*Journal of the Nigerian Association of Mathematical Physics*
***Volume28, No. 1,* (*November, 2014*), pp 63 – 68**
***© J. of NAMP***

# Series Solution of Mixed Convection Laminar Flow

***S.A Aliu[1] and R.O Fagbenle[2]***

***[1]Department of Mechanical Engineering, University of Benin, Benin City, Nigeria***
***[2]Department of Mechanical Engineering, University of Ibadan, Ibadan, Nigeria***

***Abstract***

***The boundary layer equations for the mixed forced and free convection laminar flow are transformed into dimensionless coordinates. Series solution of transformed equations is sought resulting in a hierarchy of ordinary differential equations of the universal functions. A scheme comprising of fourth order Runge-Kutta, parameter differentiation and shooting methods was devised to obtain wall derivatives of the universal functions.Using the wall derivatives, local skin friction and Nusselt numbers (Nu) can be estimated for different values of Prandtl number (Pr), pressure gradient (Λ) and shape parameter (ω) for flow over different shapes.***

**Keywords:** Series solution, Convection, Laminar Flow, Runge-Kutta, dimensionless coordinates

## 1.0 Introduction

The local similarity method of Merk [1] and Meksyn [2] was found by Fagbenle [3] to be an accurate and simple method of analyzing momentum and heat transfer in forced convection laminar boundary layer flows once the associated universal functions have been obtained. Since Chao and Fagbenle [4] set forth the corrected form of the series resulting from the local similarity " wedge parameter" method of Merk [1], the method has been successfully applied to mixed convection laminar flow by Cameron et al. [5] and Meissner et al. [6]. Kim et al. [7] also applied the method to the power law fluids in forced convection flows while Chang et al, [8] applied it to pure natural convection flow of power law fluids. The purpose of this paper is to demonstrate that simple and familiar parameters could be employed in the Merk series for the mixed convection laminar flow over two-dimensional or rotationally symmetric shapes

## 2.0 Basic Equations

Consideration is given to the steady, incompressible, laminar, constant property mixed convective boundary layer flow over two dimensional or axisymmetric bodies[9]

Continuity: $$\frac{\partial(ru)}{\partial x} + \frac{\partial(rv)}{\partial y} = 0 \qquad (1)$$

Momentum: $$\frac{u\partial u}{\partial x} + \frac{v\partial u}{\partial y} = \frac{Udu}{dx} + g\gamma\beta(T - T_\infty) + \frac{\nu d^2u}{dy^2} \qquad (2)$$

Energy: $$\frac{u\partial T}{\partial x} + \frac{v\partial T}{\partial x} = \frac{\kappa\partial^2 T}{\partial y^2} \qquad (3)$$

with the associated boundary conditions

$u(x,0) = v(x,0) = 0$; $T(x,0) = T_w$ (constant) (4)

$u(x,\infty) = U(x)$; $T(x,\infty) = T_\infty$ (constant)

The (x, y) coordinates are now transformed into (ξ, η) dimensionless coordinates in the manner of Meksyn [2] as follows:

$$\xi = \int \frac{U}{U_\infty}\frac{r^2}{L^2}\frac{dx}{L}$$

$$\eta = \left[\frac{\mathrm{Re}}{2\xi}\right]^{1/2}\frac{U}{U_\infty}\frac{r}{L}\frac{y}{L} \qquad (5)$$

Corresponding author: ***S.A Aliu***, E-mail:-, Tel.: +2348022104691

where L is the characteristic length of the body.
Re = $U_\infty$ L/ν. A dimensionless stream function f (ξ,η) is introduced as follows:

$$\psi(x,y) = U_\infty L \left(\frac{2\xi}{\mathrm{Re}}\right)^{1/2} f(\xi,\eta) \tag{6}$$

The continuity equation (1) is identically satisfied.
By defining a dimensionless temperature θ (ξ,η):

$$T(x, y) - T_w = (T_\infty - T_w)\,\theta(\xi,\eta) \tag{7}$$

Equations (2) and (3) transform into

$$f''' + ff'' + \Lambda\left[1-(f')^2\right] = 2\xi\partial(f', f)/\partial(\xi,\eta) - \left[\zeta\{Gr/\mathrm{Re}^2\}\right](1-\theta) \tag{8}$$

$$\theta'' + \Pr f\theta' = 2\xi \Pr \left[\partial(\theta, f)/\partial(\xi,\eta)\right] \tag{9}$$

with the associated boundary conditions

$$f(\xi,0) = f'(\xi,0) = 0;\ \theta(\xi,0) = 0$$

(10)

$$f'(\xi,\infty) \to 1;\quad \theta(\xi,\infty) \to 1$$

In equations (8) and (9), the following definitions hold;

