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Operation on Ideals

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Abstract

We provide basic operations on ideals such as addition, intersection, multiplication, the formation of ideal quotients, radicals, and the extensions and contractions of ideals.Our method guarantees that the ideals of non-trivial unital ring forms a complete lattice, the property which A-module does not share.

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1.0 Introduction

Throughout this work we shall carry out certain exercises set by Professor M.A. Atiyah in his Notes on Commutative Algebra. This monograph was first published in mimeograph by Mathematical Institute at Oxford University in 1965. The whole of our presentation relies heavily on Atiyah's prototype which was later published as our reference [1] by Addison Wesley in 1969.

We discuss general method by which one can determine the operation on ideals that is, behavior of the ideals in a commutative unital ring. By operations, we mean basic operations on ideals such as addition, multiplication, intersection, the formation of ideal quotients, radicals, the extensions and contraction of ideals. If$A \neq \{0\}$ be a ring, then A has a maximal ideal and a minimal prime ideals is the major objective of this work. In specific cases which have been extensively studied this questions are extremely hard to answer.

The literatures covered by this study are fairly extensive, see for example [3], [4], or [6]. We consider the formation of radicals of ideals which is a natural consideration in the context of solution of equations and the factorization of elements in commutative rings. Let a be an ideal of A . The radical of *a, r(a)*is the set of all $x \in A$, such that $x^n \in a$ for some integer $n \geq 1$ (or equivalently, it is the set of elements x in A whose image $\overline{x}$ in the factor ring A/a is nilpotent). Recently Lipman [5], Eakin et-al [8] and Sally et-al [9] have removed the assumption on characteristic. We can recover this result. Indeed, we find considerably more. Johnson [7] has conjectured that maximal ideals reduces the centralizers and operators and Eagon et-al [2] has conjectured that ideals defined by matrices and certain complex associated to them have a unique properties. We are able to show:

1.1 Definition

When we say that A is a ring, we shall mean that multiplication is commutative in A and that the multiplicative identity, denoted by 1, also belongs to A. Moreover, $1 \neq 0$, where 0 is additive identity. Also; if A, B are rings, a ring homomorphism

$$f : \mathrm{A} \to \mathrm{B}$$

Is a mapping such that whenever $x,\ y \in \mathrm{A}$, we have

$$f(x+y) = f(x) + f(y)$$
$$f(xy) = f(x)\,f(y)$$
$$f(1) = 1$$

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We shall denote the ideal of multiples of an element x by (x). That is $(x)=\{ax : a\in \mathrm{A}\}$. In general, we donate an ideal of A by notation a,b,p,m e.t.c. In this work, we discuss the basic operation on ideals, such as addition, intersection, multiplication, the formation of ideal quotients, radicals, and the extensions and contractions of ideals.
We start with the class of ideals which are by far the most important in Commutative Algebra.

By definition, an ideal p is to be the prime ideal of A if:

(i) $p\neq(1),\ (=\mathrm{A})$, and

(ii) $xy\in p\Rightarrow x\in p \ \ or\ \ y\in p$.

Part of the reason for the importance of prime ideals lies in the following proposition which we state without proof:

1.2 Proposition

An ideal of A is a prime ideal if and only if its associated quotient ring A/a is an integral domain.

An ideal m in A is said to be maximal if:

(i) $m\neq(1)\ (=\mathrm{A})$ and

(ii) a is an ideal in A such that $m\subseteq a\subseteq \mathrm{A}$; then either $a=m\ \ or\ \ a=\mathrm{A}$.

We prove that every maximal ideal is a prime ideal by obtaining the following results:

1.3 Proposition

An ideal a of A is maximal if and only if its associated quotient A/a is a field.

Proof

Suppose a is a maximal ideal of A, then $a\neq \mathrm{A}$ and so $\mathrm{A}/a\neq\{0\}$, the zero ring. For any $x\in \mathrm{A}$, we write $\overline{x}=x+a=\{x+y : y\in a\}$ and suppose that $\overline{x}\neq\overline{0}$ in A/a. To find its inverse, we note that $a+(x)$ is an ideal such that $a\neq a+(x)$. Moreover, we have $a\subseteq a+(x)\subseteq \mathrm{A}$; hence $a+x=(1)$. And so there exists y such that $1\equiv xy \pmod{a}$. Hence $\overline{x}\,\overline{y}=\overline{1}\in \mathrm{A}/a$. This proves that A/a is a field.

