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Weak Solutions of Boundary Value Problems

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Abstract

The concept of a weak solution of a boundary value problem is investigated. The existence and uniqueness of a weak solution is established. Weak solutions of some boundary value problems are obtained using direct variational method.

Keywords: Weak Solutions, Boundary Value Problems, Sobolev Space**,** Boundary Conditions, Distributional Derivative, Classical Solutions.

1.0 Introduction

The study of Weak Solutions of Boundary Value Problems is an interesting one, both to engineers, scientists and mathematicians. This work is designed to study the solutions of boundary value problems with discontinuous data and do not have continuous partial derivative of order 2k in a given domain Ω.

The classical theory of partial differential equation studies the existence and uniqueness of solutions of boundary value problems if the given data are smooth. This classical theory fails if the given data fail to be smooth. For this reason, it is necessary to generalize the concept of differentiation. This concept of generalization is called distributional or weak derivatives. It leads to the weak solutions of boundary value problems.

By a boundary value problem we mean the problem of finding a solution of a differential equation in a given domain $\Omega \subset R^n$ which satisfies prescribed conditions on the boundary δΩ.

Boundary value problems are of common occurrence in Engineering and Science. To be useful in application, boundary value problems should be well posed. A boundary value problem is said to be well posed if the solution:

i) Exists
ii) Is unique, and
iii) Depends continuously on the given data.

The solution of a boundary value problem is said to be stable if any small change in the given data produces a small change in the solution [1,2,3].

Boundary value problems are well posed for elliptic partial differential equations.

(a)ELLIPTIC EQUATION

The equation:

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij} \frac{\partial u}{\partial x_j} \right) + cu = f$$

is said to be elliptic at x_0 if the Eigen-values of the matrix

$$A = \begin{pmatrix} a_{11} & a_{12}\ldots & a_{1n} \\ a_{21} & a_{22}\ldots & a_{2n} \\ a_{n1} & a_{n2}\ldots & a_{nn} \end{pmatrix},$$

are of the same sign.

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(b) UNIFORMLY ELLIPTIC EQUATION

We consider the partial differential equation [4]

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij} \frac{\partial u}{\partial x_j} \right) + c(x)u = f(x) \qquad (1.1)$$
$$a_{ij}(x) = a_{ji}(x)$$

in the domain $\Omega \subset R^n$ where $f(x) \in L_2(\Omega)$ and the functions $a_{ij}(x)$ together with their first order partial derivatives as well as the function C(x) are continuous in the closed domain $\bar{\Omega}$

We also assume that the inequalities

$$\sum_{i,j=1}^{n} a_{ij}(x)\alpha_i\alpha_j \geq p\sum_{i=1}^{n}\alpha_i^2 \qquad (1.2)$$
$$p > 0$$

where P is a positive constant and

$$C(x) \geq 0. \qquad (1.3)$$

hold.

The partial differential equation (1.1) is said to be uniformly elliptic if (1.2) is satisfied.

For the partial differential equation (1.1), we consider three different types of boundary conditions:

$$u = 0 \quad \text{on} \quad \delta\Omega \qquad (1.4a)$$
$$Nu = 0 \quad on \quad \delta\Omega. \qquad (1.4b)$$
$$Nu + \sigma u = 0 \ , \quad (\sigma(s) \geq \sigma_0 \geq 0) \quad \text{on} \quad \delta\Omega \qquad (1.4c)$$

where $Nu = \sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_j} n_i$ (1.4d)

$n_i = \cos(n_i, x_i)$, i = 1,2,…n are direction cosines of the outward normal to the boundary $\delta\Omega$.

The expression (1.4d) is frequently called the derivative in the direction of the co-normal.

For instance, in the Poisson's equation [5]

$$\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = f \qquad (1.5)$$

where

$$a_{ij} = \begin{cases} 1 & j=i \\ 0 & j \neq i \end{cases} \qquad (1.6)$$

$$Nu = \sum_{i,j=1}^{n} \frac{\partial u}{\partial x_j} n_i = \frac{\partial u}{\partial n} \qquad (1.7)$$

2.0 Materials and Methods

Both the data and the solutions of boundary value problems in partial differential equations are functions defined on certain domain and spaces. In order to formulate precise theorems of existence, uniqueness and continuous dependence of the solution of boundary value problems, it is essential to specify the spaces in which these functions lie and to give precise meaning of convergence in these spaces.

L_2 – SPACE: A real function u(x) is said to be Lebesgue Square Integrable in the domain Ω if the integral

$$\int_{\Omega} |u|^2 \, dx \qquad (2.1)$$

converges.