$\Lambda = \dfrac{2\xi}{U}\dfrac{dU}{d\xi}$, the wedge angle or the pressure gradient parameter

Gr = $g\beta(T_w - T_\infty)\dfrac{L^3}{\nu^2}$, the Grashof number

$\zeta = \dfrac{2\xi\gamma}{\left(\dfrac{U}{U_\infty}\right)^3\left(\dfrac{\gamma}{2}\right)^2}$, a modified dimensionless coordinate.

Pr = ν/κ, the Prandtl number.
The primes denote differentiation with respect to η.

$\dfrac{\partial(f', f)}{\partial(\xi,\eta)}$ and $\dfrac{\partial(\theta, f)}{\partial(\xi,\eta)}$ are Jacobians which were differentiated by parameter differentiation[10].

The velocity components are given by

$$u = \frac{L}{r}\frac{\partial\psi}{\partial y} = Uf'$$

$$v = -\frac{L}{r}\frac{\partial\psi}{\partial x} = -\frac{L}{r}U(2\xi\,\mathrm{Re})^{-1/2}\left[f + 2\xi\frac{\partial f}{\partial\xi} + \left(\Lambda + \frac{2\xi}{r}\frac{dr}{d\xi} - 1\right)\eta f'\right]$$

## 3.0 Solution Procedure

Define the perturbation parameters

$$\varepsilon_1 = \zeta\frac{Gr}{\mathrm{Re}^2} \ and \ \varepsilon_2 = 2\xi\frac{d\Lambda}{d\xi}$$

In view of the form of Equation (8), we seek series solutions of the form:

$$f(\xi,\eta) = f_0(\Lambda,\eta) + \varepsilon_1 f_1(\Lambda,\eta) + \varepsilon_2 f_2(\Lambda,\eta) + \varepsilon_1^2 f_3(\Lambda,\eta) + 4\xi^2\left(\frac{d^2\Lambda}{d\xi^2}\right) f_4(\Lambda,\eta) + \tag{11}$$

$$\varepsilon_2^2 f_5(\Lambda,\eta) + \varepsilon_1\varepsilon_2(\Lambda,\eta) + \ldots\ldots\ldots\ldots$$

$$\theta(\xi,\eta)=\theta_0(\Lambda,\eta)+\varepsilon_1\theta_1(\Lambda,\eta)+\varepsilon_2\theta_2(\Lambda,\eta)+\varepsilon_1^2\theta_3(\Lambda,\eta)+4\xi^2\left(\frac{d^2\Lambda}{d\xi^2}\right)\theta_4(\Lambda,\eta) \tag{12}$$

$$+\varepsilon_2^2\theta_5(\Lambda,\eta)+\varepsilon_1\varepsilon_2\theta_6(\Lambda,\eta)+\ldots\ldots\ldots\ldots\ldots\ldots$$

Substituting (11) and (12) into (8) and (9) yields the following hierarchy of differential equations for the first five terms:

$$f_0'''+f_0 f_0+\Lambda\left[1-(f_0')^2\right]=0 \tag{13a}$$

$$f_1'''+f_0 f_1''+(2-\Lambda-2\omega)\,f_0'f_1'+(3-3\Lambda-3\omega)\,f_0''f_1=\theta_0-1 \tag{13b}$$

$$f_2'''+f_0 f_2''-(2\Lambda+2)\,f_0'f_2'+3f_0''f_2=\partial(f_0',f_0)/\partial(\Lambda,\eta) \tag{13c}$$

$$f_3'''+f_0 f_3''+(4-4\Lambda-4\omega)\,f_0'f_3'+ \tag{13d}$$
$$(5-6\Lambda-4\omega)\,f_0''f_3=(2-2\Lambda-2\omega)(f_1)^2-(3-3\Lambda-2\omega)f_1 f_1''+\theta_1$$

