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A Generalized Laplace Decomposition Method for Differential Equations

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Abstract

In this paper, a new algorithm that generalizes the modified Laplace decomposition method is developed that further facilitate easy computation and improves accuracy. A comparative study between the new algorithm and the modified Laplace decomposition method are carried out with illustrative examples where the efficiency and power of the new techniques are shown for wide classes of problems in mathematical physics. The solutions obtained are better than the existing results.

Keywords: Laplace transform, Adomian polynomial, decomposition, algorithm, accuracy.

1.0 Introduction

In the last few years, so many mathematical methods that are aimed at solving non-linear differential and integral equations arising in engineering has been the subject of extensive analytical and numerical study by many scientist or researchers.However most of them require a tedious analysis or a large computer memory to handle this problem. Also, most of the problems do not have analytic or exact solution in closed form. However, a related phenomena was recently established to facilitate the consequence of the solutions or to make savings in the computational work when applying the decomposition method.

Adomian and Rach [1], Adomian [2], Wazwaz [3]and Wazwaz [4] introduced the noise terms phenomena which was further strengthened byWazwaz[5], where he developed a necessary condition that is essentially needed to ensure the reliability of the noise terms in the non-homogenous equations whenever they appear.

Recently,Khuri [6,7] and Yusufoglu [8] introduced the concept of Laplace Decomposition method which involves a Laplace transformation numerical scheme, based on the decomposition method for solving non-linear differential equations. Recall that Laplace transform is an elementary but useful technique for solving linear ordinary differential equations that is widely used by scientists and engineers for tackling linearisedmodels. In fact, the Laplace transform is only one of the few methods that can be applied to linear systems with periodic or discontinuous driving outputs. Despite its highlighted usefulness above, it cannot handle non-linear problems which necessitated the works in Khuri [7].

This Laplace decomposition method has been used by Yusufoglu [8] to solve Duffing equation, Elgasery[9] applied the technique in solving Falkner-skan equation just to mention a few.

Most recently, a powerful modification of Laplace decomposition was proposed by Hussain and Khan [10]. The new algorithm demonstrates a rapid convergence of the series when compared with the work of Khuri [6]. The modified method has been shown to be computational efficient and accurate in several examples that are vital to the researcher in applied fields.

However, the success of the modified Laplace decomposition method depends on the proper selection of the term for u_0 (initial guess or initial approximation) after splitting the zero component into two parts, which is based mainly on trial criteria only.

In this work, we introduced a new algorithm which extends or generalizes the works of Khuri [5] and Hussain and Khan [10]. Several examples are tested and the results obtained suggest that this newly developed algorithm introduces a promising tool that will be effective in scientific field.

2.0 Laplace Decomposition Method

Consider the general form of second order homogenous differential equation of the form:

$$L[u(t)] + [Ru(t)] + [Nu(t)] = h(t) \qquad (2.1)$$

$$u(0) = f(t),\ u'(t) = g(t) \qquad (2.2)$$

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where$L$ is the second order derivative which is assumed to be easily invertible, R is the linear differential operator of less order than L, Nu represents the non-linear term and $h(t)$ is the source term. The methodology consists of applying Laplace transform to both sides of equation (2.1), we obtain

$$\mathcal{L}L[u(t)] = \mathcal{L}[Ru(t)] + \mathcal{L}[Nu(t)] = \mathcal{L}[h(t)] \quad (2.3)$$

Using the differentiation property of Laplace transform and applying the initial conditions (2.2), we get

$$\mathcal{L}[u(t)] = \frac{f(t)}{s} + \frac{g(t)}{s^2} - \frac{1}{s^2}\mathcal{L}[Ru(t)] - \frac{1}{s^2}\mathcal{L}[Nu(t)] + \frac{1}{s^2}\mathcal{L}[h(t)] \quad (2.4)$$

The next step of the method is to represent the solution $u(t)$ by an infinite series of the form:

$$u(t) = \sum_{n=0}^{\infty} u_n(t) \; (2.5)$$

and decompose the nonlinear form in equation (2.4) as

$$N[u(t)] = \sum_{n=0}^{\infty} A_n \; (2.6)$$

whereA_n are the Adomian polynomials of $u_0, u_1, u_2, \cdots, u_n$ that are given by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{n=0}^{\infty} \lambda^n u_n\right)\right]_{\lambda=0} \quad (2.7)$$

