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On the Nilpotency Class and the Irreducible Representation for Finite Non Abelian Groups Using the Centre

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Abstract

In this paper we use the centre to determine the minimum and maximum nilpotency classand the corresponding irreducible representations for finite non abelian groups. A linear relationship between the centre,nilpotency class and the irreducible representation of non abelian groups was derived.

1.0 Introduction

The definitions and theorems required for the understanding of the topic are covered here. This forms the background for the results in this paper.

1.1 Definition

The centre $Z(G)$ of a group is the set of elements z in $Z(G)$ that commute with every element q in G. That is:

$Z(G) = \{z \in G : zq = qz\}$, for all $q \in G$.

We note that $Z(G)$ is a commutative normal subgroup of G. The quotient of G by $Z(G)$ is isomorphic to the inner automorphism of G.

The group G has trivial centre if $Z(G) = \{e\}$ where e is the identity element of G and the centre is said to be minimum. It is maximum when G is abelian and we have Z(G) = G..We call the elements $x \in Z(G)$ central elements and the elements $y \in G- Z(G)$ non central elements. However our work is based on non abelian groups. That is, where:

$\{e\} < Z(G) < G$.

1.2 Definitions

(i) If $a, q \in G$, we say that a is conjugate to q if there exists an element $g \in G$ such that $gq = ag$. The conjugacy class of a denoted by $C(a)$ is the set of all elements of G that are conjugates to a. That is:

$C(a) = \{g^{-1}ag : g \in G\}$.

We observe that the conjugacy classes of a group are disjoint and the union of all the conjugacy classes forms the group.

(ii) The centralizer $C_G(q)$ of an element q in G is the set of all elements $g \in G$ that commute with q. Equivalently we write:

$C_G(q) = \{g \in G : gq = qg$, for any $q \in G\}$

The index of $C_G(q)$ in G is the size of the conjugacy class $C(q)$ of q in G. That is:

$|C(q)| = |G: C_G(q)|$.

In particular $|C(q)|$ divides $|G|$. If $q \in Z(G)$ then $C_G(q) = G$.

From [1], we have:

1.3 Theorem

If G is finite and H is a subgroup of G then $|H|$ divides $|G|$. More over the number of distinct left cosets of H in G is denoted by $|G:H|$and

$|G:H|= |G|/|H|$.

The next theorem is from [2]

1.4 Theorem

$$|G| = \sum|G: C_G(q_i)| \qquad (i),$$

where the sum runs over the elements from each conjugacy class of G.

From 1.2(ii), equation (i) becomes

$$|G|= |Z(G)| + \sum|G: C_G(q_i)| \qquad (ii)$$

The sum in (ii) runs over q_i from each conjugacy class such that q_i is not an element of $Z(G)$. From 1.2 and equation (ii) above we have:

$$|G|=|Z(G)| + \sum|C(q_i)| \qquad (iii)$$

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1.5Definition

The subgroup $G^{/}$ or [G,G] of a group G generated by the elements of the form $sts^{-1}t^{-1}$, for all s, t ∈ G is called the derived group or commutator subgroup of G. We write $[s,t] = sts^{-1}t^{-1}$ and call this the commutator of s and t. Thus:

$G^{/} = \{ [s,t] : s, t \in G \}$.

The commutator [s,t] is an element of G that measures the failure of the elements s and t to commute. The derived subgroup $G^{/}$ is normal in G and the quotient $G/G^{/}$ is called the abelianization of G which we denote by G_{ab} where ab stands for abelianization. It is the largest abelian quotient of G.

In a group the centre and the commutator subgroups play dual roles. Any subgroup of G that is contained in Z(G) is normal and abelian in G. Since the centre is abelian. While any subgroup of G that contains [G,G] is normal in G since G/[G,G] is abelian .

For a non abelian group a measure of how close the group is to being abelian could be based on how close the centre is to G or how close the commutator subgroup is to the identity. The bigger $G^{/}$ is the less abelian G is.

