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Validation of Kaplan-Yorke Conjecture for some Attractors

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Abstract

Chaos is a term common in nonlinear dynamics. In fact, most researches are done to investigate or exploit the concept of chaos in dynamical systems. Various tools for investigating the behaviours of dynamical systems, without solving analytically, have been developed over the years. One of the common techniques is the computation of the Dimensions.

Capacity Dimensions, Information Dimensions, Lyapunov Dimensions, etc. are different types of dimensions and they are named according to how they are computed.

Kaplan and Yorke (1979) conjectured that the Lyapunov dimension and Information dimension are equal. This paper verified this conjecture for the Lorenz and the Duffing attractors.

Keywords: Chaos, Variational Equations, Lyapunov Exponents, Dimensions

1.0 Introduction

An attractor is a set towards which a trajectory tends as the system evolves in time. When there is a stable attracting solution, motions that start close enough to the attractor remain close to it as time goes to infinity. The trajectory can be periodic or chaotic but there is no restriction on the trajectories other than the fact that they remain close to the attractor. An attractor can be a point (in which case it is called a point attractor), a set of points, lines, surfaces, volumes or a complicated set of fractal structure (strange attractors). When the geometry of an attractor is not easily defined by the classical analytical methods, then such an attractor is considered a *strange attractor*.

A system is chaotic if a small change in the initial conditions would cause a great change in the behaviour of the system. This sensitive dependence on the initial condition is referred to as *butterfly effect*.

Several techniques have been developed to investigate the behaviours of dynamical systems. These methods include the Melnikov approach, Normal form theory [1], by experiment ([2] and [3]), investigation of numerical integration by varying parameters [4], modern topological analysis [5], dimension, Lyapunov exponents etc.

Some of the different types of dimensions include Capacity Dimension, Information Dimension, and Lyapunov Dimension (also referred to as Kaplan-Yorke Dimension). Although there is no relationship in the calculation of Lyapunov dimension and Information Dimension, Kaplan and Yorke [6] conjectured that the results are equal.

This paper therefore investigates the conjecture for two common attractors in nonlinear dynamics; Lorenz and Duffing attractors.

2.0 Variational Equation and Characteristic Multiplier

If the starting points of two trajectories differ by an infinitesimal quantity, then the rate at which they attract or repel can be measured by the *Variational equation*.

The Variational equation for the initial value problem (IVP)

$$\dot{x} = f(x,t), \quad x(t_0) = x_0 \qquad (1)$$

with solution $\phi(t,t_0,x_0)$ is

$$\frac{\partial}{\partial t}\Phi(t,t_0,x_0) = \frac{\partial}{\partial x} f(x,t)\Phi(t,t_0,x_0), \qquad \Phi(t_0,t_0,x_0) = I_n \qquad (2)$$

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Where

$$\Phi(t_0,t_0,x_0)=\frac{\partial}{\partial x_0}\phi(t,t_0,x_0).$$

The solution to this Variational equation is found by appending the Variational equation to the original equation so that we have a system of the form [8]

$$\begin{Bmatrix}\dot{x}\\ \Phi(t,t_0,x_0)\end{Bmatrix}=\begin{Bmatrix}f(x,t)\\ \frac{\partial}{\partial x}f(x,t)\Phi(t,t_0,x_0)\end{Bmatrix} \qquad (3)$$

with the initial condition

$$\begin{Bmatrix}x(t_0)\\ \Phi(t_0,t_0,x_0)\end{Bmatrix}=\begin{Bmatrix}x_0\\ I_n\end{Bmatrix}. \qquad (4)$$

The eigenvalues of the Variational equation is called the *Characteristic multiplier* (also called *Floquet multipliers*).

For an autonomous system, the Variational equation at an equilibrium point $\hat{x}$ will be

$$\frac{\partial}{\partial t}\Phi(t,\hat{x})=\frac{\partial}{\partial x}f(\hat{x})\Phi(t,\hat{x}),\quad \Phi(t_0,\hat{x})=I_n \qquad (5)$$

and with a solution matrix

$$\Phi(t,\hat{x})=e^{\frac{\partial}{\partial x}f(\hat{x})t} \qquad (6)$$

and supposing the eigenvalues of $\frac{\partial}{\partial x}f(\hat{x})$ to be $\lambda_1,\lambda_2,\ldots,\lambda_n$ then λ_i and m_i are related by the formula

$$m_i=e^{\lambda_i T} \qquad (7)$$

where $T>0$ is the period of the corresponding non-autonomous system.