substituting equation (2.5) and equation (2.6) into equation (2.4), we have

$$\mathcal{L}\left[\sum_{n=0}^{\infty} u_n(t)\right] = \frac{f(t)}{s} + \frac{g(t)}{s^2} - \frac{1}{s^2}\mathcal{L}[Ru(t)] - \frac{1}{s^2}\mathcal{L}\left[\sum_{n=9}^{\infty} A_N\right] + \frac{1}{s^2}\mathcal{L}[h(t)] (2.8)$$

equation (2.8) can be rewritten as

$$\mathcal{L}\left[\sum_{n=0}^{\infty} u_n(t)\right] = \frac{f(t)}{s} + \frac{g(t)}{s^2} - \frac{1}{s^2}\mathcal{L}[Ru(t)] - \frac{1}{s^2}\mathcal{L}\left[\sum_{n=0}^{\infty} A_n\right] + \frac{1}{s^2}\mathcal{L}[h(t)] (2.9)$$

On comparing both sides of the equation (2.9), we obtained the recurrence relations

$$\mathcal{L}[u_0(t)] = \frac{f(t)}{s} + \frac{g(t)}{s^2} + \frac{1}{s^2}\mathcal{L}[h(t)] (2.10)$$

$$\mathcal{L}[u_{n+1}(t)] = -\frac{1}{s^2}\mathcal{L}[Ru_n(t)] - \frac{1}{s^2}\mathcal{L}[A_n] \quad n \geq 1 \quad (2.11)$$

Applying the inverse Laplace transform to both sides of equations (2.10 – 2.11), we obtained a recurrence relation

$$u_0(t) = Q(t) (2.12)$$

$$u_{n+1}(t) = -\mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[Ru_n(t)]\right] - \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[A_n]\right], \quad n \geq 0 \quad (2.13)$$

Where $Q(t)$ represents the term arising from source term and prescribed initial conditions.
Evaluating the Laplace transform of the quantities on the right hand side of equations (2.12 – 2.13). Then applying the inverse Laplace transform, we obtain the values of $u_1, u_2, u_3, \cdots$ recursively [6].

3.0 The Modified Laplace Decomposition Method

Hussain and Khan [10], proposed a modification to the assumption made by Khuri [6], Yusufoglu [8]. The modification was based on the assumption that if the zeroth component $u_0(t) = Q(t)$ can be divided into the sum of two parts, namely $Q_0(t)$and $Q_1(t)$, therefore we get $Q(t) = Q_0(t) + Q_1(t)$.
The suggestion was that only the part $Q_0(t)$ be assigned to the zeroth component of u_0 while $Q_1(t)$ is combined with other terms of equation (2.13). Thus, the recursive relation (2.12 – 2.13) becomes

$$u_0(t) = Q_0(t) (2.14)$$

$$u_1(t) = Q_1(t) - \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[Ru_0(t)] + \frac{1}{s^2}\mathcal{L}[A_0]\right] (2.15)$$

$$u_{n+2}(t) = -\mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[Ru_{n+1}(t)] + \frac{1}{s^2}\mathcal{L}[A_{n+1}]\right], \quad n \geq 0 \quad (2.16)$$

Observation shows that the slight reduction in the terms of u_0 reduces the computational work and may give exact solution by using two iterations only without necessarily using the Adomian polynomial. However, the success of the above algorithm depends on proper selection of the function $Q_0(t)$ and $Q_1(t)$ which is based mainly on trial criterias only.

4.0 A new algorithm of Laplace Decomposition Method

In view of the shortcomings highlighted in Hussain and Khan [10], we hereby propose a new algorithm based on replacing the component of $Q(t)$ by a series of infinite components where $Q(t)$ is expressed in Taylor's series expansion

$$Q(t) = \sum_{n=0}^{\infty} Q_n(t) \qquad (2.17)$$

This results into a new recurrence relation

$$u_0(t) = Q_0(t)(2.18)$$

$$u_{n+1}(t) = Q_{n+1}(t) - \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[Ru_n(t)] + \frac{1}{s^2}\mathcal{L}[A_n]\right], \quad n \geq 0 \qquad (2.19)$$

The above new algorithm can be seen as a generalization of equations (2.12 – 2.13) and equations (2.14 – 2.16) respectively. If u_0 contains only one term, the relation (2.18 – 2.19) becomes (2.12 – 2.13) and if u_0 contains only two terms $Q_0$and Q_1, then, equation (2.18 – 2.19) becomes equations (2.14 – 2.16).

5.0 Numerical Examples

To give a clear overview of our study and to illustrate the effectiveness of the new algorithm, we will consider five examples on differential equations.

