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Primary Decomposition of Ideals and Submodules

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Abstract

In this work our main aim is to develop new method of proves for the first and second Uniqueness Theorems for the primary decomposition of modules over commutative unital rings. The first asserts essentially that prime ideals which occur in the set of ideals $\{(N:x); x \in M\}$ are independent of the particular decomposition of N. The last result asserts that the minimal primary decomposition of N of isolated set of prime ideal is independent of the decomposition.Our method guarantees primary decomposition of modules over commutative unital rings.

Keywords: Rings of Fractions, Modules of Fraction, Isomorphism, RingsHomomorphism, Endomorphism, Nipotent, Noetherian Modules.

1.0 Introduction

The theory of commutative unital ring holds such an important place in Algebra that it is not surprising that our literature is rich its generalization. To see how one might be led to the kind of generalization to be discussed here, we state that an arbitrary submodule needs not to have a primary decomposition. However, in this paper, we shall only be interested in those that have, for example Neatherian module. Robbiano et al [1] studied and obtained excellent results on primary power of prime ideal. Kunz et al [2] proved that Cohen – MaCaulay rings and ideal have invariants of algebraic groups. Danilov [3] presented the concept of the group of ideal classes of a complete ring where ideal operations remained crisp. Yamamoto [4] and Lipman [5] took a departure from the earlier mentioned by considering the decomposition fields of difference sets and the Jacobian ideal of the module difference. Bazzoni [6], Lidia
et al [7] and Lipman [8] studied the concept of modules from different approaches and obtained interesting results. Inspired by these successful approaches, we showed that if$\underline{a}$ be a decomposable ideal and $a = \bigcap_{i=1}^{n} \underline{q_i}$ be a minimal primary decomposition of $\underline{a}$.

Let $\underline{p} = r(\underline{q_i}), 1 \le i \le n$. Then p_i are precisely the prime ideals which occur in the set of radicals $r(\underline{a} : x), (x \in \mathrm{A})$, and hence are independent of the particular decomposition of $\underline{a}$.

and if $\underline{a}$ be a decomposable ideal, let $\underline{a} = \bigcap_{i=1}^{n} \underline{q_i}$ be a minimal primary decomposition of $\underline{a}$, let $\{\underline{p}_{i_1}, \ldots, \underline{p}_{i_m}\}$ be isolated set of prime ideal of $\underline{a}$. then $\bigcap_{j=1}^{m} \underline{q}_{i_j}$ is independent of decomposition.

We shall restate and prove the analogues of the following standard results for ideals of commutative rings.

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1.1 Theorem (First Uniqueness Theorem)

Let $\underline{a}$ be a decomposable ideal and $a = \bigcap_{i=1}^{n} \underline{q_i}$ be a minimal primary decomposition of $\underline{a}$. Let $\underline{p}$ $= r\left(\underline{q_i}\right), 1 \le i \le n$. Then p_i are precisely the prime ideals which occur in the set of radicals $r\left(\underline{a} : x\right), \left(x \in \mathrm{A}\right)$, and hence are independent of the particular decomposition of $\underline{a}$.

1.2 Proposition

Let S be a multiplicatively closed subset of a commutative ring A, and let $\underline{q}$ be $\underline{p}$ – primary ideal. Then

(i) If $S \cap \underline{p} \ne \phi$, $S^{-1}\underline{q} = S^{-1}\mathrm{A}$

(ii) If $S \cap \underline{p} \ne \phi$, $S^{-1}\underline{q}$ is $S^{-1}\underline{p}-$ primary and its contraction in A is $\underline{q}$. Hence primary ideals correspondence $\left(\underline{a} \leftrightarrow S^{-1}\underline{a}\right)$ between ideals in $S^{-1}\mathrm{A}$ and contracted ideals in A.

1.3 Proposition

Let S be a multiplicatively closed subset, $\underline{a}$ a decomposable ideal, $\underline{a} = \cap\, \underline{q_i}$ a minimal primary decomposition of $\underline{a}$. Let $\underline{p_i} = r\left(\underline{q_i}\right)$ and suppose the $\underline{q_i}$ numbered so that S meets $\underline{p}_{m+1}, \ldots, \underline{p}_n$ but not $\underline{p}_n, \ldots, \underline{p}_m$. Then $\underline{a}^e = S^{-1}\underline{a} = \bigcap_{i=1}^{m} S^{-1}\underline{q_i}$, $\underline{a}^{ec} = \bigcap_{i=1}^{m} \underline{q_i}$ and these are minimal primary decompositions.

