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Analytical Solutions of Laplace's equation in Parabolic Cylinder and Prolate Spheroidal Coordinate System

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Abstract

In this article, analytical solutions of Laplace's equation in Parabolic Cylinder and Prolate spheroidal coordinate system were constructed. These two coordinate system can be used to described an interesting physical systems that are shaped like a parabolic Cylinder and the solution of these coordinate systems were not seen available in any literature.

Keywords: Analytical Solution, Laplace's equation, Parabolic, Cylinder, Prolate, Spheroid

1.0 Introduction

The choice of a particular coordinate system is motivated by the geometrical form of the body under study and can result in a considerably simplified analysis of the problem [1].

The solution of Laplace's equation in Cartesian and Polar coordinate systems were seen available in the literature while that of Parabolic Cylinder and Prolate Spheroidal coordinate system were not seen available in any literature.

In other words, to express boundary condition in a reasonably simple way, one must have coordinate surface that fit the physical boundaries of the problem [2, 3]. For instance, in calculating the effect of introducing a dielectric sphere into an electric field, one uses spherical Polar coordinates. Thus, the range of field problems that can be handled by a physicist will depend upon the number of coordinate systems with which the person is familiar [3].

A spheroid is obtained by rotating an ellipse about one of its principal axis. If the ellipse is rotated about its major axis, a prolate spheroid is formed, while an oblate spheroid is formed if the ellipse is rotated about its minor axis. Spheroidal coordinates eliminate the cumbersome mathematical expressions obtained with rectangular coordinates and allow the simple determination of areas and volumes. They offer an obvious generalization of physical processes described in spherical coordinates and in addition yield interesting limiting cases of the infinitely thin, finite "wire" and the infinitely thin circular disk [4].

Thus, in this study we solve Laplace's equation in these two coordinate systems.

2.0 An Analytical Solution of Laplace's equation in Parabolic Cylinder coordinate

If r, φ, z are cylindrical polar coordinates, the coordinater ξ and η may be defined by

$$\xi = \sqrt{(2r)\cos\frac{1}{2}\varphi}, \quad \eta = \sqrt{(2r)\sin\frac{1}{2}\varphi}$$

Also, ξ and η are related to the Cartesian Coordinates by the following relations

$$x = \frac{1}{2}(\xi^2 - \eta^2), \quad y = \xi\eta$$

The ξ = constant, and η = constant are orthogonal parabolic cylinders and ξ, η, z are called parabolicic cylindrical coordinates. It is found that the direction cosines and the element of distance in parabolic cylinder coordinates are given respectively [5]

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$$h_1 = h_2 = \sqrt{\xi^2 + \eta^2}\,, \quad h_3 = 1$$

and

$$ds^2 = \left(\xi^2 + \eta^2\right)\left(d\xi^2 + d\eta^2\right) + dz^2$$

Thus, the differential operator $\nabla^2 V$ is given as [5]

$$\nabla^2 V = \frac{1}{\xi^2 + \eta^2}\left(\frac{\partial^2 V}{\partial \xi^2} + \frac{\partial^2 V}{\partial \eta^2}\right) + \frac{\partial^2 V}{\partial z^2} \quad (2.1)$$

If Φ is the electromagnetic scalar potential, then Laplace's equation $\nabla^2 \Phi = 0$ can be written in parabolic cylinder coordinate as

$$0 = \frac{1}{\xi^2 + \eta^2}\left(\frac{\partial^2 V}{\partial \xi^2} + \frac{\partial^2 V}{\partial \eta^2}\right) + \frac{\partial^2 V}{\partial z^2} \quad (2.2)$$

Let us now seek a solution separable of equation (2.2) of the form

$$\Phi(\xi, \eta, z) = U(\xi)P(\eta)Q(z) \quad (2.3)$$

Thus (2.2) becomes

$$0 = \frac{1}{\xi^2 + \eta^2}\left(\frac{1}{U}\frac{\partial^2 U}{\partial \xi^2} + \frac{1}{P}\frac{\partial^2 P}{\partial \eta^2}\right) + \frac{1}{Q}\frac{\partial^2 Q}{\partial z^2} \quad (2.4)$$

