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A Modified Armijo – Type Strategy for Solving Nonlinear Systems of Equations

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Abstract

The prominent method for eliminating the well known shortcomings of Newton's method for solving systems of nonlinear equations is Chord Newton's method. Nevertheless, the method mostly requires high number of iterations and the convergence will even be lost especially as the dimension of the system increases. This paper, proposed a new line search strategy, by using the steps of backtracking in the Armijo – type. The anticipation has been to improve the overall performance of Chord Newton's. Our scheme was implemented on some benchmark nonlinear systems which show that, the proposed approach is very encouraging compared to some other Newton's – like methods.

Keywords: Newton's method, systems, multi-points, Fixed Newton's, Chord Newton's

1.0 Introduction

Let us consider the problem

$$F(x) = 0, \qquad (1)$$

with $F : R^n \to R^n$.

Where the mapping F is assumed to satisfy the following standard assumptions:

1. F is continuously differentiable in an open convex set Φ;
2. There exists a solution x^* of (1) in Φ such that $F(x^*) = 0$ and $F'(x^*) \neq 0$;
3. The Jacobian $F'(x)$ is locally Lipschitz continuous at x^*.

The most widely used iterative method in solving such problems is Newton's method [1]; yet in recent past many other promising iterative methods for handling nonlinear equations have appeared in the literature via Taylor's series, interpolating polynomials, etc. The attractive features of Newton's method are that it is simple to implement and it converges rapidly. Newton's method produces an iterative sequence $\{x_k\}$ from any given initial guess x_0 in the neighbourhood of solution, through the following steps:

Algorithm 1(Newton's method)

For $k = 0,1,2,\ldots$ where $F'(x_k)$ is the Jacobian matrix of F:

Step 1: Solve $F'(x_k)s_k = -F(x_k)$

Step 2: Update $x_{k+1} = x_k + s_k$

where s_k is the Newton correction.

Nevertheless, Newton's method has some numerous shortcomings, which attract the attention of many researchers over time. This includes; the computation and storage of Jacobian matrix as well as solving n linear equations in each iteration. It is well known that, Jacobian computation entails first-order derivatives of the systems, but some functions derivatives are

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quite costly and sometimes not available or could not be done precisely. In this case Newton's method cannot be used directly. Therefore, to do away with this crucial issue some effort need to surface.
It has been suggested that the Jacobian matrix be evaluated either once and for all or once every few iterations, instead of at every iteration as is strictly required [2]. The simplest method to tackle this very crucial issue is the Chord Newton's method. This method saves a lot of the computational burdens of the Jacobian matrix $F'(x_k)$, by approximating the Jacobian with the Jacobian at x_0 (Initial guess) i.e

$$F'(x_k) \approx F'(x_0) \qquad \forall k = 0,1,2,\dots$$

The Chord Newton's method generates an iterative sequence $\{x_k\}$ via the following algorithm.

Algorithm 2 (Chord Newton's method)

Given x_0

Solve $F'(x_0)s_k = -F(x_k)$ for s_k $k = 0,1,2,\dots$

Update $x_{k+1} = x_k + s_k$.

Moreover, there are quite numerous approaches proposed by researchers to handle the shortcomings of Newton's method, which include; quasi-Newton's method and inexact Newton's [3-9]. Despite the fact that, Chord Newton's is the simplest variant of Newton's method, still the convergence is not guaranteed especially as the system dimension increases and requires high number of iterations. In contrast, this paper presents an improved variant of Chord Newton's method using a new line search strategy, via the steps of backtracking in the Armijo – type line search. The anticipation has been to improve the overall performance of Chord Newton's.

We organized the rest of the paper as follows: Section 2 presents the proposed method, Convergence Analysis is given in Section 3, Numerical experiments are reported in section 4, and finally conclusion is given in section 5.

2.0 Methodology for proposed method (CNLS)

In this section we present the new line search (Armijo – Type) and apply it to Chord Newton's method. The method generates a sequence of points $\{x_k\}$ via

$$x_{k+1} = x_k - \alpha_k \left(F'(x_0)\right)^{-1} F(x_k) \qquad (2)$$

where $F'(x_0)$ is a fixed Jacobian matrix and α_k is a step length. The Armijo rule is among the inexact line search methods which guarantees a sufficient degree of accuracy to ensure the algorithm convergence. Nevertheless, the Armijo line search requires high floating points operations and function call. To this end, we proposed a simple line search strategy, which has less computational cost, less floating points operations as well as less CPU time consumptions compared to classical Armijo line search.