Example 1:

Consider the nonlinear partial differential equation

$$u_t + uu_x = x + xt^2 \qquad (3.1)$$

$$u(x,0) = 0 \qquad (3.2)$$

Applying the Laplace transform to both side of the equation (3.1), we have

$$s\, u(x,s) - u(x,0) = \mathcal{L}[x + xt^2] - \mathcal{L}[uu_x] \qquad (3.3)$$

Using the initial condition equation (3.2), equation (3.3) becomes

$$u(x,s) = \frac{x^2}{s^2} + \frac{x(2!)}{s^4} - \frac{1}{s}\mathcal{L}[u\, u_x] \qquad (3.4)$$

Taking the inverse Laplace transform of equation (3.4)

$$u(x,t) = xt + x\frac{t^3}{3} - \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}[uu_x]\right]$$

We decompose the solution as an infinite sum given by Khuri [6] and the nonlinear term by Adomian[2].

$$\sum_{n=0}^{\infty} u_n(x,t) = xt + x\frac{t^3}{3} - L^{-1}\left[\frac{1}{s}L\left[\sum_{n=0}^{\infty} A_n\right]\right]$$

By applying the MIDM, i.e. using the recurrence relation (2.14 – 2.16)

$$u_0(x,t) = xt$$

$$u_1(x,t) = x\frac{t^3}{3} - L^{-1}\left[\frac{1}{s}L[u_0 u_x]\right] = 0$$

$$u_{n+1}(x,t) = 0, \qquad n \geq 1.$$

Therefore, using Wazwaz[5],

$$u(x,t) = 0, n \geq 1.$$

REMARK

Observe that if $x\frac{t^3}{3}$ is chosen as $u_0(x,t)$, the result obtained does not converge. This justifies the need for the new algorithm proposed in this paper.

Example 2:

Consider the non-homogenous differential equation

$$u''(t) + u(t) = t \qquad (3.5)$$

$$u(0) = u'(0) = 1 \qquad (3.6)$$

Applying the Laplace transform to both side of the equation (3.5) and using the initial conditions (3.6) just like previous manner, we get

$$\mathcal{L}[u(t)] = \frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^4} - \frac{1}{s^2}\mathcal{L}[u(t)] \qquad (3.7)$$

Taking the inverse Laplace transform of equation (3.7)

$$u(t) = 1 + t + \frac{t^2}{3!} - L^{-1}\left[\frac{1}{s^2}L[u(t)]\right]$$

We decompose the solution as an infinite series as done in the previous example

$$\sum_{n=0}^{\infty} u_n(t) = 1 + t + \frac{t^3}{3!} - L^{-1}\left[\frac{1}{s^2}L[u_n(t)]\right]$$

Now, applying the new algorithm using the recurrence relation (2.18 – 2.19), we have

$$u_0(t) = 1$$
$$u_1(t) = t - L^{-1}\left[\frac{1}{s^2}L[u_0(t)]\right]$$
$$u_2(t) = \frac{t^3}{3!} + L^{-1}\left[\frac{1}{s^2}L[u_1(t)]\right], \qquad n \geq 2.$$

Thus,

$$u_0(t) = 1$$
$$u_1(t) = t - \frac{1}{2}t^2$$
$$u_2(t) = \frac{t^4}{4!}$$
$$u_3(t) = -\frac{1}{6!}t^6$$
$$\vdots = \vdots$$
$$u(t) = \sum_{n=0}^{\infty} u_n(t) = 1 + t - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \frac{1}{6!}t^6 + \cdots$$

Hence, $u(t) = t + cos\, t$ which is the exact solution.

Example 3:

Consider the nonlinear partial differential equation

$$u''(t) + u(t)u'(t) = \frac{1}{2}sin\, 2t \qquad (3.8)$$
$$u(0) = 0, \qquad u'(t) = 1 \qquad (3.9)$$

Applying the Laplace transform to both sides of equation (3.8) and using the initial conditions (3.9) just like in previous examples, we have

$$\mathcal{L}[u(t)] = \frac{1}{s^2} + \frac{1}{s^2(s^2+4)} - \frac{1}{s^2}\mathcal{L}[u(t) + u(t)u'(t)] (3.10)$$

Taking the inverse Laplace transform of both sides of equation (3.10) and after simplifications, we get

$$u(t) = \frac{5t}{4} - \frac{1}{8}sin\, 2t - \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[u(t) + u(t)u'(t)]\right]$$