1.6Definition

A normal series for a group G is a chain of subgroups

$$G = G_0 \supset G_1 \supset ... \supset G_k = \{1\} \text{ -------- (i)}$$

in which $G_{i+1} \lhd G_i$ or

$$\{1\} = G_1 \subset G_2 \subset ... \subset G_k = G \text{ --------- (ii)}$$

In which $G_1 \lhd G_{i+1}$

The quotient groups G_i / G_{i+1} and G_{i+1} / G_i for the normal series (i) and (ii) respectively are called factors of the normal series. The series in (i) is said to be a descending series and the one in (ii) ascending. A normal series is abelian if all the factors are abelian and of prime order. If G admits a normal series whose factors are abelian, so does any subgroup or quotient group of G. Where G and its image $\bar{G}$ admit the same then so is their product. The next theorem characterises the above series based on the characterisation of the commutator subgroups.

From [3], we get that:

1.7Theorem

The series in 1.6(ii) is abelian if and only if [$[G_i, G_i] \subset G_{i+1}$ and the one in 1.1(i) is abelian if and only if $[G_{i+1}, G_{i+1}] \subset G_i$ for all i. Furthermore, (i) and (ii) are called central series (cs) if and only if $[G, G_i] \subset G_{i+1}$ *and* $[G, G_{i+1}] \subset G_i$ respectively. For ascending central series it is required that $[G_i \lhd G]$ is necessary for G / G_i to be a group. Also $G_{i+1} / G \subset Z(G / G_i)$ implies that G_{i+1} / G_i is abelian. The converse is false as abelian subgroups need not lie in Z(G).

1.8 Remark

A normal series is really a 'filtration' of G rather than 'decomposition' of G. This is so as it is a way of filling up G rather than breaking G apart. However, any group that is a direct product of finitely many groups admits a normal series whose factors are (isomorphic to) the subgroups appearing in the direct products. For finite abelian groups, if $G_1 \lhd G_{i+1}$ are successive terms in a composition series, the quotient G_{i+1} / G_i is abelian simple group. Any central series is an abelian series as $G_{i+1} / G \subset Z(G / G_i)$ implies G_{i+1} / G_i is abelian. A normal series can be refined by inserting subgroups between the subgroups already in the series to a point of no repetition. That is to the point when the factors are simple or trivial. Such a series is said to be unrefinable.

1.9Definition

An unrefinable normal series that includes no repetition is called a composition series.Any two composition series of a finite group G have the same length and the composition factors obtained from the series are the same (up to order of occurrence and isomorphism). That is a group determines its composition series or factors (up to isomorphism).

1.10 Definition

The series $G = G^{(0)} \supset G' \supset G'' \supset ... \supset \{1\}$ is called the derived series of G. If it is abelian then $G^{/}=\{1\}$ with the property that $G^{(0)} = G, \quad G^{(1)} = G^{/} = [G, G]$,

$G^2 = G'' = [G', G']$ and $G^{(i+1)} = (G')' = [G^{(i)}, G^{(i)}]$. If $G^{(i)} = G^{(i+1)}$ then $G^{(i)} = G^{(j)}$ for all i>j.

From [3] we have:

1.11 Theorem

If $G = G_0 \supset G_1 \supset ... \supset G_k = \{1\}$ is an abelian series for G then, $G^{(i)} \subset G_i$ for all i. In particular if G has an abelian normal series then the derived series is a normal series.

Proof:

We have $G^{(o)} = G_0 = G$, since G / G_i is abelian (by hypothesis) $G^{'} \subset G_i$. Assume that $G^{(i)} \subset G_i$, since G_i / G_{i+1} is abelian, we have that $G_i^{'} \subset G_{i+1}$ so $G^{(i+1)} = (G^{'})^i \subset G_i{}' \subset G_{i+1}$

Suppose G has a normal series with abelian factor we index the series as in 1.6(i) since normal series have only finitely many subgroups in it. Then for all i, $G^{(i)} \subset G_i$ so $G^{(ii)}$ is trivial for large i since G_i is trivial for large i. Therefore the derived series of G reaches the identity and is a normal series.