Conversely, suppose that A/a is a field and $a\subseteq b\subseteq \mathrm{A}$ for any ideal b of A, the first inclusion being strict. Let $x\in \mathrm{A},\ y\in b,\ y\notin a$. Then $\overline{y}\neq\overline{0}$ in A/a. Since A/a is a field, one can find Σ such that $\overline{x}=\overline{y}\,\overline{z}\Rightarrow x-yz\in a\subseteq b\Rightarrow x\in b$, since $y\in b$, by hypothesis. This proves that $\mathrm{A}\subseteq b\subseteq \mathrm{A}\Rightarrow b=\mathrm{A}$. Hence a is maximal. $Q.\ E.\ D.$.

Combining (1.2) and (1.3), it is clear that every maximal ideal is prime. The converse is obviously false, since $\{0\}$ is prime in $\mathbb{Z}$, the ring of integers, without being maximal.

Next, to demonstrate the abundance of prime ideal, we prove the following:

1.4 Proposition

Let $\mathrm{A}\neq\{0\}$ be a ring, then A has a maximal ideal.

Proof

Let S be the set of ideal $a\neq \mathrm{A}$ of the ring A. By hypothesis, $(0)=\mathrm{A}$ and so S is non-empty. We can therefore order S by inclusion. Consider any ascending chain $\{b_i : i\in I\}$ in S, so that for any $i,j\in I$ either $b_i\subseteq b_j\ \ or\ \ b_j\subseteq b_i$. Consider the set $b=\bigcup_{i=I} b_i$, we claim that b is an ideal.

Indeed, if $x\in b$, then $x\in b_i$ for some $i\in I$. Hence, $a=\mathrm{A}\Rightarrow ax\in b_i\Rightarrow \mathrm{A}b\subseteq b$. Furthermore, if $x,\ y\in b$, then $y\in b_j$ for some $j\in I$ and without loss of generality, we may assume $b_i\subseteq b_j\Rightarrow x,y\in b_j\Rightarrow x\pm y\in b_j\subseteq b$.

Thus, our claim has been established. Moreover, since $i \in I \Rightarrow i \notin b_i$ by hypothesis, we deduce that $i \notin b$. Hence $b \in S$. Thus, any ascending chain in S has upper bound in S and so by zorn's lemma S has a maximal element, m say. This proves our proposition.

1.5 Proposition

If $\mathrm{A} \neq \{0\}$ is a ring, then A has a minimal prime ideal.

Proof

Let Σ be the set of prime ideals in A. Σ is non-empty by (1.4). Let $\{p_i\}$ be a chain of prime ideals in Σ. Their intersection is an ideal $a = \bigcap_{i=I} p_i = \bigcap_{p_i \subseteq p_j} p_i$ for some $j \in I$.

To prove that a is indeed a prime ideal, suppose that $xy \in a,\ y \notin a$, then $y \notin p_j$ for some $j \in I$. Since $xy \in p_i,\ i \in I$, by the supposition that $xy \in a$ and by hypothesis p_j is prime, it follows that $x \in p_j$.

Moreover, if $p_i \subseteq p_j$, then $y \notin p_j \Rightarrow y \in p_i$, and by the argument, we have just used, we deduce that $x \in p_i$. Thus $x \in a = \bigcap_{p_i \subseteq p_j} p_i$.

Clearly $a \neq (1)$, since $i \in I \Rightarrow 1 \notin p_i$. Thus, a is a prime ideal. Moreover, since $p = \bigcap_{i \in I} p_i \subseteq p_i$ for all i, it follows that any chain in Σ has a lower bound in Σ and by zorn's lemma, Σ has a minimal element. This proves our proposition.

Next we turn to some results related to the formulation of ideals quotients.

Let a, b be ideals in a commutative ring A, the ideal quotient of a by b written $(a:b)$ is defined by $(a:b) = \{x \in \mathrm{A} : xb \subseteq a\}$.