We decompose the solution as an infinite sum to obtain $u(t) = \sum_{n=0}^{\infty} u_n(t)$ and the nonlinear term $u(t)u'(t)$ is represented by Kaya [11]

$$A_n(u) = u_n u_0' + u_{n-1}u_1' + u_{n-2}u_2' + \cdots u_1 u_{n-1}' + u_0 u_n', \qquad n \geq 0$$
$$u(t) = \sum_{n=0}^{\infty} u_n(t) = \frac{5t}{4} - \frac{1}{3}sin\, 2t - L^{-1}\left[\frac{1}{s^2}L\left[u_n + \sum_{n=0}^{\infty} A_n\right]\right] = 0$$

Based on the suggestion of the new algorithm (2.17), we express $Q(t) = \frac{5t}{4} - \frac{1}{8}sin\, 2t$ in Taylor's series expansion to get

$$Q(t) = t + \frac{1}{6}t^3 - \frac{1}{30}t^5 + \frac{1}{315}t^7 - \frac{1}{5670}t^9 + \cdots$$

And using the recurrence relation (2.18 – 2.19), we get

$$u_0(t) = t$$
$$u_1(t) = \frac{1}{6}t^3 - \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[u_0(t) + A_0]\right] = \frac{1}{6}t^3 - \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[2t]\right] = -\frac{1}{3!}t^3$$
$$u_2(t) = -\frac{1}{30}t^5 - \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[u_1(t) + A_1]\right] = \frac{1}{5!}t^5$$
$$u_3(t) = \frac{1}{315}t^7 - \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[u_2(t) + A_2]\right] = -\frac{1}{7!}t^7$$
$$u_4(t) = \frac{1}{5670}t^9 - \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[u_3(t) + A_3]\right] = \frac{1}{9!}t^9$$

Continuing in the same manner, all other component of the series can be obtained using Maple or Wolfram Mathematica, thus

$$u(t) = t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \frac{1}{7!}t^7 + \cdots$$

and in a closed form is $u(t) = lim_{t\to\infty}\, u_n(t) = sin\, t$ which is the exact solution.

Example 4:

Consider the nonlinear fourth order initial value problem

$u^4(t) = -u(t)u''(t) + u'^2(t)$(3.11)

$$u(0) = 0 \quad (3.12)$$

$$u'(0) = u''(0) = u'''(0) = 1 \quad (3.13)$$

Applying the Laplace transform to both side of the equation (3.11), we have

$$s^4\mathcal{L}\, u = u(0)s^3 - u'(0)s^2 - u''(0)s - u'''(0) = \mathcal{L}[-uu''] + \mathcal{L}[u']^2$$

Using the initial condition equation (3.12) to (3.13),

$$\mathcal{L}\, u = \frac{1}{s^2} + \frac{1}{s^3} + \frac{1}{s^4} + \frac{1}{s^4}\mathcal{L}[-uu''] + \mathcal{L}[u']^2 \quad (3.14)$$

Taking the inverse Laplace transform of equation (3.14), we get

$$u(t) = t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \mathcal{L}^{-1}\left[\frac{1}{s^4}\mathcal{L}[-uu''] + \mathcal{L}[u']^2\right] \quad (3.15)$$

According to the Laplace decomposition method, we assume that a series solution of the unknown function $u(t)$ is given by $u(t) = \sum_{n=0}^{\infty} u_n(t)$ while the nonlinear terms $(u')^2$ and uu'' can be decomposed into the infinite series of polynomials given as

$$(u')^2 = \sum_{n=0}^{\infty} A_n \text{ and } \quad uu'' = \sum_{n=0}^{\infty} B_n$$

where the components of $u(t)$ will be determined recursively and the A_n's, B_n's are the Adomian polynomials of Wazwaz[12]. Few terms of A_n and B_n are given below.

$A_0 = {u_0'}^2$

$A_1 = 2u_0'u_1'$

$A_2 = 2u_0'u_2' + {u_1'}^2$

$A_3 = 2u_0'u_1' + 2u_1'u_2'$

$\vdots$

and

$B_0 = u_0u_0''$

$B_1 = u_0u_1'' + u_1u_0''$

$B_2 = u_0u_2'' + u_1u_1'' + u_2u_0''$

$B_3 = u_0u_3'' + u_1u_2'' + u_2u_1'' + u_3u_0''$

equation (3.15) now becomes

$$\sum_{n=0}^{\infty} u_n(t) = t + \frac{t^2}{2!} + \frac{t^3}{3!} + L^{-1}\left[\frac{1}{s^4}L\left[\sum_{n=0}^{\infty} A_n + \sum_{n=0}^{\infty} B_n\right]\right]$$