Or

$$\frac{1}{\xi^2 + \eta^2}\left(\frac{1}{U}\frac{\partial^2 U}{\partial \xi^2} + \frac{1}{P}\frac{\partial^2 P}{\partial \eta^2}\right) = -\frac{1}{Q}\frac{\partial^2 Q}{\partial z^2} \quad (2.5)$$

Since the left hand side is a function of only ξ and η, and the right hand side is a function of only z, therefore equality holds,

L.H.S = R.H.S = constant,

Let us choose the constant to be m^2 , then

$$\frac{d^2 Q}{dz^2} + m^2 Q = 0 \quad (2.6)$$

and

$$\frac{1}{\xi^2 + \eta^2}\left(\frac{1}{U}\frac{d^2 U}{d\xi^2} + \frac{1}{P}\frac{d^2 P}{d\eta^2}\right) = m^2 \quad (2.7)$$

For a special case in which m= 0, equation (2.7) yields

$$\frac{1}{U}\frac{d^2 U}{d\xi^2} = \omega^2 = -\frac{1}{P}\frac{d^2 P}{d\eta^2} \quad (2.8)$$

Where ω^2 is a constant

Equation (2.6) has solutions of the form

$$Q(z) = \begin{cases} e^{i\omega z} \\ e^{-i\omega z} \end{cases} \quad (2.9)$$

It was worthnoting that, for Q to be single-valued, ω must be an integer i.e $\omega \in$

From (2.8),

$$\frac{d^2 U}{d\xi^2} - \omega^2 U = 0 \quad (2.10)$$

With solution set given as

$$U(\xi) = \begin{cases} e^{-i\omega\xi} \\ e^{i\omega\xi} \end{cases} \quad (2.11)$$

Also from (2.8)

$$\frac{d^2P}{d\eta^2}+\omega^2P=0 \quad (2.12)$$

with solution as

$$P(\eta)=\begin{cases} e^{i\omega\eta} \\ e^{-i\omega\eta} \end{cases} \quad (2.13)$$

Thus equations (2.9), (2.11) and (2.13) can be substituted into (2.3) to yield solutions to Laplace's equation (2.2).

1. An Analytical Solution of Laplace's equation in Prolate Spheroidal coordinates

Prolate Spheroidal Coordinates (u, v, φ) are related to the Cartesian Coordinates by [5]

$$x = l\cosh u\cos v,\ y = l\sinh u\sin v\cos\varphi,\ z = l\sinh u\sin v\sin\varphi$$

Where $u \geq 0$, $0 \leq v \leq 2\pi$, $0 \leq \varphi \leq 2\pi$

Also the direction cosines are

$$h_1 = h_2 = l\sqrt{\cosh^2 u - \cos^2 v},\ h_3 = l\sinh u\sin v \quad (3.1)$$

Also the Laplacian operator is defined as [5]

$$\nabla^2 V = \frac{1}{l^2\sinh u\sin v(\cosh^2-\cos^2 v)}\left\{\frac{\partial}{\partial u}\left(\sinh u\sin v\frac{\partial V}{\partial u}\right)+\frac{\partial}{\partial v}\left(\sinh u\sin v\frac{\partial V}{\partial v}\right)+\frac{1}{l^2\sinh^2 u\sin^2 v}\frac{\partial^2 V}{\partial\varphi^2}\right\}$$

(3.2)

The alternative coordinates are defined as

$$\xi = \cosh u,\ \eta = \cos v$$

And thus (3.2) becomes

$$\nabla^2 V = \frac{1}{l^2(\xi^2-\eta^2)}\left\{\frac{\partial}{\partial\xi}\left((\xi^2-1)\frac{\partial V}{\partial\xi}\right)+\frac{\partial}{\partial\eta}\left((1-\eta^2)\frac{\partial V}{\partial\eta}\right)+\frac{1}{l^2(\xi^2-1)(1-\eta^2)}\frac{\partial^2 V}{\partial\varphi^2}\right\} \quad (3.3)$$

If Φ is the electromagnetic scalar potential then we can write Laplace's equation in Prolate Spheroidal Coordinates as:

$$0 = \frac{1}{l^2(\xi^2-\eta^2)}\left\{\frac{\partial}{\partial\xi}\left((\xi^2-1)\frac{\partial\Phi}{\partial\xi}\right)+\frac{\partial}{\partial\eta}\left((1-\eta^2)\frac{\partial\Phi}{\partial\eta}\right)+\frac{1}{l^2(\xi^2-1)(1-\eta^2)}\frac{\partial^2\Phi}{\partial\varphi^2}\right\} \quad (3.4)$$