1.12Remark

In 1.11 we have that derived series controls the decay of any abelian series from below. It also shows that when G admits an abelian normal series its derived series is its shortest descending series and $G_k = \{1\}$ implies $G^{(k)} = \{1\}$. So no abelian series can reach the identity before the derived series does.If the series in 1.6 is an abelian series for G then $G^{(i)} \subset G_i$ for all i. In particular if G has an abelian normal series then the derived series is a normal series.

Next is a classical definition

1.13 Definition

The ascending series of subgroups:

$\{1\} = Z_0(G) \subset Z_2(G) \subset ... \subset Z_k(G)$ with $Z_1(G) \lhd G$ for all i such that $Z_i(G) \subset Z_{i+1}(G)$ forall i is called upper central series (ucs) of G. If G is abelian then $Z_1(G) = G$ and the union of the UCS is called hypercentre.

If in 1.6, (ii) is a centralseries for G then $G_i \subset Z_i(G)$ for all i. In particular if G has a normal central series (ncs) then the ucs is a normal series and we have that $G_i \subset Z_i(G)$ implies $Z_k(G) = G$.

1.14 Definition

The series of the form $G = G_0 \supset G_i ... \supset \{1\}$ is called lower central series (lcs) of G. It has the property that the G_i are subgroups where:

$$G_0 = G, \quad G_1 = [G,G] = G', \quad G_2 = [G,G_1] = [G_1,G].$$

That is $G_{i+1} = [G,G_i] = [G_i,G], G_i > 0$ and $G_1 / G_{i+1} \subset Z(G/[G_1,G])$.

By this the G_i commute with any element in G and so $G_1 / G_{i+1} \subset Z(G/G_{i+1})$.

Observed that the G_i form a descending central series of G. Inductively, $G_1 \lhd G, G_i \lhd G$ for all i. As with derived subgroups $G^{(i)}$, the G_i might form an ascending central series for G since they may never reach {1}.

The theorem that follows is a collection of the properties of the series in 1.6according to [3].

1.15 Theorem

If 1.6(ii) is central series for G then $G_i \subset Z_i(G)$ for all i particularly if G is an ncs then the ucs is a normal series.

1.16Definition

A group is said to be solvable if it has a chain of subgroups

$\{1\} = G_0 \lhd G_1 \lhd ... \lhd G_n = G$ where each $G_i \lhd G_{i+1}$

with abelian quotient G_{i+1}/G_i. If subgroup H of G isnormalin G then G is solvable if and only if H and G/H are solvable. This implies that G is not solvable if there exist subgroups $H \lhd K \leq G$ where G/H is non abelian. From this we have that solvable groups are diametrically opposed to non abelian simple groups. Equivalently a group G is solvable if it satisfied the following equivalent conditions:

(a) G has an abelian normal series;

(b) $G^{(i)}$ is trivial for some i.

1.17 Remark

The minimum value r such that $G^{(r)}=\{1\}$ is called the derived length or solvable length of G where G is solvable and ris the number of factors in the derived series such that $G^{(r)}$ is trivial. The trivial group is the only group for which r=0. For non trivial abelian groups r=1. However for non abelian group r=2. The commutator subgroup of such groups is abelian so also the centre.

Generally for n>0, G is solvable of length r≤n where there is an abelian subgroup H not necessarily Z(G) such that G/H has solvable length ≤n-1. The derived length of subgroup or quotient group is less than that of the group. However the derived length of a direct product is the maximum length of the factors.

For $G=S_n$, G is solvable if n≤4. If $G=D_{2n}$ G is solvable for n ≥2. In particular G has solvable length of 2 for n>3. We state the next theorem from [3] for reference.

1.18 Theorem

Solvability is closed under subgroup, homomorphic images, quotient groups and direct products. Moreover if N is normal in G then G is solvable if and only if N and G/N are.

1.19 Definition

Let *p* be a prime number. Then *G* is called a*p* - group if the order of *G* is a positive power of *p*.

1.20Definition

A group G is nilpotent if it has a normal series

$\{1\} = G_1 \subset G_2 \subset ... \subset G_k = G$ where $G_i / G_{i+1} \leq Z(G_1 / G_{i+1})$.