From the above definition, it can be seen that every function that is continuous in the closure of the domain Ω is square integrable [6, 7].

Convergence in the Space L_2 (Ω)

We say that a sequence of functions $u_n(x)$ in $L_2(\Omega)$ converges in the space $L_2(\Omega)$ to a function $u \in L_2(\Omega)$ if [8]

$$\lim_{n\to\infty} \|u_n - u\| = 0$$

that is, $$\lim_{n\to\infty} \sqrt{\int_\Omega \left|u_n(x)-u(x)\right|^2} dx = 0$$

or $$u_n(x) \underset{n\to\infty}{=} u(x) \quad in \;\; L_2(\Omega)$$

Cauchy Sequence

A sequence of functions $u_n \in L_2(\Omega)$ is called Cauchy sequence in $L_2(\Omega)$ if [8]

$$\lim_{\substack{n\to\infty \\ m\to\infty}} \|u_n - u_m\| = 0$$

Every sequence that is convergent in $L_2(\Omega)$ is a Cauchy sequence. A normed vector space is complete if every Cauchy sequence in the space converges to a vector in the space. $L_2(\Omega)$ space is complete [5]

INNER PRODUCT SPACE

Let X be a vector space over a scalar K (real or complex). An inner product on X is a scalar valued function

$$P : X \times X \to K$$

that associates each pair of vectors x and y in X, a scalar $\langle x, y\rangle$ in K called the inner product of x and y such that for all x, y, z ϵ X and α ϵ K has the following properties[9,10].

$$\left.\begin{aligned} &\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle \\ &\langle \alpha x, y\rangle = \alpha\langle x, y\rangle \\ &\langle x, y\rangle = \overline{\langle y, x\rangle} \\ &\langle x, x\rangle \ge 0 \end{aligned}\right\} \qquad (2.2)$$

and $\langle x, x\rangle = 0$ if and only if $x = 0$

A vector space X with an inner product $\langle .,. \rangle$ defined on it is called an inner product space that is $(X, \langle .,. \rangle)$. An inner product space is also called Pre-Hilbert space.

HILBERT SPACES

An inner product space X is complete if every Cauchy sequence $\{x_n\}$ in X converges to a vector x in X. A complete inner product space is called a Hilbert space

OPERATORS IN HILBERT SPACE

Let H_1, H_2 be Hilbert spaces and $S : H_1 \times H_2 \to K$ a bounded sesquilinear form. S has a representation of the form:

$$S(x, y) = \langle Ax, y\rangle \qquad (2.3)$$

where $A : H_1 \to H_2$ is a bounded linear operator. A is uniquely determined by S and

$$\|S\| = \|A\| \qquad (2.4)$$

Let $b : H \times H \to R$ be a bounded bilinear form. Then there exists a bounded linear operator $A : H \to H$ such that

$$b(x, y) = \langle Ax, y\rangle \quad \text{and} \qquad (2.5)$$

$$\|b\| = \|A\| \qquad (2.6)$$

Theorem (Lax - Milgram) [5, 11]:

Let $b : H \times H \to R$ be a bilinear form which is

i) Continuous
ii) Coercive in H.

Given any continuous linear function $f : H \to R$, there is a unique vector u ϵ H such that:

$$b(u,x) = f(x) \tag{2.7}$$

$$and \quad \|u\|_H \leq \frac{1}{\alpha}\|f\| \tag{2.8}$$

SOBOLEV SPACE

Let Ω be a non-empty open subset of R^n. Then for any non-negative integer m and for any real number p, $1 \leq p < \infty$, we define Sobolev space of order m denoted by $W^{m,p}(\Omega)$ as follows:

$$W^{m,p}(\Omega) = \left\{u \in L_p(\Omega) : D^\alpha u \in L_p(\Omega) \forall |\alpha| \leq m\right\}.$$

$$where\ D^\alpha u\ are\ weak\ derivatives. \tag{2.9}$$

The norm defined for $u \in W_p^m(\Omega)$ is as follows:

$$\|u\|_{m,p} = \left(\sum_{|\alpha|\leq m} \|D^\alpha u\|^p\right)^{\frac{1}{p}} \quad \text{for } 1 \leq p < \infty \tag{2.10}$$

$$\|u\|_{m,\infty} = \max_{|\alpha|\leq m} \|D^\alpha u\|_\infty \quad \text{for } p = \infty \tag{2.11}$$