1.4 Theorem: (Second Uniqueness Theorem)

Let $\underline{a}$ be a decomposable ideal, let $\underline{a} = \bigcap_{i=1}^{n} \underline{q_i}$ be a minimal primary decomposition of $\underline{a}$, let $\left\{\underline{p}_{i_1}, \ldots, \underline{p}_{i_m}\right\}$ be isolated set of prime ideal of $\underline{a}$. then $\bigcap_{j=1}^{m} \underline{q}_{i_j}$ is independent of decomposition. In particular.

1.5 Corollary

The isolated primary components (i.e. primary components $\underline{q_i}$ corresponding to the minimal prime ideals $\underline{p_i}$) are uniquely determined by $\underline{a}$.

Assume that M is a fixed A -module, and suppose that N, P is submodules of M. Consider the set $\left(N : P\right) = \left\{a \in \mathrm{A} : aP \subseteq N\right\}$. Then it is clear that if $a, b \in \left(N : P\right)$, and $x \in p$ then $\left(a \pm b\right)x = ax \pm bx \in N$, since $aP \subseteq N, bP \subseteq N$. Moreover, for any $u \in \mathrm{A}, a \in \left(N : P\right)$, we have $uaP = u\left(aP\right) \subseteq uN \subseteq N$. Thus $\left(N : P\right)$ the ideal quotient of the submodule of N, P.

Let N be a submodule of M, the radical of $N\left(in\ M\right)$ is the ideal $r_m\left(N\right) = \left\{a \in \mathrm{A} : a^q M \subseteq N \text{ for some } q > 0\right\} = r\left(N : M\right)$.

Let $a \in \mathrm{A}$, a defines and endomorphism $\phi_a : M \to M$ namely $\phi_a\left(x\right) = ax$. The element a is nilpotent in M if ϕ_a is a nilpotent, and a is a zero-divisor in M if ϕ_a is a zero-divisor in the ring of homomorphisms of M.

N is a primary submodule of M if

(i) $N \ne M$;

(ii) Every zero-divisor in M/N is nilpotent.

First we prove the following:

1.6 Lemma

(i) If N is a primary submodule of M, then $r_M(N)$ is a prime ideal, $\underline{p}$, of A; and if

(ii) $N_i, 1 \le i \le n$ are $\underline{p}-$primary submodules, then so is their intersection.

Proof

(i) Suppose N is primary, we prove that $r_M(N)$ is a prime ideal as follows:

(a) We first claim that $r_M(N) \ne \mathrm{A}$. Indeed $N \ne M \Rightarrow$ there exists $x \in M,\ x \notin N$. But then, this entails that $1 \notin r_M(N) = r(N : M)$.

(b) Suppose $ab \in r_M(N),\ b \notin r_M(N)$. Then $a^q b^q M \subseteq N,\ b^q M \subseteq N$. But then $\phi a^q(b^q u + N) = a^q b^q u + N = N$ for any $u \in M$. Hence, since $b^q u + N \ne N$, it follows that a^q is a zero-divisor in M/N and so must be nilpotent, since N is primary. That for some $p > 0$, we have $\phi a^{pq} : M/N \to M/N$ is the zero endomorphism. That is $a^{pq} M \subseteq N$ and whence $a \in r_M(N)$, proving that this ideal is some prime ideal, p, of A.

(ii) First of all, it is clear that $\bigcap_{i=1}^{n} N_i \ne M$, since no N_i equal M by hypothesis.

Furthermore, it is clear that $r\left(\bigcap_{i=1}^{n} N_i : M\right) = \left\{a \in \mathrm{A} : a^q M \subseteq N_i, 1 \le i \le n \text{ for some } q > 0\right\}$

$= \bigcap_{i=1}^{n} r(N : M) = \bigcap_{i=1}^{n} \underline{p} = \underline{p}$

Set $N = \bigcap_{i=1}^{n} N_i$ and let a be a zero-divisor in M/N. Then there exists some $x \notin N$ and $ax + N = N$. That is, $ax \in N,\ x \notin N$. But then, $x \in N \Rightarrow$ there exists some i, with $x \notin N_i$ but $ax \in N_i$. Since N_i is primary, we see that $\underline{a}$ is a zero-divisor in M/N and so must be nilpotent. Whence, there exists $p > 0$ and $a^p M \subseteq N_i \Rightarrow a \in r(N_i : M) = \underline{p} = r_M(N)$ by what we have proved. This proves in turn that $N = \bigcap_{i=1}^{n} N_i$ is $\underline{p}-$primary as required.