This series is called central series of length n. The minimal such n is called nilpotency class of G. Observe from above that nilpotent groups are solvable since G_i / G_{i+1} is abelian. Furthermore nilpotent groups have non trivial centres. Consequently p - groups are nilpotent groups.

From [4] we have the famous Burnside theorem which says that groups of order divisible by atmost two primes are solvable in the next theorem. The details of the proof is given by[5].

1.21Theorem

Every group of order $p^a q^b$ where p and q are distinct primes is solvable.

Proof

If a=0 or b=0 G is solvable. By induction on |G| we need only proof that if a non abelian group of order $p^a q^b$, for a,b>0 contains a proper normal subgroups N≠{1} then N and G/N both have orders of the form $p^a q^b$. Thus N and G/N are solvable by induction hypothesis hence G is solvable.

The conclusion of Burnside's is generally false when the order of G has three prime factors. By Sylow theorems a group G of order $p^a q^b$ has subgroups P and Q of orders p^a and q^b respectively with the property that if $P \cap Q$ is trivial, then G=PQ .

This leads us to the next theorem from [3] which is a generalization of Burnside's $p^a q^b$ theorem using nilpotent subgroups in place of prime power subgroups.

1.22Theorem

If G is a finite group and G=MN where M and N are nilpotent then G is solvable. More generally a finite group G is solvable if and only if there are nilpotent subgroups $N_1, N_2, ..., N_r$ such that $G= N_1 N_2 ... N_r$ with $N_i N_j=N_j N_i$

Next is Feit Thompson's theorem (1963) from [6]. Thisis the deepest result about solvability of finite groups and illustrates the special role of the prime 2 in group theory.

1.23Theorem

Groups of finite odd order are solvable.

1.24 Definition

For prime p if the size of a group G is $p^k m$ with p not dividing m, then subgroups of index p^k are called p sylow complements or sylow complements. This is justified by the fact that if T is a p sylow subgroup and H is a p sylow complement then $T \cap H$ is trivial and the set TH coincides with the whole group.

Thetheorem that follows shows the relationship between solvability and Sylow theorems proved by [3].

1.25Theorem

(i) A finite group is solvable if and only if every Sylow subgroup of G has a complement.
(ii) A finite group is nilpotent if and only if every Sylow subgroup of G has a normal complement.

1.26Theorem

If a finite group satisfies the converse of the Lagrange's theorem then it is solvable.

Proof

Such a group contains p-Sylow complements for every prime p dividing the order of G so the group is soluble from 1.25(i)

1.27Remark

If all the factors of the normal series of a group are abelian then G is abelian. A group satisfying the converse of Lagrange's theorem is called Lagrangian. Every nilpotent group is Lagrangian and the Lagrangian groups lie strictly between nilpotent and soluable finite groups. S_4 is lagrangian but not nilpotent, A_4 is soluble but not lagrangian . In 1.25 if we insist that the complement to the Sylow subgroup in i to be normal than just to exist then it will provide a characteristic of finite nilpotent groups.

1.28 Definition

The chain of subgroups $\{1\} = Z_0(G) \subset Z_1(G) \subset ... \subset ...$ with $Z_0(G) = \{1\}$ is called the acs or ucs of G. Given $Z_i(G)$, then $Z_{i+1}(G)$ is the normal subgroup of G corresponding to the centre of $Z(G)/Z_i(G)$. So that

$Z_{i+1}(G)/Z_i(G)=Z(G/Z_i(G))$. If $Z_i(G)= Z_{i+1}(G)$ then $Z_i(G) = Z_j(G)$ for all j>i. In this case $Z(G)/Z_i(G)$ is trivial.
From [7] and [8] nilpotent groups are characterized as follows.