By using the properties of norm of $L_p(\Omega)$, it follows that $W_p^m(\Omega)$ is a normed vector space. If p=2, then $W_2^m(\Omega) = H^m(\Omega)$ where $H^m(\Omega)$ is an inner product space. Given $u, v \in H^m(\Omega)$, we define an inner product as follows:

$$\langle u, v\rangle = \sum_{|\alpha|\leq m} \int_\Omega D^\alpha u(x) D^\alpha v(x) dx \tag{2.12}$$

For $m = 1$

$$\|u\|_{1,P} = \left(\int_\Omega |u|^p + \sum_{i=1}^n \left|\frac{\partial u}{\partial x_i}\right|^p dx\right)^{\frac{1}{p}} \tag{2.13}$$

For $m = 2$

$$\|u\|_{2,P} = \left(\int_\Omega (|u|^p + \sum_{i=1}^n \left|\frac{\partial u}{\partial x_i}\right|^p + \sum_{ij=1}^n \left|\frac{\partial^2 u}{\partial x_i \partial x_j}\right|^p) dx\right)^{\frac{1}{p}} \tag{2.14}$$

$W^{m,p}(\Omega)$,$1 \leq p < \infty$ is a Banach space [11].

In the classical theory of partial differential equations (PDEs), we seek the solution u which satisfies some relations including its partial derivatives in some domain $\Omega \subset R^n$. The existence and uniqueness of the solution u of well known problems of PDEs can be proved if the given data are smooth.

This classical theory fails if the given data fail to be smooth. To take care of this situation, it is necessary to generalize the idea of differentiation. This concept of generalization is called distributional or weak derivative defined as follows:

Let f, g ϵ $L_{1,}$ loc(Ω) (class of measurable functions which are Lebsegueintegrable) then g = D^α f is called the distributional (weak) derivatives of f of order α if

$$\int_\Omega g\varphi dx = (-1)^{|\alpha|} \int_\Omega f D^\alpha \varphi dx \tag{2.15}$$

$$\varphi \in C_0^\infty(\Omega)$$

We observe immediately that the right hand side of equation (2.15) is defined even when f is not differentiable but just locally integrable [5,9].

3.0 Weak Solutions of Boundary Value Problems

3.1 Basic Definition of Weak Solution

Let $v \in C_0^\infty(\Omega)$, that is v is a function with compact support. Functions that are infinitely differentiable with compact support in Ω are equal to zero on some neighbourhood of the boundary of the domain.

We consider the partial differential equation:

$$-\Delta u = f \quad \text{in } \Omega \tag{3.1}$$

If we multiply (3.1) by $v \in C_0^\infty(\Omega)$ and integrate over the domain Ω with respect to x, we have:

$$-\int_\Omega \left[\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2}\right] v dx = \int_\Omega f v dx \tag{3.3}$$

Using Green's theorem, []

$$\int_\Omega \frac{\partial g_1}{\partial x_i} g_2 dx = \int_{\partial\Omega} g_1 g_2 \eta_i ds - \int_\Omega g_1 \frac{\partial g_2}{\partial x_i} dx$$

$$\int_\Omega v \frac{\partial^2 u}{\partial x_1^2} = \int_{\partial\Omega} v \frac{\partial u}{\partial x_1} \eta_1 ds - \int_\Omega \frac{\partial v_2 \partial u}{\partial x_1 \partial x_1} dx$$

(Renardy and Rogers, 1980).

Since v = 0 on ∂Ω, it implies that

$$\int_\Omega v \frac{\partial^2 u}{\partial x_1^2} = \int_\Omega \frac{\partial v}{\partial x_1} \frac{\partial u}{\partial x_1} dx$$

Similarly,

$$\int_\Omega v \frac{\partial^2 u}{\partial x_2^2} = \int_\Omega \frac{\partial v}{\partial x_2} \frac{\partial u}{\partial x_2} dx$$

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$$\int_\Omega v \frac{\partial^2 u}{\partial x_n^2} = \int_\Omega \frac{\partial v}{\partial x_n} \frac{\partial u}{\partial x_n} dx$$

Together (3.2) becomes

$$\sum_{i=1}^{n} \int_\Omega \frac{\partial v}{\partial x_i} \frac{\partial u}{\partial x_i} dx = \int_\Omega f v dx \tag{3.4}$$

If $f \notin C(\Omega)$, that is f is not a function that is continuous over the domain Ω then (3.1) does not have a classical solution but a weak solution.