1.6 Proposition

Let a, b and c be ideals of ring A, then

(i) $a \subseteq (a:b)$

(ii) $(a:b)b \subseteq a$

(iii) $((a:b):c) = (a:b\ c) = ((a:c):b)$

(iv) $\left(\bigcap_i a_i : b\right) = \bigcap_i (a_i : b)$

(v) $\left(a : \sum_i b_i\right) = \bigcap_i (a : b_i)$

Proof

(i) $a \subseteq (a:b)$. By definition of ideal, $x \in a \Rightarrow xb \subseteq a$ (because a is an ideal) $\Rightarrow x \in (a:b)$ by definition of $(a:b) \Rightarrow a \subseteq (a:b)$.

(ii) By definition $(a:b)b$ is generated by products of the form xy where $x \in (a:b)$ and $y \in b$. But then $x \in (a:b),\ y \in b \Rightarrow xy \in a$. Hence, each generator of $(a:b)b$ lies in a and so $(a:b)b \subseteq a$ as required.

(iii) Let $x \in ((a:b)c)$ and consider any generator yz of bc.

Then $z \in c \Rightarrow zx \in (a:b) \Rightarrow yzx \in a$, since $y \in b$. Hence multiplication by x transforms every generator of bc into an element of a.

Hence $xbc \subseteq a \Rightarrow x \in (a:bc) \Rightarrow ((a:b):c) \subseteq (a:bc)$

Next, let $u \in (a:bc)$. For any element $v \in c,\ w \in b$, we have

$vw \subseteq b = bc \Rightarrow uvw \in a \Rightarrow uw \in (a:c) \Rightarrow u \in ((a:c)b) \Rightarrow (a:bc) \subseteq ((a:c)b)$.

Let $s \in ((a:c)b),\ t \in b, r \in c$.

Then $st \in (a:c) \Rightarrow str \in a \Rightarrow srb \subseteq a \Rightarrow sc \subseteq (a:b) \Rightarrow s \in ((a:b)c)$

$\Rightarrow ((a:c)b) \Rightarrow ((a:b)c)$ proving equality of the giving ideals.

(iv) Let $x \in \left(\bigcap_i a_i : b\right)$; then $xb \subseteq a_i$ for each $i \Rightarrow x \in (a_i : b)$ for each $i \Rightarrow x \in \bigcap_i (a_i : b)$

$\Rightarrow \left(\bigcap_i a_i : b\right) \subseteq \bigcap_i (a_i : b)$.

Conversely, $y \in \bigcap_i (a_i : b) \Rightarrow y \in (a_i : b)$ for each

$i \Rightarrow yb \subseteq \left(\bigcap_i a_i : b\right) \Rightarrow y \in \left(\bigcap_i (a_i : b)\right) \Rightarrow \bigcap_i (a_i : b) \subseteq \left(\bigcap_i a_i : b\right)$.

This proves equality

(v) Let $x \in (a_i : \Sigma b_i) \Rightarrow (\Sigma b_i) \subseteq a$. In particular, $b_i \subseteq \Sigma b_i \Rightarrow xb_i \subseteq a \Rightarrow x \in (a : b_i)$ for each $i \Rightarrow x \in \bigcap_i (a : b_i)$. Hence $(a : \Sigma b_i) \subseteq \bigcap_i (a_i : b)$. Since an element of Σb_i is of the form $u = y_1 + y_2 + \ldots + y_n$, where $y_i \in b_{ij}$, then

$z \in \bigcap_i (a : b_i) \Rightarrow zy \subseteq a \Rightarrow z \in (a : \Sigma b_i) \Rightarrow \bigcap_i (a : b_i) \subseteq (a : \Sigma b_i)$. By Axiom of Extension

$(a : \Sigma b_i) = \bigcap_i (a : b_i)$.

Next, we consider the formation of radicals of ideals, which is a natural consideration in the context of solution of equations and the factorization of elements in commutative rings.