1.29Definition

Let G be a group then:
(1) G is nilpotent if it is a direct product of its Sylow subgroups,
(2) If G is nilpotent then any proper subgroup of G is properly contained in its normalisers,
(3) If G is a p - group then it is nilpotent.
Consequently G is nilpotent if $Z_i(G)=G$ or $L_i=\{1\}$ for some i>1. That is the lcs reached 1 in a finite number of steps or the ucs reaches G in a finite number of steps. The minimal i for which this occur is called the nilpotency class of G. That is the number of factors in the ucs and lcs. The trivial group is the only group of nilpotency class i=0 while for abelian group i=1. For non abelian group i=2. Such groups satisfy $[G,G] \subset Z(G)$, that is G/Z(G) is non trivial and abelian. If a group has nilpotency class i=n then G/Z(G) has nilpotency i=n-1,for n>0.
We note here that solvable and nilpotent groups are groups in which the ucs (equivalently the lcs) or derived series are actually normal series. That is these subgroups reach the end.Non trivial nilpotent groups must have non trival centres. So $Z_i(G)\neq Z_{i+1}(G)$ as there exists some i such that $Z_i(G)=G$. A group G of order p^k has nilpotency class of k-1 since every $|G| =p^2$ is abelian.
The theorems in 1.30 and 1.31 proved by[8], show that solvability and nilpotency behave well under standard constructions.

1.30Theorem

Nilpotency is closed with respect to subgroups, quotient groups and direct products.

Proof

Applying the view of lcs, if $H \subset G$ and $N \triangleleft G$ then by induction $L_i(H) \subset L_i(G)$ and $L_i(G/N) \subset (NL_i(G))/N$ for all i. For groups G *and* $\tilde{G}$, $L_i(G\times\tilde{G}) = L_i(G)\times L_i(\tilde{G})$ for all i. Therefore if the lcs of G *and* $\tilde{G}$ reach the identity,so does the lcs for any subgroup of G, quotient group of G and $G\times\tilde{G}$.

1.31 Corollary

If $H, K \triangleleft G$ such that G/H and G/K are nilpotent, then so is $G / H \cap K$

Proof

The direct product G/H $\times$G/K is nilpotent from 1.30. The diagonal map G $\rightarrow$ G/H $\times$G/K has kernel H$\cap$K, so G/ H $\cap$K is isomorphic to a subgroup of nilpotent group and thus is nilpotent.
What follows is a corollary from [3].

1.32 Corollary

For a nilpotent group G, with K a normal subgroup of G, then K and G/K are nilpotent.
The converse is not true in general for consider $G=S_3$, $Z(S_3)=1$, so its acs has $Z_i(G)=1$ for all i. Hence it is not nilpotent but A_3 is normal in S_3 and $S_3/A_3=C_2$ are both nilpotent. The group S_3 is the smallest non nilpotent group.It is however, solvable.

1.33Remark

The study of finite nilpotent groups essentially reduces to the study of p -groups which are a rich source of nilpotent groups especially groups of prime power order. The group D_{2n} is nilpotent if and only if 2n is a power of a prime.
The solvability of N and G/N implies solvability of G gives a conceptual role for the class of all solvable groups.
Nilpotency of N and G/N does not implies nilpotency of G in general. However the next theorem from [3] provides a condition under which it holds. That is where nilpotency of N and G/N implies nilpotency of G.

1.34Theorem

Let G be a group,N a normal subgroup of G such that for some i $N \subset Z_i(G)$ where $Z_i(G)$ is a member of the ucs for G. If G/N is nilpotent then G is nilpotent.

Proof

Since $G/Z_i(G)$ is a quotient group of G/N, then from 1.30 $G/Z_i(G)$ is nilpotent. And since
$Z_i(G) \subset Z_{i+1}(G) \subset ... \subset G$. Writing mod $Z_i(G)$ we have:
$Z_i(G)/ Z_i(G) \subset Z_{i+1}(G)/ Z_i(G) \subset Z_{i+2}(G)/ Z_i(G) \subset ... \subset G/ Z_i(G)$.
This is the ucs for $G/ Z_i(G)$. If $G/ Z_i(G)$ is nilpotent we must have $Z_j(G)/ Z_i(G)= G/ Z_i(G)$ for some $j \geq i$, so $Z_j=G$ for some j. Thus G is nilpotent.N is nilpotent since $N\subset Z_i(G)$ where $Z_i(G)$ is nilpotent.
We note that 1.34 is close to the quotient lifting property that is if N and G/N' are nilpotent then G is nilpotent.
What follows is a generalization of the property of finite p - groups to all nilpotent groups according to [1] and [3].