Let us assume that f ϵ $L_2(\Omega)$. This only makes sense if each of $\frac{\partial u}{\partial x_i} \in L_2(\Omega)$. This is only possible if $u \in W_2^1(\Omega)$. This fact makes it possible to introduce the following definition sufficiently general for our purpose of the weak solution of (3.1).

3.1.1 Definition (3.1)

Let $u \in W_2^1(\Omega)$, $f \in L_2(\Omega)$. If the condition (3.4) is satisfied for the function u(x), then we say that u(x) is the weak solution of (3.1).

By definition, the weak solution is a function $u \in W_2^1(\Omega)$ which needs not have in Ω derivatives of the second order, but has only generalized derivatives of the first order.

Nevertheless, if $u \in C^2(\Omega)$ and $f \in C(\Omega)$ and U(x) is the weak solution of (3.1), that is condition (3.4) is satisfied for $v \in C_0^\infty(\Omega)$ then u(x) is a classical solution [6,1, 10].

3.1.2 General Case

Let $$A = \sum_{|i||j| \le k} (-1)^{|i|} D^i (a_{ij} D^i) \quad (3.5)$$

be the differential operator of order 2k, then

$$\sum_{|i||j| \le k} (-1)^{|i|} D^i (a_{ij} D^i u) = \mathrm{f} \quad (3.6)$$

is the partial differential equation of order 2k. Let $v \in C_0^\infty(\Omega)$, the space of infinitely differentiable functions with compact support or test functions.

We multiply (3.6) by $V \in C_0^\infty(\Omega)$ and integrate over the domain Ω with respect to x. Using Green's theorem, we have:

$$\sum_{|i||j| \le k} \int_\Omega a_{ij} D^j u D^i v \mathrm{dx} = \int_\Omega \mathrm{fv.dx} \quad (3.7)$$

If $f \notin C(\Omega)$,(that is f is not a function that is continuous over the domain Ω), then (3.6) does not have a classical solution but a weak solution.

Let $f \in L_2(\Omega)$. This only makes sense if each of $D^j u \in L_2(\Omega)$. This is only possible if $u \in W_2^k(\Omega)$.

Hence in general terms, we define the weak solution of Boundary Value Problems as follows:

Definition (3.2)

Let $u \in W_2^k(\Omega)$, $\mathrm{f} \in \mathrm{L}_2(\Omega)$. If the condition (3.7) is satisfied for the function u(x), then we say that u(x) is the weak solution of (3.6).

4.0 Existence And Uniqueness Of Weak Solution Of Boundary Value Problem

The existence and uniqueness of weak solution of boundary value problem is seen clearly under the assumption that the bilinear form ((u, v)) is of the v-elliptic type, and is proved by Lax-Milgram theorem which is a generalization of Riesz representation of linear functional.

3.3.1 Theorem

Let H be a Hilbert space with the inner product (u, v). Let ((v, u)) be a bilinear form of the differential operator of the partial differential equation (3.8) defined for $V \in H,\ u \in H$ and such that there exist constants $k > 0$, $\alpha > 0$ independent of u and v such that $\forall\ V \in H,\ u \in H$;

$$\|((v,u))\| \le k \|v\| \|u\| \quad (4.1)$$

$$((\mathrm{v},\mathrm{v})) \ge \alpha \|v\|^2 \quad (4.2)$$

then every bounded linear functional can be represented in the form

$$Fv = ((v,z)) \quad , \quad v \in H \quad (4.3)$$

where z is an element of the space H uniquely determined by the functional F and

$$\|z\| \le \frac{\|F\|}{\alpha} \quad \text{holds} \quad (4.4)$$

The proof of this theorem is based on Riesz representation theorem which makes it possible to express every bounded linear functional Fv in H as [1,6]

$$FV = (v,t) \quad (4.5)$$

where $\|\mathrm{t}\| = \|F\|$

3.3.2 Definition (3.4) - (V-Ellipticity)

Let the bilinear form ((v, u)) and the space V be given. The form ((v, u)) is called v-elliptic if there exists a constant $\alpha > 0$ such that for
$\forall\ v \in H$ we have

$$((\mathrm{v},\mathrm{v})) \geq \alpha \|v\|^2 \qquad (4.6)$$

The definition of V-ellipticity of the bilinear form ((v, u)) and Lax-Milgram theorem makes it possible to formulate an existence and uniqueness theorem on the weak solution of boundary value problems [6].

