_{y=-h_0}$ (h_0 is the mean depth of the fluid)

5.0. Discussion and analysis of the Governing equations

In considering small perturbations of this steady flow, we let the velocity potential be

$$\left.\begin{aligned}\phi &= \phi_1\\ \phi &= \phi_2\end{aligned}\right\} \begin{aligned} y &> 0\\ y &< 0\end{aligned} \tag{12c}$$

and we let the interfaces have form

$$y = \eta(x,t) \tag{12d}$$

as in Fig. 2

In the unperturbed state,

$$\begin{aligned}\phi_1 &= u_1 x;\\ \phi_2 &= u_2 x\\ \eta &= 0\end{aligned} \tag{12e}$$

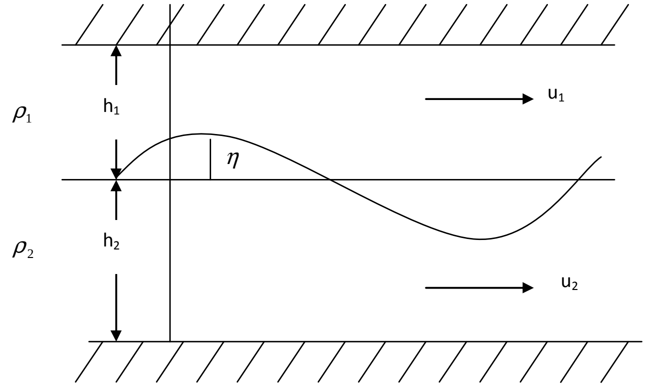

Fig. 2: Two superposed liquids at the interface

Since the perturbations of ϕ and η are small enough to satisfy the assumptions on the boundary conditions and their linearization above, then, we shall require functions $\phi(x,y,t)$ and $\eta(x,t)$ [7], satisfying

$$\phi_{xx} + \phi_{yy} = 0$$
$$\phi_y(x, h_1, t) = 0 \qquad (12f)$$
$$\phi_y(x, -h_2, t) = 0$$

At the interface $\eta - y = 0,$ so applying (8) gives

$$\frac{\partial \eta}{\partial t} + u_1 \frac{\partial \eta}{\partial x} = \frac{\partial \phi_1}{\partial y}$$
$$and$$
$$\frac{\partial \eta}{\partial t} + u_2 \frac{\partial \eta}{\partial x} = \frac{\partial \phi_2}{\partial y} \qquad (12g)$$

on $y = 0$.

Also, when the velocity is approximately u in the x-direction, eqn. (4a) takes the form

$$q_t + uq_x + \frac{1}{\rho}\nabla p + \nabla W = 0 \qquad (12h)$$

so that

$$\varphi_t + u\varphi_x + \frac{1}{\rho} p + W = C \qquad C \text{ is a constant} \qquad (12i)$$

Neglecting surface tension, then we have a continuous pressure across the interface [8, 9], and so on $y=0$,

$$\rho_1\left(\frac{\partial \varphi_1}{\partial t} + u_1 \frac{\partial \varphi_1}{\partial x} + g\eta\right) = \rho_2\left(\frac{\partial \varphi_2}{\partial t} + u_2 \frac{\partial \varphi_2}{\partial x} + g\eta\right) \qquad (12j)$$

for which we seek harmonic wave solutions

$$\varphi_1 = u_1 x + A_1 \cosh k(y - h_1) \exp i(kx - \omega t)$$
$$\varphi_2 = u_2 x + A_2 \cosh k(y + h_2) \exp i(kx - \omega t) \qquad (12k)$$
$$\eta = ia \exp i(kx - \omega t)$$

We immediately obtain the dispersion relation by substituting (12j) into (12k) and eliminating A_1, A_2, and a:

$$kg(\rho_2 - \rho_1) = \rho_1(u_1 k - \omega)^2 \coth kh_1 + \rho_2(u_2 k - \omega)^2 \coth kh_2 \qquad (13)$$

For short waves,

$$kh_1 >> 1 \text{ and } kh_2 >> 1$$

and so, (13) reduces to:

$$kg(\rho_2 - \rho_1) = \rho_1(u_1 k - \omega)^2 + \rho_2(u_2 k - \omega)^2 \qquad (14)$$

With (13) and (14), we can then introduce the reference frame in which the total momentum is zero, which moves with velocity $(\rho_1 u_1 + \rho_2 u_2)/\rho_1 + \rho_2$ [10]. The stream velocities with respect to this frame are:

$$U_1 = -\frac{\rho_2(u_2 - u_1)}{\rho_1 + \rho_2}$$
$$U_2 = -\frac{\rho_1(u_2 - u_1)}{\rho_1 + \rho_2} \qquad (15)$$

and so (14) reduces to $(\rho_1 + \rho_2)^2 \omega^2 + \rho_1\rho_2(u_2 - u_1)^2 k^2 = (\rho_2^2 - \rho_1^2) gk \qquad (16)$

The dispersion curve which results is an ellipse as shown in Fig. 3.

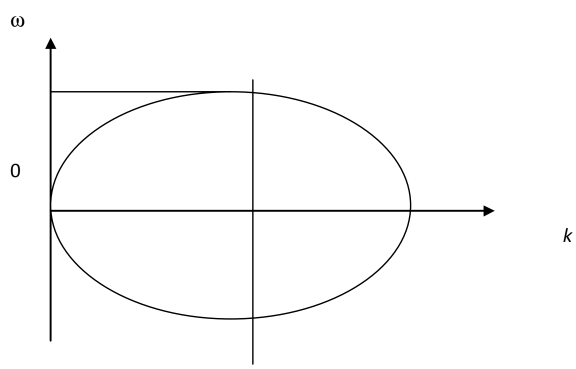

Fig. 3: The dispersion curve from two superposed liquid

6.0 Conclusion

When gravity waves are propagated along the common surface of two superposed liquids, the system will only transmit undistorted, unattenuated harmonic waves when ω and k satisfy

$$|\omega| < \frac{\rho_2 - \rho_1}{2\sqrt{(\rho_1\rho_2)}} \frac{g}{u_2 - u_1} \qquad (17)$$

and

$$0 \le k \le \frac{\rho_2^2 - \rho_1^2}{\rho_1\rho_2} \frac{g}{(u_2 - u_1)^2} \qquad (18)$$

When $k > \frac{\rho_2^2 - \rho_1^2}{\rho_1\rho_2} \frac{g}{(u_2 - u_1)^2}$, ω is pure imaginary and the time dependence of η is exponential, and as such, there can be no oscillations.

k has the form

$$k = \tfrac{1}{2}\left(\frac{\rho_2^2 - \rho_1^2}{\rho_1\rho_2} \frac{g}{(u_2 - u_1)^2}\right) \pm i\mu \qquad (19)$$

when $\omega > \frac{\rho_2 - \rho_1}{2\sqrt{(\rho_1\rho_2)}} \frac{g}{u_2 - u_1}$ so that the waves may be attenuated or may increase and become unstable which can then be treated as a non-linear problem.

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