Now let us seek a separable solution of Laplace's equation (3.4) of the form

$$\Phi = W(\xi,\eta)Q(\varphi) \quad (3.5)$$

Then equation (3.4) becomes

$$-\frac{1}{Q}\frac{\partial^2 Q}{\partial\varphi^2} = \frac{(\xi^2-1)(1-\eta^2)}{(\xi^2-\eta^2)}\left\{\frac{1}{W}\frac{\partial}{\partial\xi}\left((\xi^2-1)\frac{\partial W}{\partial\xi}\right)+\frac{1}{W}\frac{\partial}{\partial\eta}\left((1-\eta^2)\frac{\partial W}{\partial\eta}\right)\right\} \quad (3.6)$$

Thus, we can write

$$\frac{1}{Q}\frac{\partial^2 Q}{\partial\varphi^2} = m^2 \quad (3.7)$$

and

$$\frac{(\xi^2-1)(1-\eta^2)}{(\xi^2-\eta^2)}\left\{\frac{1}{W}\frac{\partial}{\partial\xi}\left((\xi^2-1)\frac{\partial W}{\partial\xi}\right)+\frac{1}{W}\frac{\partial}{\partial\eta}\left((1-\eta^2)\frac{\partial W}{\partial\eta}\right)\right\} = m^2 \quad (3.8)$$

Where m^2 is a constant of separation. Now let $W(\xi,\eta)$ be of the form:

$$W(\xi,\eta) = U(\xi)P(\eta) \quad (3.9)$$

Then equation (3.8) becomes

$$\frac{(\xi^2-1)(1-\eta^2)}{(\xi^2-\eta^2)}\left\{\frac{1}{U}\frac{\partial}{\partial\xi}\left((\xi^2-1)\frac{\partial U}{\partial\xi}\right)+\frac{1}{P}\frac{\partial}{\partial\eta}\left((1-\eta^2)\frac{\partial P}{\partial\eta}\right)\right\}-\frac{m^2(\xi^2-\eta^2)}{(\xi^2-1)(1-\eta^2)} = 0 \quad (3.10)$$

The solution of (3.7) is of the form

$$Q(\varphi)=\begin{cases} e^{im\varphi} \\ e^{-im\varphi} \end{cases} \quad (3.11)$$

Now consider a special case of (3.10) in which m = 0, then

$$\frac{d}{d\xi}\left((\xi^2-1)\frac{dU}{d\xi}\right)=\alpha^2 U \quad (3.12)$$

and

$$\frac{d}{d\eta}\left((1-\eta^2)\frac{dP}{d\eta}\right)=-\alpha^2 P \quad (3.13)$$

Where α^2 is a separation constant,

Equation (3.12) can be written as

$$\frac{d}{d\xi}\left((\xi^2-1)\frac{dU}{d\xi}\right)-\alpha^2 U=0 \quad (3.14)$$

Let $\alpha^2 = l(l+1)$, where l is arbitrary, then (3.14) becomes

$$\frac{d}{d\xi}\left((\xi^2-1)\frac{dU}{d\xi}\right)-l(l+1)U=0 \quad (3.15)$$

With solution as

$$U(\xi)=\begin{cases} (\xi^2-1)^{\frac{l}{2}} \\ (\xi^2-1)^{-\frac{(l+1)}{2}} \end{cases} \quad (3.16)$$

Also equation (3.13) can be written as

$$\frac{d}{d\eta}\left((1-\eta^2)\frac{dP}{d\eta}\right)+l(l+1)P=0 \quad (3.17)$$

Equation (3.18) is simply the ordinary Legendre's equations of order l. Hence the solutions are the Legendre polynomials

$$\{P_l\}_{l=0}^{\infty} \quad (3.18)$$

Conclusion

With the solution of Laplace's equation in Prolate spheroidal and Parabolic cylinder Coordinates, interesting problems in electrodynamics can be solved in these systems. For instance, spheroidal antennas can be used to model a variety of different antenna shapes, from wire antennas through cylindrical antennas, to disk antennas. Thus, for antennas that are long and then, prolate spheroidal coordinates fit the geometry more closely [6], [7].

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