Let a be an ideal of A. The radical of $a,\ r(a)$ is the set of all $x \in \mathrm{A}$, such that $x^n \in a$ for some integer $n \geq 1$ (or equivalently, it is the set of elements x in A whose image $\bar{x}$ in the factor ring A/a is nilpotent).

1.7 Proposition

Let a, b be ideals of a ring A and p be a prime ideal of A. Then

(i) $r(a) \supseteq a$

(ii) $r(r(a)) = r(a)$

(iii) $r(ab) = r(a \cap b) = r(a) \cap r(b)$

(iv) $r(a) = (1) \Leftrightarrow a = 1$

(v) $r(a+b) = r(r(a) + r(b))$

(vi) if p is a prime, $r(p^n) = p$ for some $n > 0$

Proof

(i) if $x \in a$, then taking $n = 1$, we have $x = x' \in r(a)$. Hence $a \subseteq r(a)$.

(ii) By (i), $r(a) \subseteq r(r(a))$.

Conversely, $x \in r\left(r\left(a\right)\right) \Rightarrow x^n \in r\left(a\right)$ for some $n > 0 \Rightarrow \left(x^n\right)^m \in a$ for some $m > 0 \Rightarrow x^{nm} \in a$ for $nm > 0$. That is, $r\left(r\left(a\right)\right) = r\left(a\right)$.

(iii) $ab \subseteq a \cap b \Rightarrow r\left(ab\right) \subseteq r\left(a \cap b\right)$. Also, let $x \in \left(a \cap b\right)$, then $x^n \in a \cap b$ for some $n > 0 \Rightarrow x^n \in a,\ x^n \in b \Rightarrow x \in r\left(a\right),\ x \in r\left(b\right) \Rightarrow x \in r\left(a\right) \cap r\left(b\right) \Rightarrow r\left(a \cap b\right) \subseteq r\left(a\right) \cap r\left(b\right)$. Finally, let $y \in r\left(a\right) \cap r\left(b\right)$, then $y^n \in a,\ n > 0$ and $y^{m+n} = y^m y^n \in ab \Rightarrow y \in r\left(ab\right)$.

Hence, $r\left(a\right) \cap r\left(b\right) \subseteq r\left(ab\right)$ and we have the chain of inclusion

$r\left(ab\right) \subseteq r\left(a \cap b\right) \subseteq r\left(a\right) \cap r\left(b\right) \subseteq r\left(a \cap b\right)$.

By axiom of extension, we deduce that

$r\left(ab\right) = r\left(a \cap b\right) = r\left(a\right) \cap r\left(b\right)$

(iv) $r\left(a\right) = \left(1\right) \Rightarrow 1 = 1^n \in a$ for some $n > 0 \Rightarrow a \supseteq \left(1\right) \supseteq a \Rightarrow a = \left(1\right)$.
$a = \left(1\right) \Rightarrow \left(1\right) = a \subseteq r\left(a\right)$ by $\left(i\right)$ $\Rightarrow 1 = a \subseteq r\left(a\right) \subseteq \left(1\right) \Rightarrow r\left(a\right) = 1$.

(v) $a + b \subseteq r\left(a\right) + r\left(b\right)$ by $r\left(a+b\right) \subseteq r\left(r\left(a\right) + r\left(b\right)\right)$.
Conversely, let $y \in r\left(r\left(a\right) + r\left(b\right)\right)$, then $y^n \in \left(r\left(a\right) + r\left(b\right)\right)$ for some $n > 0$.
Suppose $y^n = u + v$, where $u \in r\left(a\right),\ v \in r\left(b\right)$ with indices t, s; that is, $u^t \in a,\ v^s \in b$ for $t > 0,\ s > 0$. Hence $y^{n\left(t+s-1\right)} = \left(u+v\right)^{t+s-1} = \Sigma u^1 v^k$ where it is impossible for $1 < s$ and $k < t$ simultaneously. Hence $\Sigma u^1 v^k \in a + b$. Thus $y \in r\left(a+b\right)$.