$$f_4'''+f_0 f_4''+(2\Lambda+4)\,f_0'f_4'+4f_0''f_4=2\left(f_0'f_2'-f_0''f_2\right) \tag{13e}$$

$$\theta_0''+\Pr\,f_0\theta_0'=0 \tag{14a}$$

$$\theta_1''+\Pr f_0\theta_1'-(2-3\Lambda-2\omega)\Pr f_0'\theta_1=-(3-3\Lambda-2\omega)\Pr f_1\theta_0' \tag{14b}$$

$$\theta_2''+\Pr\,f_0\theta_2'-2\Pr\,f_0'\theta_2=\Pr\partial(\theta_0,f_0)/\partial(\Lambda,\eta)-3\Pr\,f_2\theta_0' \tag{14c}$$

$$\theta_3''+\Pr f_0\theta_3'-(4-6\Lambda-4\omega)\Pr f_0'\theta_3=\Pr[(2-3\Lambda-2\omega)f_1'\theta_1 \tag{14d}$$
$$-(3-3\Lambda-2\omega)f_1\theta_1-(5-6\Lambda-4\omega)f_3\theta_0'$$

$$\theta_4''+\Pr f_0\theta_4'-4\Pr f_0'\theta_4=\Pr\left[f_0'\theta_2-f_2\theta_0'-5f_4\theta_0'\right] \tag{14e}$$

Eqns. 13 and 14 (a), (c) and (e) above show a non-dependence on θ and ω.

In these equations, $\omega=\left(\frac{2\xi}{r}\right)\left(\frac{dr}{d\xi}\right)$ and for two dimensional flows, ω = 0. The associated boundary conditions for the above equations are:

$$f_i(\Lambda,0)=f_i'(\Lambda,0)=0;\ \theta_i(\Lambda,0)=0,\ i\geq 0$$

$$f_0'(\Lambda,\infty)\to 1,\ \theta_0(\Lambda,\infty)\to 1 \tag{15}$$

$$f_i(\Lambda,\infty)\to 0,\ \theta_i(\Lambda,\infty)\to 0,\ i>0$$

Solutions of the set of simultaneous differential equations for the $f_i$'s and $\theta_i$'s subject to the boundary conditions (15) were obtained using a fourth-order RungeKutta integration procedure. The roots were arrived at using shooting methods and Jacobiansin Equations (13c) and (14c) were evaluated by using the technique of parameter differentiation. Computer programs (details in [11])were used to calculate wall derivatives of the dimensionless universal velocity functions $f_i''$ and the temperature functions $\theta_i{}'$, (i = 0, 1, 2, 3, 4) for values of $\Lambda$ ranging from -0.15 to 1.0 for different Prandtl numbers and as well for $\omega$ in the case of rotationally symmetric flows.

## 4.0 Results

The functions $f_i$ and $\theta_i$ are regarded as universal functions and useful wall derivatives of these functions can be tabulated for various values of $\Lambda$ and $\omega$.Table 1 shows the tabulation of $f_o'', f_2''\ and\ f_4''$ with respect to $\Lambda$ since they are independent of Prandtl number whilst Tables 2 and 3 presents the tabulation of $f_1''and\ f_3''$ with respect to both $\Lambda$ and Pr respectively. Table 4 presents the tabulation of $\theta_i{}'$, (i = 0, 1, 2, 3, 4) for $\Lambda$ and Pr = 0.7.

**Table 1**.Wall derivatives of universal velocity functions

| $\Lambda$ | $f_0''$ | $f_2''$ | $f_4''$ |
|---|---|---|---|
| -0.15 | 0.2164 | -0.3480 | 0.0803 |
| -0.10 | 0.3193 | -0.2226 | 0.0529 |
| -0.05 | 0.4003 | -0.1666 | 0.0394 |
| 0.00 | 0.4696 | -0.1333 | 0.0310 |
| 0.10 | 0.5870 | -0.0946 | 0.0211 |
| 0.20 | 0.6867 | -0.0723 | 0.0154 |
| 0.30 | 0.7748 | -0.0579 | 0.0118 |
| 0.40 | 0.8544 | -0.0479 | 0.0093 |
| 0.50 | 0.9277 | -0.0394 | 0.0070 |
| 0.60 | 0.9958 | -0.0348 | 0.0062 |
| 0.70 | 1.0598 | -0.0304 | 0.0052 |
| 0.80 | 1.1203 | -0.0268 | 0.0044 |
| 0.90 | 1.1777 | -0.0239 | 0.0038 |
| 1.00 | 1.2326 | -0.0215 | 0.0033 |