3.3.3 Theorem

Let the boundary value problems be as given

$$\sum_{|i||j|} (-1)^{|i|} D^i (a_{ij} D^i u) = \mathrm{f} \qquad (4.7)$$

with bounded measurable coefficient and bounded bilinear form ((v, u)).
let

$$\mathrm{f} \in \mathrm{L}_2(\Omega) \qquad (4.8)$$

Given the operators $B_1,\ ...B_p$ on the individual parts $\partial\Omega_1,\ ...\partial\Omega_p$ of the boundary of Ω; the space

$$V = \{v : V \in W_2^k(\Omega)\ \ ,\ \ \mathrm{B_r\, v} = 0\ \text{ on }\ \delta\Omega\} \qquad (4.9)$$

the functions $g_p(s) \in L_2(\delta\Omega)$ and $\omega \in W_2^k(\Omega)$ (characterizing the stable boundary condition) satisfying

$$B_p w = g_p(s)\ \text{ on }\ \delta\Omega; \qquad (4.10)$$

the function $h_{k-p}(s) \in L_2(\delta\Omega)$ (characterizing the unstable boundary condition); then the function $u \in W_2^k(\Omega)$ is called the weak solution of the boundary value problem given by (4.7) to (4.10).

If the form ((v, u)) is v-elliptic, then the given problem has exactly one weak solution, $u \in W_2^k(\Omega)$ and there exists a positive constant C independent of the given data a_{ij}, f, h_{pl}, w, etc such that

$$\|u\| \leq C\left\{\|f\| + \|w\| + \sum_{p=1}^{r}\sum_{l=1}^{k-p}\|h_{pl}\|\right\} \qquad (4.11)$$

where $\mathrm{u}, \mathrm{w} \in W_2^k(\Omega)$ holds.

5.0 Construction of Weak Solution of Boundary value. Problem with Homogeneous Boundary Conditions

By Boundary value problem, with homogeneous boundary conditions we mean the problem (4.7) – (4.10) for which
$w \in W_2^k(\Omega),\ w = 0\ on\ \delta\Omega,\ h_{pl} \in L_2(\delta\Omega),\ h_{pl} = 0\ on\ \delta\Omega,\ p = 1,...r,\ l = 1,\ldots,\mathrm{k-p}$. Then the weak solution of such problem is a function $u \in W_2^k(\Omega)$ such that

$$u \in V \qquad (5.1)$$

$$((v,u)) = \int_\Omega vf dx\ \ ,\ \ \forall\ v \in V \qquad (5.2)$$

According to the assumption of the existence and uniqueness of weak solution of boundary value problems, that the bilinear form
((u, v)) satisfies the inequalities

$$((u,v)) \le k\|u\|\,\|v\| \tag{5.3}$$

$$v, u \in W_2^k(\Omega) \quad (\text{boundedness})$$

$$((v,u)) \ge \alpha\|v\|^2 \tag{5.4}$$

holds for $\alpha > 0$, $\forall\, v \in V$ (v - ellipticity) and

also symmetric

$$((u,v)) = ((v,u)) \tag{5.5}$$

Then the set of functions $v_1(x)$, $v_2(x)$ …$v_n(x)$ forms a base in the space V with the inner product

$$((u,v)) = ((v,u)) \tag{5.6}$$

where their Gram determinant

$$\begin{vmatrix} ((v_1, v_1)), \ldots, & ((v_1, v_n)) \\ ((v_n, v_1)), \ldots, & ((v_n, v_n)) \end{vmatrix} \ne 0 \tag{5.7}$$

4.2.1 Theorem

Consider the problem (5.1) – (5.2). Let the form ((u, v)) satisfy the assumptions (5.3) – (5.5) and the functions $V_1(x)$, $V_2(x)$ … $V_n(x)$ constitute a base in V. Let the system of equations be formed for all positive integer n

$$\left.\begin{array}{l} ((v,v_1))C_{n1} + \ldots((v,v_n))C_{nn} = \int_\Omega v_1 f dx \\ ------------------------------ \\ ((v_n,v_1))C_{n1} + \ldots + ((v_n,v_n))C_{nn} = \int_\Omega v_n f dx \end{array}\right\} \tag{5.8}$$

in n unknowns $C_{n1}, \ldots, C_{nn}$.

Then the sequence of functions

$$u_n(x) = \sum_{i=1}^{n} C_{ni} v_i(x) \tag{5.9}$$

converges in the space V to the weak solution U(x) of the problem (5.1) – (5.2).

Proof

Since the determinant of equation (5.8) is not equal to zero, (5.8) is uniquely solvable thus C_{ni} ,(i =1, …n) are uniquely determined.