Given $\varepsilon \in (0,1)$ and $\sigma < 1$, the proposed approach find the appropriate α, such that

$$\|F(x_k + \alpha_k d_k)\| \le \sigma \|F(x_k)\| \qquad (3)$$

In addition, the new approach is implemented in an iterative way using a fixed initial value of α as follows:

Algorithm: Armijo – Type

Step 0: Set $k = 0, \alpha_0 > 0$ and $\sigma < 1$,

Step 1: $\|F(x_{k+1})\| \le \sigma \|F(x_k)\|$, choose α_k as the step size; STOP. Else $\alpha_{k+1} = \dfrac{\alpha_k}{2}, k = k+1.$

This approach is often called as backtracking scheme. We implement our strategy on Chord Newton's method anticipating to overcome the shortcomings of the method under study. In the following, we present the algorithm of the modified Chord Newton's method as follows:

Algorithm: CNSL

Step 1: Choose an initial guess x_0 and $\alpha_0 > 0$ compute $F'(x_0)$ and let $k := 0$

Step 2: Computer $F\left(x_k\right)$ $\left\|F\left(x_k\right)\right\| \le 10^{-4}$ stop

Step 3: Compute $d = -F\left(x_k\right) * \left(F'\left(x_0\right)\right)^{-1}$

Step 4: If $\left\|F\left(x_k + \alpha_k d_k\right)\right\| \le \sigma \left\|F\left(x_k\right)\right\|$, retain α_k and goto 5. Else set $\alpha_{k+1} = \dfrac{\alpha_k}{2}$ and repeat 4 until $\left\|F\left(x_k + \alpha_k d_k\right)\right\| \le \sigma \left\|F\left(x_k\right)\right\|$ is satisfied

Step 5: Let $x_{k+1} = x_k + \alpha_k d_k$

Step 6: Set $k := k+1$ and goto 2

3.0 Numerical Results

In order to present the performance of the proposed methods, we apply the methods to solve three (3) benchmarks problems. Three Newton's – like methods are compared:

1. Newton's Method (NM)
2. Classical Chord Newton's Method (CNM)
3. CNLS method proposed in this paper.

The comparison was based on Number of iterations, CPU time in seconds, storage requirement and floating points operations. We present all the results using performance profiles indices in terms of robustness, efficiency and combined robustness and efficiency as proposed in [10]. All the computational experiments were carried out in double precision computer, the termination point is considered to be

$$\left\|s_k\right\| + \left\|F\left(x_k\right)\right\| \le 10^{-4} \tag{4}$$

We designed the program to terminate whenever:

i. The number of iteration is at least 500 but no point of x_k satisfying (4) is obtained;
ii. CPU time in second reaches 500;
iii. Insufficient memory to initiate the run.

" __" represents a failure due to any of i – iii.

In the following we present some details of the benchmarks test problems as follows:

Problem 1 System of n nonlinear equations: (Artificial problem)

$$f_i(x) = \left(\sum_{i=1}^{n} x_i + 1\right)\left(x_i - 1\right) + \exp\left(x_i - 1\right) - 1.$$

$$i = 1, 2, \ldots, n \quad \text{and} \quad x_0 = (3, 3, 3 \ldots, 3).$$

Problem 2 Spares function of Byeong [8]

$$f_i(x) = x_i^2 - 1$$

$$i = 1, 2, \ldots, n \quad \text{and} \quad x_0 = (0.5, 0.5, 0.5 \ldots 0.5).$$

Problem 3 System of n nonlinear equations: (Artificial problem)

$$f_i(x) = x_i\left(\cos x_i - \frac{1}{n}\right) - x_n\left[\sin x_i - 1 - \left(x_i - 1\right)^2 - \frac{1}{n}\sum_{i=1}^{n} x_i\right]$$

$$i = 1, 2, \ldots, n \quad \text{and} \quad x_0 = (7, 7, 7 \ldots 7).$$

The numerical results of the methods when solving the benchmarks problems with different dimensions are reported in Table 1. It shows that CNLS method outperforms the NM and CNM methods in terms of number of iterations, CPU time and floating point operations respectively. This is due to the fact that the modified line search strategy has enhanced the performance of Chord Newton's method. Moreover, this strategy requires very less computational effort and low memory requirement. Indeed, we observed that CNLS method has 100% of successes rates (convergence to the solution) when compared with NM method having 59% and CNM method with 44% respectively.

Table 1: Numerical Results of NM, CNM and CNLS Methods.

Prob	Dim	NM		CNM		CNLS	
		NI	CPU	NI	CPU	NI	CPU
1	25	7	0.045	83	0.041	55	0.026
2	25	6	0.016	__	__	5	0.011
3	25	7	0.032	69	0.031	45	0.016
1	50	7	0.015	89	0.047	60	0.027
2	50	6	0.032	__	__	5	0.016
3	50	6	0.062	47	0.031	32	0.031
1	100	7	0.214	95	0.140	66	0.064
2	100	6	0.421	__	__	5	0.112
3	100	6	0.187	33	0.062	23	0.031
1	300	7	0.515	104	0.281	75	0.213
2	300	6	0.442	__	__	5	0.218
3	300	5	0.359	20	0.125	15	0.109
1	500	7	1.014	108	0.780	79	0.654
2	500	6	0.718	__	__	5	0.969
3	500	5	0.796	17	0.281	13	0.274

4.0 Conclusion

This paper proposed a new line search strategy, by using the steps of backtracking in the new Armijo – Type line search. The anticipation has been to improve the overall performance of Chord Newton's. It is also worth mentioning from the numerical experiments that, the proposed method is capable of significantly reducing the execution time (CPU time), number of iterations and floating points operations as compared to NM and CNM methods, while maintaining good accuracy of the numerical solution to some extent. Another fact that makes the Armijo – Type strategy more attractive is that, it can be applied to any iterative method for solving nonlinear systems. Finally, we conclude that our approach is a good alternative to Newton's – type methods for solving large-scale nonlinear equations.

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