**Table 2**. Wall derivatives of universal velocity functions $f_1''$ **($\Lambda$,0)**

| | Prandtl No | | | |
|---|---|---|---|---|
| $\Lambda$ | 0.7 | 1.0 | 7.0 | 10 |
| - -0- | | | | |
| -0.15 | 1.2543 | 1.1762 | 0.7579 | 0.6886 |
| -0.10 | 1.0298 | 0.9965 | 0.6269 | 0.5706 |
| -- | | | | |
| 0.05 | 0.9269 | 0.8700 | 0.5657 | 0.5153 |
| 0.00 | 0.8639 | 0.8108 | 0.5277 | 0.4808 |
| 0.10 | 0.7869 | 0.7383 | 0.4803 | 0.4379 |
| 0.20 | 0.7399 | 0.6938 | 0.4507 | 0.4109 |
| 0.30 | 0.7074 | 0.6629 | 0.4296 | 0.3917 |
| 0.40 | 0.6835 | 0.6400 | 0.4136 | 0.3770 |
| 0.50 | 0.6651 | 0.6222 | 0.4009 | 0.3654 |
| 0.60 | 0.6505 | 0.6080 | 0.3904 | 0.3557 |
| 0.70 | 0.6388 | 0.5964 | 0.3816 | 0.3476 |
| 0.80 | 0.6294 | 0.5869 | 0.3741 | 0.3406 |
| 0.90 | 0.6217 | 0.5792 | 0.3676 | 0.3345 |
| 1.00 | 0.6158 | 0.5730 | 0.3624 | 0.3293 |

**Table 3**.Wall derivatives of universal velocity functions $f_3''$ ($\Lambda$, 0)

| $\Lambda$ | 0.7 | 1.0 | 7.0 | 10 |
|---|---|---|---|---|
| -0.15 | -2.3038 | -2.0105 | -0.8040 | -0.6589 |
| -0.10 | -0.8860 | -0.7783 | -0.3270 | -0.2710 |
| -0.05 | -0.5254 | -0.4630 | -0.1995 | -0.1662 |
| 0.00 | -0.3698 | -0.3263 | -0.1423 | -0.1189 |
| 0.10 | -0.2327 | -0.2051 | -0.0899 | -0.0753 |
| 0.20 | -0.1720 | -0.1510 | -0.6550 | -0.0549 |
| 0.30 | -0.1384 | -0.1207 | -0.5120 | -0.0429 |
| 0.40 | -0.1171 | -0.1013 | -0.4180 | -0.0349 |
| 0.50 | -0.1022 | -0.0874 | -0.3480 | -0.0289 |
| 0.60 | -0.0905 | -0.0765 | -0.2910 | -0.0241 |
| 0.70 | -0.0795 | -0.0662 | -0.0240 | -0.0198 |
| 0.80 | -0.0667 | -0.0542 | -0.0186 | -0.0153 |
| 0.90 | -0.0455 | -0.0366 | -0.0117 | -0.0096 |
| 1.00 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |

**Table 4**.Wall derivatives of universal temperature functions Pr = 0.7

| $\Lambda$ | $\theta_0'$ | $\theta_1'$ | $\theta_2'$ | $\theta_3'$ | $\theta_4'$ |
|---|---|---|---|---|---|
| -0.15 | 0.3644 | 0.3852 | -0.0143 | -1.4822 | 0.0105 |
| -0.10 | 0.3870 | 0.2592 | -0.0171 | -0.5189 | 0.0058 |
| -0.05 | 0.4022 | 0.2055 | 0.0029 | -0.2876 | 0.0034 |
| 0.00 | 0.4139 | 0.1742 | 0.0051 | -0.1918 | 0.0020 |
| 0.10 | 0.4314 | 0.1376 | 0.0069 | -0.1110 | 0.0005 |
| 0.20 | 0.4444 | 0.1156 | 0.0074 | -0.0767 | -0.0002 |
| 0.30 | 0.4547 | 0.1002 | 0.0075 | -0.0581 | -0.0006 |
| 0.40 | 0.4632 | 0.0881 | 0.0073 | -0.0463 | -0.0008 |
| 0.50 | 0.4705 | 0.0775 | 0.0074 | -0.0375 | -0.0010 |
| 0.60 | 0.4768 | 0.0675 | 0.0068 | -0.0297 | -0.0009 |
| 0.70 | 0.4824 | 0.0569 | 0.0066 | -0.0208 | -0.0009 |
| 0.80 | 0.4873 | 0.0445 | 0.0063 | -0.0065 | -0.0009 |
| 0.90 | 0.4918 | 0.0276 | 0.0060 | 0.0343 | -0.0009 |
| 1.00 | 0.4959 | 0.0000 | 0.0058 | 0.0000 | -0.0009 |

More results, including those for rotationally symmetric shapes which depend on ω are contained in [11]. These results can be used to determine the local skin friction and Nusselt numbers (Nu) for different values of Prandtl number, pressure gradient and shape parameter for the study geometries of flat plate, wedge shape, horizontal cylinderand sphereas can be foundin [4,9,11 and12].

## 5.0 Conclusion

The corrected Merk series of Chao and Fagbenle [4] has been applied to the mixed convection flow problem for the general two-dimensional or rotationally symmetric boundary layer flow. The resulting universal functions have been obtained for the first five terms of the series in the case of two-dimensional and rotationally symmetric flows. The method has been applied with good results to the mixed convection flow over flat plate, wedge shape, horizontal cylinder and sphere. The first three terms being found to be sufficient in most cases.

## References

[1] Merk, H.J., (1959) Rapid calculations for boundary-layer transfer using wedge solutions and asymptotic expansions, J. Fluid Mech, Vol. 5, pp. 460-480

[2] Meksyn, D., (1961) New Methods in Laminar Boundary-Layer Theory, Pergamon Press, Oxford, Chapters 6, 7 and 9

[3] Fagbenle, R.O., (1973) PhD Thesis, Department of Mechanical Engineering, University of Illinois, Urbana Champaign.

[4] Chao, B.T., and Fagbenle, R.O., (1974) On Merk's methods of calculating boundary layer transfer, Int. J. Heat Mass Transfer, Vol. 17, pp. 223-240

[5] Cameron, M.R., Jeng, D.R., and Dewitt, K.J., (1991) Mixed convection from two dimensional and axisymmetric bodies of arbitrary contour, Trends in Heat, Mass & Momentum Transfer, Vol.1, pp. 1-15

[6] Meissner, D.L., Jeng, D.R., and Dewitt, K.J., (1994) Mixed convection to power law fluids from two dimensional or axisymmetric bodies, Int. J. Heat Mass Transfer, Vol. 37, No 10, pp. 1475-1485

[7] Kim, H.W., Jeng, D.R., and Dewitt, K.J., (1983) Momentum and heat transfer in power law fluid flow over two-dimensional or axisymmetric bodies, Int. J. Heat Mass Transfer, Vol. 26, pp.245-258

[8] Chang, T.A., Jeng, D.R., and Dewitt, K.J.,(1988) Natural convection to power law fluids from two dimensional or axisymmetric bodies of arbitrary contour, Int. J. Heat Mass Transfer, Vol. 31, No 3, pp. 615-624

[9] Aliu, S.Aand Fagbenle, R.O (2013)"Laminar mixed convection over two dimensional and axysymmetric bodies. Proceedings of the American Society of Mechanical Engineering (ASME) International Mechanical Engineering Congress & Exposition, IMECE 2013, San Diego, USA. 15-21 November 2013 Paper No. IMECE2013-63447, 12pp.

[10] Na, T.Y.(1979).*Computational methods in engineering boundary value problems.* New York:Academic Press.

[11] Aliu, S, A. 2014. Laminar mixed convection boundary layer flow over twodimensional and axisymmetric bodies. Thesis. Mechanical Engineering, Technology. University of Ibadan.

[12] Falana, A. 2013.Forced convection thermal boundary layer transfer for non isothermal surfacesusing the modified Merk series. Thesis. Mechanical Engineering, Technology. University of Ibadan.