Using the new algorithm given in equation (2.18 – 2.19), we get

$u_0(t) = t$

$$u_1(t) = \frac{1}{2!}t^2 - \mathcal{L}^{-1}\left[\frac{1}{s^5}\right] = \frac{t^2}{2!} + \frac{t^4}{4!}$$

$$u_2(t) = \frac{1}{3!}t^3 - \mathcal{L}^{-1}\left[\frac{1}{s^6} - \frac{1}{s^8}\right] = \frac{t^3}{3!} + \frac{t^5}{5!}$$

$$u_3(t) = \frac{1}{6!}t^6 - \mathcal{L}^{-1}\left[\frac{1}{s^4}\mathcal{L}[A_2 - B_2]\right] = \frac{t^6}{6!} + \frac{t^8}{8!}$$

$$u_4(t) = \frac{1}{7!}t^7 - \mathcal{L}^{-1}\left[\frac{1}{s^3}\mathcal{L}[A_3 - B_3]\right] = \frac{t^7}{7!} + \frac{t^9}{9!}$$

Then, putting it together in series gives

$$u(t) = t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \frac{t^6}{6!} + \frac{t^7}{7!} + \frac{t^8}{8!} + \frac{t^9}{9!} + \cdots$$

Hence, $u(t) = lim_{n\to\infty} u_n(t) = e^t - 1$ which can be verified through substitution to be the exact solution of (3.11 – 3.13).

Example 5:

Consider the linear wave (telegraph) equation

$$3u_t(x,t) + u_{tt}(x,t) = u_{xx}(x,t) + 3(1 + x^2 + t^2) \quad (3.16)$$

subject to the initial conditions

$$u(x,0) = x, \qquad u_t(x,0) = 1 + x^2, \qquad x \in \Re \quad (3.17)$$

$$u(0,t) = t + \frac{t^3}{3}, \qquad u_x(0,t) = t, \qquad t \in \Re \quad (3.18)$$

Applying the Laplace transform to both sides of the equation (3.16), we get

$s^2u(x,t) - su(x,0) - u_t(x,0) = \mathcal{L}[3x^2 + 3t^2 + 3] + \mathcal{L}[u_{tt} - 3u_t]$

Using the initial condition equation (3.17)and simplifying, we get

$$u(x,t) = \frac{x}{s} + (1+x^2)\frac{1}{s^2} + \frac{1}{s^2}\left[\frac{3x^2}{s} + \frac{6}{x^3} + \frac{3}{s}\right] + \frac{1}{s^2}\mathcal{L}[u_{tt} - 3u_t] \qquad (3.19)$$

Taking the inverse Laplace transform of both sides of equation (3.19)

$$u(x,t) = \mathcal{L}^{-1}\left[\frac{x}{s}\right] + (1+x^2)\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] + \mathcal{L}^{-1}\left[\frac{1}{s^2}\left(\frac{3x^2}{s} + \frac{6}{x^3} + \frac{3}{s}\right)\right] + \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[u_{tt} - 3u_t]\right]$$

Then applying the new algorithm given in equations (2.18 – 2.19) in a similar manner, we get

$$u_0(t) = x$$
$$u_1(t) = t$$
$$u_2(t) = x^2 t - 3\frac{t^2}{2}$$
$$u_3(t) = \frac{3}{2}t^2 + \frac{11}{6}t^3 - \frac{3}{2}x^2t^2$$
$$u_4(t) = \frac{3}{2}x^2t^2 - \frac{3}{2}t^3 - \frac{13}{8}t^4 + \frac{3}{2}x^2t^3$$
$$u_5(t) = 13\frac{t^4}{8} - \frac{3}{2}x^2t^3 - \frac{9}{8}t^5 - \frac{9}{8}x^2t^4$$
$$u_6(t) = -\frac{9}{8}t^5 - \frac{51}{80}t^6 + \frac{9}{8}x^2t^4 + \frac{27}{40}x^2t^5, \qquad \text{and so on.}$$

Hence, $u(x,t) = u_0(t) + u_1(t) + u_2(t) + u_3(t) + u_4(t) + \cdots = x + t + x^2t + \frac{t^3}{3}$, which can be verified through substitution to be the exact solution of equations (3.16 – 3.18).

6.0 Conclusion

A new algorithm of high accuracy has been developed and demonstrated to be useful in solving linear as well as nonlinear differential equations of different kinds. It is also worth noting that the new algorithm displays a fast convergence of the solution by carefully observing phenomenon of cancelling noise terms shown with some illustrative examples when compared with existing methods like Taylor matrix method, Runge-kutta method, Picard method just to mention a few. The results obtained indicate that the algorithm is efficient and accurate.

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