A primary decomposition of submodule N of M is, an expression of N as a finite intersection of primary submodules, say $N = \bigcap_{i=1}^{n} N_i$. If moreover

(i) the $\underline{p}_i = r_M(N)$ are all distinct, and

(ii) No proper subfamily of $\{N_i\text{'}S\}$ generates N, that is, $N_i \not\supseteq \bigcap_{j \ne i} N_i,\ (1 \le i \le n)$, the primary decomposition $N = \bigcap_{i=1}^{n} N_i$ is said to be <u>minimal</u> (or <u>irredundant</u> or <u>reduced</u> or <u>normal</u>$\cdots$).

1.7 Remark

It is clear from the above lemma, that if a primary decomposition exists, we can achieve (i) and then we can omit only the superfluous terms to achieve (ii) thus any primary decomposition can be normalized to achieve a minimal decomposition.

1.8 Lemma

If N is a $\underline{p}-$primary submodule of M, then $\underline{q}=(N:M)$ is a $\underline{p}-$ primary ideal of A.

Proof

Since N is $\underline{p}-$ primary, $N \neq M$ and thus $1 \notin q$. Suppose $ab \in \underline{q},\ b \notin \underline{q}$, then for some $x \in N,\ abx \in N,\ bx \notin N$. That is, $a(bx+N)=N$ and $bx+N \neq N$, and so a zero-divisor in M/N and is nilpotent by hypothesis. Thus there exists $n>0$ such that $a^{\ n}M \subseteq N$ for some $n>0$. Thus $a \in \underline{p}$, and thus $\underline{q}$ is $\underline{p}-$primary as required $Q.E.D$.

1.9 Lemma

(i) Let $\underline{p}_i, \ldots, \underline{p}_n$ be prime ideals and let $\underline{a}$ be an ideal contained in $\bigcup_{j=1}^{n} \underline{p}_j$. Then $\underline{a} \subseteq \underline{p}_i$ for some i.

(ii) Let $\underline{a}_1, \ldots, \underline{a}_n$ be ideals and let $\underline{p}$ be a prime ideal containing $\bigcap_{i=1}^{n} \underline{a}_j$. Then $\underline{p} \supseteq \underline{a}_i$ for some i. If $\underline{p}=\bigcap_{i=1}^{n} \underline{a}_j$, then $\underline{p}=\underline{a}_i$ for some i.

Proof

(i) This is proved by induction on n in the form $\underline{a} \not\subseteq \underline{p}_j$, $(i \le j \le n) \Rightarrow \underline{a} \not\subseteq \bigcup_{i=1}^{n} \underline{p}_j$. It is certainly true if $n=1$. If $n>1$ and the result is true for $n-1$. Then for each j, there exists $x_j \in \underline{a}$ such that $x_j \notin \underline{p}_j$. Whenever $i \neq j$. If for some j, we have $x_j \notin \underline{p}_j$,we are then through. If not, then $x_j \in \underline{p}_j$ for all j. But, then consider the element

$y=\sum_{j=1}^{n} x_1, x_2, \ldots\ x_{j-1}\ \ x_{j+1} \ldots x_n$, we have $y \in a$ and $y \notin \underline{p}_j$, $1 \le j \le n$.

Hence $a \not\subseteq \bigcup_{j=1} p_j$.

(ii) Suppose $\underline{p} \not\supseteq \underline{a}_j$ for all j. Then there exists $x_j \in \underline{a}_j,\ x_j \notin \underline{p}, 1 \le j \le n$ and therefore $y=\pi x_j \in \pi \underline{a}_j \subseteq \bigcap \underline{a}_j$, but $y \notin \underline{p}$, since $\underline{p}$ is prime and no factor of y belongs to $\underline{p}$. Hence $\underline{p} \not\supseteq \cap \underline{a}_j$. Finally, if $\underline{p}=\cap \underline{a}_j$, then for some $i, \underline{a}_i \subseteq \underline{p} \subseteq \underline{a}_i$, showing that $\underline{p}=\underline{a}_i$ for some i. Q. E. D.

1.10 Lemma

Let N be a $\underline{p}-$primary submodule of M. Then

(i) if $x \in N,\ r(N:x)=A$ and

(ii) if $x \notin N,\ r(N:x)=\underline{p}$ and $(N:x)$ is $\underline{p}-$primary.