1.35Theorem

If G is a nontrivial nilpotent group then:

(1) For everynontrivial normal subgroup N in G, $N\cap Z(G)\neq\{1\}$ and $[G,N]\neq N$

(2) For every proper subgroup H, $H \neq N_G(H)$.
(3) If G is a nilpotent group and $H \leq G$, with finite order n, then $g^n \in H$ for all g in G.

Nilpotent groups include abelian groups. The next result from [3]shows this and is true for all nilpotent groups.

1.36 Theorem

If G is a nilpotent group then:

(1) The elements of finite order in G form a subgroup of G.
(2) If $H,K \leq G$ and |G:H| is finite then$|K:H \cap K|$ is finite and it divides |G:H|.

[8] gives the next theorem.

1.37Theorem

If G is a nontrivial soluble group then any nontrivial normal subgroup of G contains a nontrivial abelian normal subgroup of G. In particular, if G is a nontrivial soluble group of nonprime size then G contains proper normal subgroup.

1.38Theorem

For a nontrivial finite group G:

(i) G is soluble if and only if any nontrivial quotient of G has a nontrivial abelian normal subgroup.
(ii) G is nilpotent if and only if any nontrivial quotient if G has a nontrivial centre.

Proof

(i) Since quotients of soluble groups are solubleand quotients of nilpotent groups are nilpotent, from 1.18 the 'only if' directions follow from nontrivial soluble groups having nontrivial abelian normal subgroups and nontrivial nilpotent groups having nontrivial centres, (and the finiteness of G is irrelevant).

Conversely suppose every quotient of G has a nontrivial abelian normal subgroup. Then G itself has a nontrivial abelian normal subgroup say G_1. If $G/ G_1$is abelian then G is soluble by the quotient lifting property. If G/ G_1 is nonabelian then G/ G_1 is atleast nontrivial and then has a nontrivial abelian normal subgroup which has the form G_2/ G_1, so $G_2 \neq G_1$ and $G_2 \triangleleft G$. Now we have thenormal series $\{1\} \triangleleft G_1 \triangleleft G_2 \triangleleft G$ where the first and second factors are abelian. If $G/ G_2$is abelian thenG is soluble. If G/ G_2 is nonabelian then it has a nontrivial abelian normal subgroup G_3/ G_2 and we can refine the normal series by inserting G_3. Continuing this procedure eventually leads to $G_i = G$ for $i \geq 0$ and G_i is an abelian normal series of G, so G is soluble.

(ii) Now suppose every nontrivial quotient of G has a nontrivial centre. Then G has a nontrivial centre $Z_1(G)$. If $Z_1(G) = G$ then G is abelian according to 1.35and thus nilpotent. If $Z_1(G) \neq G$ then $G/Z_1(G)$ is nontrivial quotient of G so it has a nontrivial centre which is exactly $Z_2(G)/ Z_1(G)$. As long as $Z_1(G) \neq G$ the quotient $G/Z_i(G)$ has a nontrivial centre so $Z_{i+1}(G) \neq Z_i(G)$. Since G is finite we eventually must have $Z_i(G) = G$ for $i>0$, so G is nilpotent.

Next is a concrete characterization of finite nilpotent groups from [3].

1.39Theorem

A finite group is nilpotent if and only if all if its Sylow subgroups are normal or equivalently the group is isomorphic to the direct product of its Sylow subgroups.

The next theorem from [3] is Carter's 1961 theorem about conjugate subgroupsin soluble groups.

1.40Theorem

Any finite soluble group contains a nilpotent subgroup H such that $N_G(H)=H$, and any two such subgroups are conjugates.

The solvability and nilpotency of a group are related in:

1.41 Theorem

Any nilpotent group is solvable (the converse is not true in general)

Proof

From 1.6, a normal series that is central is abelian. So nilpotency implies solubility.

Alternatively in terms of the special subgroup series we introduce $G^{(i)} \subset L_i$ for all i (1.11). If G is nilpotent then for large i the subgroup L_i is trivial so $G^{(i)}$ is trivial.