1. Lyapunov Exponents [7]

Lyapunov exponents determine whether a system is chaotic or not by quantifying the sensitivity to a change in the initial conditions. We define the Lyapunov exponents of an initial condition x_0 of a flow as

$$\hat{\lambda}=\lim_{t\to\infty}\tfrac{1}{t}\ln\left|m_i(t)\right| \qquad i=1,\ldots,n \text{ (provided the limit exists),} \qquad (8)$$

where m_i's are the eigenvalues of the solution $\Phi(t,x_0)$ to the Variational equation.

2. Dimensions

In classical geometry, dimension is defined as the minimum number of coordinates needed to specify a point uniquely (and they are usually integers). Meanwhile, for a dynamical system, the dimension is defined as the number of state variable needed to describe the dynamical system.

Dimension of a non-chaotic attractor is always an integer but that of a strange attractor are non-integers. A non-integer dimension is called *fractal dimension*[8].

2.1 Information Dimension [8]

Let an attractor A be covered with volume elements (lines, squares, spheres, cubes, etc.), each of side e and suppose it requires a minimum of $N(e)$ number of volume elements to cover A then we define the *information dimension* as

$$D_{\inf}=\lim_{e\to 0}\frac{H(e)}{\ln\left(e^{-1}\right)}, \qquad (9)$$

where

$$H(e)=\sum_{i=1}^{N(e)}P_i\ln P_i \qquad (10)$$

and $P_i =$ relative frequency with which a trajectory enters the *i*-th volume element and $H(e)$ is the entropy. In this case, the number of times a trajectory visits a box is also considered in the box counting procedure.

2.2 Lyapunov Dimension

This dimension was introduced by Kaplan and Yorke [6]. It uses the Lyapunov exponents to calculate the dimension. Suppose the Lyapunov exponents are λ_1, λ_2, …, λ_n and are ordered so that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Now, let *j* be the largest integer for which the sum $\lambda_1 + \lambda_2 + \cdots + \lambda_j \geq 0$, we define the Lyapunov dimension D_L

$$D_L = j + \frac{\lambda_1 + \lambda_2 + \cdots + \lambda_j}{|\lambda_{j+1}|} \tag{11}$$

• For an equilibrium point,we have that $\lambda_i < 0$ foralli, and hence, *j*=0 and D_L=0.

• For a limit cycle, $\lambda_1 = 0$, $\lambda_2 < 0$, $\cdots$ $\lambda_n < 0$ then $D_L = 1 + \frac{\lambda_1}{|\lambda_2|} = 1$.

• For chaotic attractor in 3*D*, $\lambda_1 > 0$, $\lambda_2 = 0$, $\lambda_3 < 0$ so that *j* = 2 and $D_L = 2 + \frac{\lambda_1}{|\lambda_3|}$

The Kaplan-Yorke conjecture [6] states that $D_{inf} = D_L$ but this conjecture is yet to be proved.

3.0 The Models

The two models considered in this paper are the Lorenz and the Duffing equations.

The Lorenz system is a system of three ordinary differential equations used by Lorenz Edward in 1963 to simplify the mathematical model of atmospheric convection [9]. The Lorenz equation is

$$\begin{aligned} \dot{x} &= \sigma(y - x), \\ \dot{y} &= rx - y - xz, \\ \dot{z} &= xy - \beta z. \end{aligned} \tag{12}$$

The Lorenz equations arise in simplified models for lasers, dynamos, thermosyphons, brushless DC motors, electric circuits, and chemical reactions [10, 11].

The forced Duffing oscillators [13] is given as

$$\ddot{x} + \alpha\dot{x} + \beta x + \delta x^3 = \gamma \cos \omega t, \tag{13}$$

Where $\alpha \geq 0$ is the damping constant, $\beta x + \delta x^3$ is the nonlinear elasticity and $\gamma \cos \omega t$ is the periodic forcing. Choosing $x_3 = \omega t$, the system (13) can be reduced to a system of three ordinary differential equations;

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\alpha x_2 - \beta x_1 - \delta x_1^3 + \gamma \cos x_3, \\ \dot{x}_3 &= \omega. \end{aligned} \tag{14}$$

4.0 Summary

The Lyapunov exponents of the Lorenz attractors are calculated to be 0.90, 0.00, -14.54. Using the formula (11), the Lyapunov dimension is calculated to be 2.06. The Information dimension for the Lorenz attractor is also found to be equal to 2.06.

The Lyapunov exponents computed for the Duffing attractor are 0.10, -0.15, 0.00 and the Lyapunov dimension (using formula (11)) is 2.66 and the Information dimension found to be 2.66.

The Kaplan-Yorke conjecture is therefore correct for the Lorenz and the Duffing attractors up to two decimal places.

It is important to reiterate that this is not enough to confirm that the conjecture is true in all cases. A more rigorous proof is needed to prove the Kaplan-Yorke conjecture for all cases.

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