Proof

(i) if $x \in N$, then $Ax \subseteq N$, hence $(N:x)=A$ and $r(N:x)=r(A)=A$.

(ii) if $x \notin N$ and $a \in (N:x)$, then $ax \in N$. Hence a is a zero-divisor in M/N and must be nilpotent. Put another way, a belongs to $r(N:M)=\underline{p}$. That is, $(N:x) \subseteq \underline{p}$. Since it should be clear that $(N:M) \subseteq (N:x)$, we see that $(N:M) \subseteq (N:x) \subseteq \underline{p}$. Taking radicals, we deduce

$\underline{p} = r(N:M) \subseteq r(\underline{p}) = \underline{p}$. To see that $\underline{q} = (N:x)$ is $\underline{p}-$ primary, suppose $a(bx+N) = 0 \in M/N$ or $b \in \underline{q}$. Finally, $x \notin N \to 1 \notin \underline{q}$. This completes the proof.

1.11 Lemma

Let S be a multiplicatively closed subset of A, N, P submodules of M. Then

(i) $r(S^{-1}N : S^{-1}M) = S^{-1}(r(N:M))$ and

$S^{-1}N \cap S^{-1}P = S^{-1}(N \cap P)$.

Proof

(i) *If* $\frac{a}{s} \in S^{-1}(r(N:M))$, then $a \in r(N:M)$ and so $a^n M \subseteq N$, for some $n > 0$. Thus $\frac{a^n}{s^n} \in (S^{-1}N : S^{-1}M)$ and so $\frac{a}{s} \in (S^{-1}N : S^{-1}M)$.

Conversely, if, $\frac{b}{t} \in r(S^{-1}N : S^{-1}M)$ then $\frac{b^n}{t^n} \in (S^{-1}N : S^{-1}M)$

for some $n > 0$. Whence, for any $\frac{x}{u} \in S^{-1}M$, we have $\frac{b^n x}{t^n x} = \frac{y}{v}$

for some $y \in N,\ v \in S$. That is $1(vb^n x - t^n uy) = 0$, for some $1 \in S$. Whence $(1vb)^n \in (N:M) \Rightarrow 1vb \in r(N:M)$ $\Rightarrow 1vb \in r(N:M) \Rightarrow \frac{b}{t} = \frac{1vb}{1vt} \in S^{-1}(r(N:M))$ .Thus, $S^{-1}(r(N:M)) \subseteq r(S^{-1}N : S^{-1}M) \subseteq S^{-1}(r(N:M))$.

(ii) Clearly, let $S^{-1}(N \cap P) \subseteq S^{-1}N \cap S^{-1}P$. Conversely, let $z \in S^{-1}N \cap S^{-1}P$, then $\frac{x}{s} = z = \frac{y}{t}$

for some $x \in N,\ y \in P,\ s, t \in S$. That is, $u(tx - sy) = 0$ for some $u \in S$.

Whence $utx = usy \in N \cap P \Rightarrow z = \frac{x}{s} = \frac{utx}{uts} = \frac{usy}{uts} = \frac{y}{t} \in S^{-1}(N \cap P)$.

That is, $S^{-1}(N \cap P) \subseteq S^{-1}N \cap S^{-1}P \subseteq S^{-1}(N \cap P)$. Q. E. D.

1.12 Theorem (First Uniqueness Theorem)

Let N be decomposable submodule of A-module. M, and let $N = \bigcap_{j=1}^{n} N_j$ be a minimal primary decomposition of N. Let $\underline{p}_j = r_M(N_j),\ 1 \le j \le n$. Then, the $\underline{P}_j$ is precisely the prime ideals which occur in the set of ideals $\{r(N:x);\ x \in M\}$, and hence are independent of the particular decomposition of N.

Proof

For any $x \in M$, we have $(N:x) = \left(\bigcap_{j=1}^{n} N_j : x\right) = \bigcap_{j=1}^{n}(N_j : x)$.

Taking radicals, we obtain $r(N:x) = \bigcap_{j=1}^{n} r(N_j : x) = \bigcap_{x \in N_j} \underline{p}_i$ (by 1.10). Suppose $r(N:x)$ is a prime ideal $\underline{P}$ of A, then $\underline{P} = (r(N:x)) = \bigcap_{x \notin N_i} N_i \Rightarrow \underline{P} = \underline{P}_i$ for some i (by 1.9). Hence, every prime ideal of the form $r(N:x)$ is one of the $\underline{P}_j = r(N_j : M) = r_N(N_j), 1 \le j \le n$.