Also we want to show that the unknowns C_{ni} , … $C_{nn}$are equivalent to the condition that the expression

$$\left(\left((u - \sum_{i=1}^{n} C_{ni} v_i \;,\; u = \sum_{i=1}^{n} C_{ni} v_i)\right)\right) \tag{5.10}$$

be minimal among all the expression of the form

$$\left(\left((u - \sum_{i=1}^{n} b_{ni} v_i \;,\; u = \sum_{i=1}^{n} b_{ni} v_i\right)\right) \tag{5.11}$$

where$b_{ni}$are arbitrary real numbers and u(x) be a weak solution of the problem. Expanding (5.11), we have

$$((u,u)) - \sum_{i=1}^{n} b_{ni}((u,v_i)) - \sum_{i=1}^{n} b_{ni}((v_i,u)) + \sum_{i=1}^{n} b_{ni} b_{ni}((v_i,v_i))$$

Since ((v, u)) = ((u, v)) we have

$$((u,u)) - 2\sum_{i=1}^{n} b_{ni}((u,v_i)) + \sum_{i=1}^{n} b_{ni} b_{ni}((v_i,v_i)) \tag{5.12}$$

The expression (5.12) is a quadratic function in n variables b_{ni}. For the expression (5.12) to assume the extreme (minimal) then it is necessary that its partial derivatives with respect to b_{ni} be equal to zero for $b_{ni} = C_{ni}$.

$$-2\sum_{i=1}^{n}((u,v_i))+2\sum_{i=1}^{n}C_{ni}((v_i,v_i))=0 \qquad (5.13)$$

From (5.2) we have

$$((\mathrm{v}_i,\mathrm{u}))=\int_\Omega \mathrm{v}_i\, fdx \qquad (5.14)$$

Putting (5.14) into (5.13) and dividing by 2 we have

$$-\int_\Omega v_i\, fdx+\sum_{i=1}^{n}C_{ni}((v_i,v_i))=0$$

$$\Rightarrow \sum_{i=1}^{n}C_{ni}((v_i,v_i))=\int_\Omega v_i\, fdx \qquad (5.15)$$

This now becomes (5.8). Since (5.8) is uniquely solvable, it follows from the condition that the expression (5.8) be minimal among all expressions of the form (5.9), and then $C_{ni}$are uniquely determined.

Consequently, the conditions (5.6) and (5.8) are equivalent.

From (5.2):

$$((v_i,v_i))\geq \alpha\|v_i\|^2 \qquad (5.16)$$

More over, the function $\mathrm{v}_i(\mathrm{x}),\ \mathrm{v}_2(\mathrm{x})\ldots\mathrm{v}_n(\mathrm{x})$ constitutes a base in the space V; hence the function u(x) may be approximated in this space to an arbitrary accuracy by a suitable linear combination of these functions.

From here and (5.16), it follows that $\forall\ \varepsilon>0\ \exists\ \eta_0$ such that $\forall\ n>\eta_0$, it is possible to find suitable constants bn_isuch that

$$\left(\left(u-\sum_{i=1}^{n}bn_iv_i\ \ ,u-\sum_{i=1}^{n}bn_iv_i\right)\right)<\varepsilon \qquad (5.17)$$

Since the constants C_{ni} are determined by the condition that the expression (5.8) be minimal among all expression of the form (5.9), then

$$\underset{n\to\infty}{Lim}\left(u-\sum_{i=1}^{n}C_{ni}v_i\ \ ,\ u-\sum_{i=1}^{n}C_{ni}v_i\right)=0$$

$$\Rightarrow \underset{n\to\infty}{Lim}\left\|u-\sum_{i=1}^{n}C_{ni}v_i\right\|^2=0 \qquad (5.18)$$

$$\text{but}\sum_{i=1}^{n}C_{ni}v_i=u(x)$$

$$\Rightarrow \underset{n\to\infty}{Lim}\,u_n(x)=u(x) \qquad (5.19)$$

Hence the theorem is proved (Rektorys,1977).

5.0 Summary and Conclusion

Boundary value problems governed by partial differential equations arise in many real-life situations. In most cases the data are not continuous (smooth). Classical solutions become unattainable. This work has helped to handle the failure of the classical theory of partial differential equations by providing solution for boundary value problems whose data are not smooth.

We have shown how such problems can be solved first by reducing them to variational problems. Appropriate function spaces are identified and approximate weak solutions are obtained by direct variational method (Ritz method). We hope that a wide range of engineering problems can now be solved.

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