The reason any p - group is nilpotent group is that non trivial finite p - groups have nontrivial centre so the ucs has to keep growing until it reaches the whole group. That is suppose that $|G|=p^n>1$. Then $Z_1(G)$ is nontrivial. If some $Z_i(G)$ is nontrivial and not equal to G, then $G/ Z_i(G)$ is a nontrivial finite p - group so its centre is nontrivial. Therefore $Z_{i+1}(G)$ is strictly larger than $Z_i(G)$. This cannot continue indefinitely so some $Z_i(G)$ equals G.

Next is a table from [9] showing the nilpotency of some p- groups being important source of nilpotent groups.

Table 1: Nilpotency class for $|G|= p^n$, with $p=2$, $5 \leq n \leq 7$; for $p= 3$, $5 \leq n \leq 6$

Group order	**Class**	**$Z_i(G)$**	**G_{ab}**
2^5	3	C_2	$C_2 X C_4$
2^6	4	C_2	$C_2 X C_4$
2^7	5	C_2	$C_2 X C_4$
3^5	3	C_3	$C_3 X C_9$
3^6	4	C_3	$C_3 X C_9$

For any non abelian group, the maximum size of the centre is proved by [2] as:

1.42 Theorem

If G is a finite non abelian group, then the maximum possible order of the centre of G is ¼$|G|$.That is, $|Z(G)| \leq 1/4|G|$.

From [10] we have:

1.43 Corollary

If $|G|$ is a power of a prime p then G has a non trivial centre.

Proof

Let G be the union between its centre and the conjugacy classes say J_i of size greater than 1.

Thus from equation (*iii*) of 1.4

$$|G| = |Z(G)| + \sum|C(J_i)|$$

Each conjugacy class J_i has size of a power w say of prime p such that $w \geq 1$. In this case $w = 0$ for classes whose elements are central elements. Since each conjugacy class J_i has size a power of p then $|J_i|$ is divisible by p. Furthermore as p divides |G| it follows that p also divides $|Z(G)|$. Accordingly $Z(G)$ is non- trivial.

What follows is the definition of an important concept..

1.44 Definition

A representation of a group G over a field F is a homomorphism π from G to $GL(n, F)$, the group of n by n invertible matrices with entries in F for some integer n . Here n is the degree of π, and we write: $\pi: G \rightarrow GL(n, F)$. Consequently we say that π is a representation of G if $(aq)\pi = (a)\,\pi(q)\pi$, for all $a, q \in G$.

A representation without proper sub representations is said to be an irreducible representation.

We outline the following from [11] for reference.

1.45 Theorem

(i) Every group of orderp^2 where p is a prime is abelian;

(ii) Let n_i be the degrees of the irreducible representation of G, then $|G|=\sum n_i^2$.

1.46 Proposition

(i) The number of the irreducible representations of any group G is equal to the number of conjugacy classes of G;

(ii) Every irreducible representation of an abelian group G over the set of complex numbers is one dimensional.

Proof

(i) The class functions are determined by their values on the conjugacy classes of G. These are complex vectors spaces. They have dimension equal to the number of conjugacy classes.

But irreducible characters form a basis for the same vector space. Thus the number of conjugacy classes and the number of irreducible characters are the same.

(ii) Since G is an abelian group it has $|G|$ conjugacy classes. From (*i*) above, it shows that the number of irreducible representations of$G$ is |G| and from 1.45 (ii), we have that:

$$|G| = n_1^2 + n_2^2 + ... + n_{|G|}^2.$$

It clearly shows that this can be satisfied only when n_i= 1 for all i.

From [12] we calculate the bounds for the number of irreducible representations of prime degree as follows.

1.47Theorem

Let G be a finite non abelian group of order p^w such that $|Z(G)| =p^t$ with $t<w$, r a prime number, t and w positive integers. Then G:

(i) does not have an irreducible representation of degree p greater than 1 whenever $t = 0$;

(ii) has its minimum number of irreducible representations of degree r greater than 1 whenever $w = 3$.