Conversely, for each j, there exists some $x_j \notin N_j, x_j \in \bigcap_{i \ne j} N_j$ (since the decomposition is minimal). Whence, by (1.9) again, we deduce that $r(N : x_j) = \underline{P}_j$. This completes the proof of our theorem.

1.13 Remarks

(i) Theorem 1.12 asserts that even though the primary components N_j may fail to be invariants, their associated radicals, $r(N_j : M)$ are invariants of N. Thus, in particular, the number of factors is an invariant of N.

(ii) The prime ideals $\underline{P}_1, \ldots, \underline{p}_n$ are said to belong to N or are said to be associated with N. We write $ASS(N) = \{\underline{P}_1, \ldots, \underline{p}_n\}$.

1.14 Proposition

Let S be a multiplicatively closed subset of a ring A, let N be a $\underline{P}$-primary submodule of an A-module, M. Then

(i) if $S \cap \underline{P} \ne \phi, S^{-1}N = S^{-1}M$ and

(ii) if $S \cap \underline{P} \ne \phi, S^{-1}N$ is an $S^{-1}\underline{p}-$primary submodule of $S^{-1}M$ and its contraction in M is N. Hence, primary submodules correspond to primary submodules in the correspondence $(S^{-1}N \leftrightarrow N)$ between submodules in $S^{-1}M$ and contracted submodule in M.

Proof

(i) If $u \in S \cap p$, then, $u^n \in S^{-1} \cap (N : M)$ for some $n > 0$. Hence $\frac{u^n}{1} \in (S^{-1}N : S^{-1}M)$ and $\frac{u}{1}$ is a unit of $S^{-1}\mathrm{A}$. Whence $(S^{-1}N : S^{-1}M) = S^{-1}\mathrm{A}$ and so $S^{-1}M \subseteq S^{-1}N$, proving that $S^{-1}N : S^{-1}M$.

(ii) If $S \cap p = \phi$, then $u \in S$ and $ux \in N$ both translate into the statement that $u(x+N) = N$ and u is not nilpoint in M/N.

Hence, by the primary nature of N, we deduce $ux \in N, u \notin p \Rightarrow x + N = N \Rightarrow x \in N$.

Moreover, from the claim of implications $x \in N^{ec} = (S^{-1}N)^c \Leftrightarrow \frac{x}{1} = \frac{y}{t}$ for some $y \in N$, $t \in S \Leftrightarrow u(tx - y) = 0$ for some $u \in s \Leftrightarrow sx \in N$, for some $s \in S \Rightarrow x \in N$, we deduce the implications, $x \in N^{ec} \Rightarrow \in N \Rightarrow N^{ec} \subseteq N.$ Since, in any case, $N \subseteq N^{ec}$, we conclude that $N = N^{ec}$, if $S \cap p = \phi$.

Furthermore, using (1.11)(i), we obtain $r(S^{-1}N : S^{-1}M) = S^{-1}(r(N:M)) = S^{-1}p$. Since in general the contraction of a primary submodule is itself primary, it will be sufficient to prove that $S^{-1}N$ is primary. But then to see that $S^{-1}N$ is primary, note that if $\frac{a}{s}$ is a zero-divisor in $S^{-1}M/S^{-1}N$, then for some $\frac{x}{t} \notin S^{-1}N$, we have

$\frac{ax}{st} \in S^{-1}N$. Whence, $ax \in N$ but $x \notin N$. We deduce that a is a zero-divisor in N/M and so $a^n M \subseteq N$, for some $n > 0$ (by hypothesis on N). Clearly, then $\frac{a^n}{s^n}$ must be the zero-endomorphism of $S^{-1}M/S^{-1}N$. This shows that $S^{-1}N$ is primary as desired, Q. E. D.

1.15 Proposition

Let S be a multiplicatively closed subset of A, N is a decomposable submodule of A-module M, $N = \bigcap_{j=1}^{n} N_j$, a minimal primary decomposition of N. Let $P_j = r_M\left(N_j\right)$ and suppose that N_j are numbered so that S meets $P_{m+1}, \ldots, P_n$ but not $P_1, \ldots, P_m$.

Then, $N^e = S^{-1}N = \bigcap_{i=1}^{m} S^{-1}N_i$, $N^{ec} = \bigcap_{i=1}^{m} N_i$ and these are minimal primary decompositions.