(vi) Let $x \in r\left(p\right)$, then $x^n \in p^n$ for $n > 0 \Rightarrow x \in p$, since p is prime. Hence $p \subseteq r\left(p\right) \subseteq p$. Thus $r\left(p\right) = p$ by (ii) $r\left(p^n\right) = \cap r\left(p\right) = \cap p = p$. $Q.E.D.$

Next, we consider extensions and contractions of ideals. Indeed, let A, B be commutative unital rings and let $f : \mathrm{A} \to \mathrm{B}$ be a ring homomorphism. The extension of an ideal a of A relative to f denoted by a^e, is the ideal b of B generated by $f\left(a\right)$ the image of a under f.

That is, $a^e = \mathrm{B}f\left(a\right) = \left\{y = \Sigma y_i f\left(x_i\right) : x_i \in a, y_i \in \mathrm{B}\right\}$.

Conversely, if b is an ideal in B, then the inverse image, $f^{-1}\left(b\right)$ is easily verified to be an ideal in A. This ideal is denoted by b^c and is called the contraction of b in A induced by f. That is, $b^c = f^{-1}\left(b\right)$.

1.8 Proposition

Let a_1, a_2 be ideals of A, b_1, b_2 ideals of B and $f : \mathrm{A} \to \mathrm{B}$ be any ring homomorphism.

(i) $\left(a_1 + a_2\right)^e = a_1^e + a_2^e,\ \left(b_1 + b_2\right)^c \supseteq b_1^c + b_2^c$

(ii) $\left(a_1 \cap a_2\right)^e \subseteq a_1^e \cap a_2^e,\ \left(b_1 \cap b_2\right)^c = b_1^c \cap b_2^c$

(iii) $\left(a_1 a_2\right)^e \subseteq a_1^e a_2^e,\ \left(b_1 b_2\right)^c \supseteq b_1^c b_2^c$

(iv) $\left(a_1 a_2\right)^e \subseteq \left(a_1^e : a_2^e\right),\ \left(b_1 : b_2\right)^c \supseteq \left(b_1^c : b_2^c\right)$

(v) $r(a)^e \subseteq r(a^e),\ r(b)^c \supseteq r(b^c)$

(vi) The set E of extensions in A is closed under the operation of sum and product while the set C of contractions in B is closed under the remaining three.

Proof

(i) (a) Let $x \in (a_1 + a_2)^e$, then $x = \Sigma u_i f(x_i)$ where $x_i \in a_1 + a_2$. Let $x_i = x_{1i} + x_{2i}$ for some i, so that $f(x_i) = f(x_{1i}) + f(x_{2i})$, where $x_{1i} \in a_1,\ x_{2i} \in a_2$. But then, we have

$x = \Sigma u_i f(x_i) = \Sigma u_i f(x_{1i}) + \Sigma u_i f(x_{2i}) \in (a_1 + a_2)$.

Hence $(a_1 + a_2)^e \subseteq a_1^e + a_2^e$. Then $w = \Sigma u_i f(x_i) + \Sigma v_j f(y_j)$, where

$x_i \in a_1 \subseteq a_1 + a_2,\ y_j \in a_2 \subseteq a_1 + a_2$.

Hence $w \in (a_1 + a_2)^e$. That is, $a_1^e + a_2^e = (a_1 + a_2)^e$

(b) Let $x \in b_1^c + b_2^c$, then $x = x_1 + x_2$, where $f(x_1) \in b_1$ and $f(x_2) \in b_2$.

Thus, $f(x) = f(x_1 + x_2) = f(x_1) + f(x_2) \in b_1 + b_2$.

Whence $x \in (b_1 + b_2)^c$. That is, $b_1^c + b_2^c \subseteq (b_1 + b_2)^c$.

(ii)(a) Let $x \in (a_1 \cap a_2)^e$, then $x \in \Sigma u_i f(x_i) \Rightarrow x_i \in a_1 \cap a_2$. But then

$x_i \in a_1,\ x_i \in a_2 \Rightarrow x = \Sigma u_i f(x_i) \in a_1^e$ and $x = \Sigma u_i f(x_i) \in a_2^e$

That is, $x = \Sigma u_i f(x_i) \in a_1^e \cap a_2^e$. Hence $(a_1 \cap a_2)^e \subseteq a_1^e \cap a_2^e$

(b) If $x \in (b_1 \cap b_2)^e \Rightarrow f(x) \in (b_1 \cap b_2) \Rightarrow f(x) \in b_1$,

$f(x) \in b_2 \Rightarrow x \in b_1^c,\ x \in b_2^c \Rightarrow x \in b_1^c \cap b_2^c \Rightarrow (b_1 \cap b_2)^c \subseteq b_1^c \cap b_2^c$

Conversely, let $y \in b_1^c \cap b_2^c$, then $f(y) \in b_1,\ f(y) \in b_2$.

Hence $f(y) \in b_1 \cap b_2 \Rightarrow y \in (b_1 \cap b_2)^c \Rightarrow b_1^c \cap b_2^c \subseteq (b_1 \cap b_2)^c$.

It follows that $b_1^c \cap b_2^c = (b_1 \cap b_2)^c$.

(iii)(a) Let $x \in (a_1 a_2)^e$. Then $x \in \Sigma u_i f(x_i)$, where, $x_i \in a_1 a_2$ for each i.

Since $x_i = \Sigma v_{j_i} w_{j_i},\ v_{j_i} \in a_1,\ w_{j_i} \in a_2$ and

$\Sigma u_i f(x_i) = u_i f(\Sigma v_{j_i} w_{j_i}) = \Sigma u_i f(v_{j_i}) f(w_{j_i}) \in a_1^e\, a_2^e$ for each i.

It follows that $x \in a_1^e\, a_2^e$. Hence $(a_1 a_2)^e \subseteq a_1^e a_2^e$.

Conversely, let $y \in a_1^e\, a_2^e$, then $y = \Sigma u_1 v_1$ where $u_i \in a_1$ and $v_i \in a_2$. For any i, we have $u_i = \Sigma z_j f(x_j),\ x_j \in a_1,\ z_j \in \mathrm{B},\ v_i = \Sigma w_k f(s_k),\ s_k \in a_2,\ w_k \in \mathrm{B}$ and

$u_i v_i = (\Sigma z_j f(x_j))(\Sigma w_k f(s_k)) = \Sigma z_j w_k f(x_j) f(s_k) = \Sigma z_j w_k f(x_j s_k),\ x_j \in a_1,\ s_k \in a_2$. Hence

$u_i v_i = \Sigma z_j w_k f(x_j s_k) \in (a_1 a_2)^e$ for each i. That is, $y \in (a_1 a_2)^e$. Thus $a_1^e\, a_2^e \subseteq (a_1^e a_2^e)^e$.

Finally, $\left(a_1^e a_2^e\right)^e \subseteq a_1^e\, a_2^e$.

(b) Let $x \in b_1^c\, b_2^c$, then $x \in \Sigma x_i y_i$, where $f\left(x_i\right) \in b_1$ and $f\left(y_i\right) \in b_2$. That is, $f\left(x\right) = f\left(\Sigma x_i y_i\right) = \Sigma f\left(x_i\right) f\left(y_i\right) \in b_1 b_2 \Rightarrow x \in \left(b_1 b_2\right) \Rightarrow b_1^c b_2^c \subseteq \left(b_1\, b_2\right)^c$.

(iv)(a) Let $x \in \left(a_1 : a_2\right)^e$, then $x = \Sigma u_i f\left(x_i\right)$ where $u_i \in \mathrm{B}$ and $x_i \in \left(a_1 : a_2\right)$. Thus $x_i \in a_2 \subseteq a_1$, for all i. Let $y \in a_2^e$, then $y = \Sigma v_j f\left(y_j\right)$ where $v_j \in \mathrm{B},\ y_j \in a_2$.