Proof

$N^e = S^{-1}N = \bigcap_{j=1}^{m} S^{-1}N_j$ (by induction and (1.11)). $\bigcap_{i=1}^{m} S^{-1}N_i$ and $S^{-1}N_i$ is $S^{-1}P-$primary (by 1.14) $1 \le i \le m$. Since the P_i is distinct, the $S^{-1}p_i$ are distinct $1 \le i \le m$. Hence the decomposition of $S^{-1}N$ is a minimal primary decomposition. Contracting both sides, we have

$N^{ec} = \left(\bigcap_{i=1}^{m} S^{-1}N_i\right)^c = \bigcap_{i=1}^{m}\left(S^{-1}N_i\right)^c = \bigcap_{i=1}^{m}\left(N_i\right)$ (by 3.14 again). Q. E. D.

Next, a subset Σ of a set $\{p_1, \ldots, p_n\}$ of prime ideal associated with a decomposable submodule N denoted by $\Sigma \subseteq ASS(N)$ is isolated if it satisfies the following conditions: $p \in \Sigma$ and $p' \in ASS(N)$ and $p \supseteq p' \Rightarrow p' \in \Sigma$. In particular, if p is a minimal prime ideal belonging to N, then $\{p\}$ is isolated.

Let Σ be any isolated subset of $ASS(N)$ and let $S = \mathrm{A} - \bigcup_{p\in\Sigma} p$. Then, S is multiplicatively closed, since $1 \in S$ because $1 \notin \cup p$ and $s \in S, t \in S \Rightarrow t, s \notin p$ for any $p \Rightarrow st \in p$ for any p, since each p is prime.

Furthermore, an important property of S is the following: for any $p' \in ASS(N)$, we have $\underline{p}' \in \Sigma \Rightarrow p \cap s = \phi$; $\underline{p}' \notin \Sigma \Rightarrow \underline{p} \not\subseteq \bigcup_{p\in\Sigma} p$ (by Lemma 1.9 (ii)) $\Rightarrow \underline{p} \cap s \neq \phi$. We now use this type of S and proposition 1.15 to obtain

1.16 Theorem (Second Uniqueness Theorem)

Let N be a decomposable ideal, let $N = \bigcap_{j=1}^{n} N_j$ be a minimal primary set of prime ideals of N. Then, $\{\underline{p}_{j_1}, \ldots, \underline{p}_{j_m}\}$ be an isolated set of prime ideals of N. Then, $\bigcap_{i=1}^{m} N_{j_i}$ is independent of the decomposition.

Proof

$\bigcap_{i=1}^{m} N_{j_i} = N^{ec}$, hence depends only on N (since the $\underline{p}_j$ depends only on N).

1.17 Corollary

The isolated primary components (i.e. the primary components N_j corresponding to maximal prime ideals $\underline{p}_j$) are uniquely determined by N.

Proof
This is a particular case of our second uniqueness theorem.

1.18 Concluding Remarks

1. Our discussion of primary decomposition of A-modules (and ideals of commutative rings as a special case), helps to explain the supreme importance of prime ideals in commutative Algebra. Intuitively one can think of prime sub-modules as the basic distinctive building blocks of modules in much the same role as prime numbers in Number Theory.

2. We have used fractional modules in a significant way to prove the main results of this paper which are to be found in (1.12) and (1.16).

References

[1] Robbiano L. & Valla G. (1976), Primary power of a prime ideal, pacific J. math. 63 No.1 491-498.
[2] Kunz R. E. (1974), Cohen-MaCaulay rings and ideal theory in rings of invariants of algebraic groups, TAMS 194, 491 – 498.
[3] Danilov V.I., The group of ideal classes of a completed ring, math USSR-Sbornik 6(1968), 493-500.
[4] Yamamoto K. (1963), Decomposition fields of difference sets, Pacific J. Math. 6, 337 – 352.
[5] Lipman J. (1969), On the Jacobian ideal of the module difference, proc. Amer. Math. Soc. 21, 422 – 426.
[6] Bazzoni S., Cotilting modules are pure-injective, Proc. Amer. Math. Soc.131 (2003), no. 12, 3665–3672 (electronic), 10.1090/S0002-9939-03-06938-7. MR 1998172 (2004f:16049) .
[7] Lidia Angeleri Hügel and Dolors Herbera, Mittag-Leffler conditions on modules, Indiana Univ. Math. J. 57 (2008), no. 5, 2459–2517, 10.1512/iumj.2008.57.3325. MR 2463975 (2009i:16002) .
[8] Lipman J. (1965), Free derivation modules on algebraic varieties, Amer. J. Math. 87, 874 – 898.