Thus $xy = \left\{\Sigma u_i f\left(x_i\right)\right\}\left\{\Sigma v_j f\left(y_j\right)\right\}$. Since $x_i y_j \in a_1$, for all i and it follows that

$$xy = \Sigma u_i v_j f\left(x_i y_j\right) \in a_1^e \Rightarrow a_2^e \subseteq a_1^e \Rightarrow x \in \left(a_1^e : a_2^e\right) \Rightarrow \left(a_1 : a_2\right)^e \subseteq \left(a_1^e \subseteq a_2^e\right).$$

(b) Let $y \in \left(b_1 : b_2\right)^c \Rightarrow f\left(y\right) \in \left(b_1 : b_2\right)$. Let $x \in b_2^c$, then

$$f\left(x\right) \in b_2 \Rightarrow f\left(y\right) f\left(x\right) \in b_1 \Rightarrow f\left(xy\right) \in b_1 \Rightarrow yx \in b_1^c \Rightarrow b_2^c y \subseteq b_1$$

$$\Rightarrow y \in \left(b_1^c : b_2^c\right) \Rightarrow \left(b_1 : b_2\right)^c \subseteq \left(b_1^c : b_2^c\right)$$

(v)(a) Let $y \in r\left(a\right)^e$, then $y = \Sigma u_i f\left(y_i\right)$ where $y_i \in r\left(a\right)$ for each i.

$y^{ni} \in a$ for some $ni > 0 \Rightarrow f\left(y^{ni}\right) = \left\{f\left(y_i\right)\right\}^{ni} \in a^e \Rightarrow f\left(y\right) \in r\left(a^e\right)$ for each i.

Hence $y = \Sigma u_i f\left(y_i\right) \in r\left(a^e\right)$. Thus $r\left(a\right)^e \subseteq r\left(a^e\right)$.

(b) $y \in r\left(b\right)^c \Rightarrow f\left(y\right) \in r\left(b\right) \Rightarrow f\left(y\right)^n \in b$ for some $n > 0$

$$\Rightarrow y^n \in b^c \Rightarrow y \in r\left(b^c\right) \Rightarrow r\left(b^c\right) \subseteq r\left(b\right)^c$$

Conversely, if $x \in r\left(b^c\right)$, then $x^m \in b^c$ for some $m > 0$.

Thus $f\left(x^m\right) \in b \Rightarrow f\left(x\right)^m \in b \Rightarrow f\left(x\right) \in r\left(b\right) \Rightarrow x \in r\left(b^c\right) \Rightarrow r\left(b^c\right) \subseteq r\left(b^c\right)$.

Hence $r\left(b\right)^c = r\left(b^c\right)$.

(vi)(a) The set E of extensions is closed under the operations of sum and product. Indeed,

(i)(a) $a_1^e + a_2^e = \left(a_1 + a_2\right)^e$ and this ensures that the sum of two extensions is itself an extension. Moreover

(ii)(a) $a_1^e a_2^e = \left(a_1 a_2\right)^e$ shows the product of two extensions is itself an extension.

The set C of contractions is closed under intersection by virtue of

(ii)(b) $b_1^c \cap b_2^c = \left(b_1 \cap b_2\right)^c$ and is closed under the formation of radicals by virtue of

$\left(v\right)\left(b\right),\ r\left(b^c\right) = r\left(b\right)^c$.

To prove that C is closed under the formation of ideal quotient, we first note that for any ideal a in A, we have $a \subseteq a^{ec}$ and also for $b \in \mathrm{B}$, we have $b^{cec} = b^c$. Hence, we have the equality, $\left(b_1^c : b_2^c\right) = \left(b_1^{ce} : b_2^{ce}\right)^e$.

2.0 Conclusion

Our discussion of operation on ideals (and ideals of commutative rings as a special case), helps to explain the supreme importance of prime ideals in commutative Algebra. Intuitively we consider the formation of radicals of ideals, which is a natural consideration in the context of solution of equations and the factorization of elements in commutative rings.

The extension of an ideal a of A relative to f denoted by a^e, is the ideal b of B generated by $f(a)$ the image of a under f.

That is, $a^e = \mathrm{B}f(a) = \{y = \Sigma y_i f(x_i) : x_i \in a, y_i \in \mathrm{B}\}$.

Conversely, if b is an ideal in B, then the inverse image, $f^{-1}(b)$ is easily verified to be an ideal in A. This ideal is denoted by b^c and is called the contraction of b in A induced by f. That is, $b^c = f^{-1}(b)$.

Our results (1.4) and (1.5) shows that the ideals of non-trivial unital ring form a complete lattice. This is a property which A-module does not share.

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