Proof

(i) If $t = 0$ the centre is trivial. From 1.43 it follows that G cannot have a trivial centre when $w> 1$. Accordingly G has $|G|$ conjugacy classes and $|G|$ irreducible representations each of which according to 1.46 cannot be greater than 1 whenever t = 0.

(ii) Since G is non abelian, there exists an element $g \in G$ such that the degree of $p> 1$ (from 1.46). Let s be the number of irreducible representations of degree r with $s = t<w$. Then from 1.42,

$$|Z(G)| \leq 1/4\, |G|.$$

That is$p^t \leq 1/4\, p^w$. This implies that:

$$4 \leq p^{w-t} \text{ or } 2^2 \leq p^{w-t}.$$

To find the minimum irreducible representation we let p take its minimum value which is 2 (minimum prime). Then we have that $t = w - 2$. It follows that:

$s = t = w - 2$.

However $w \neq 0, 1, 2$ since $t> 0$ and $s> 0$. Accordingly, the minimum value of w is 3 which gives the corresponding minimum values of s and t as $t = 1$, $s = 1$ and $p> 1$.

What follows is atheorem from [12] which measures the bound for the maximum number of irreducible representations.

1.48 Theorem

Let G be a finite non abelian group of order p^w whose centre is of size p^t. Where p is a prime number. Then the maximum number s of irreducible representation of degree r such that $p> 1$ is:

(*i*) $s = w - 1$ where $s = t<w$ and

(*ii*) $w \geq 4$ when $t> 1$

Proof:

(*i*) Now by the definition of *G*, $t \neq w$, else we will be in the abelian environment. But from hypothesis $t<w$, this implies that the maximum value that *t* can attain is *w* - 1, since *t* and *w* are positive integers. Accordingly

$$s = w - 1.$$

(*ii*) In 1.47(*ii*) we have that the least number *s* of irreducible representation of degree $p> 1$ is 1 when the value of *w* is 3 and $t = 1$. But for *t* to be greater than 1, given that *w* must be greater than *t*, then $w \geq 4$.

2.0 Our Results

2.1 Theorem

If G is a finite non abelian group of order p^w and centre of order p^t, where t<w and *p* is prime, then:

The number s of irreducible representation of degree *p* is less than the nilpotency class *d* of G if the value of w is a minimum;

Proof

From the hypothesis, $|G|=p^w$ and $|Z(G)|=p^t$. When $w=0$ then $d=0$,from 1.28. Now from 1.47we have that if $w\leq2$, *G* is abelian . Therefore *G* has nilpotency class $d=1$. In this case $s=0$.

However, *G* is non abelian when $w\geq3$. In this case $d\geq2$. In particular $d=2$ when $w=3$. From the theorem in 1.47(ii), we have $s=w-2$ for min $(w)=3$. Accordingly for min(w), $s = w-2$ and $d = w-1$. That is $d>s$.

2.2 Theorem

For a finite non abelian group with property that $|G|=p^w$ and $|Z(G)|=p^t$ such that $t<w$, given that *t* and *w* are integers and *p* is prime, then the following hold.

(a) If *w* is greater than its minimum value then the number of irreducible representation of degree *p* is greater than the nilpotency class of *G*;

(b) The number of irreducible representations of degree *p* and the nilpotency class of *G* are related by the linear equation $s – d=1$. Equivalently *d*, *s* and *w* are connected by $w=1/2(d+s+3)$

Proof

(a) For $w\geq3$, where min(w)=3, $d\geq2$, from table 1. Specifically, s=w-2 and d=w-1. But for $w\geq4$, s=w-1 from 1.48.However table 1 shows that $d=w-2$. Consequently, $s>d$ whenever $w\geq4$.

(b) From the theorem in 2.1 and (a) above we have

$$S=w-1 \quad \text{(i)}$$

and $$d=w-2 \quad \text{(ii)}$$

equation (i) gives

$$w=s+1 \quad \text{(iii)}$$

and (ii) gives $w=d+2$ (iv).

From equations (iii) and (iv):

$S+1=d+2$. This leads us to $s – d=1$.

Adding equations (iii) and (iv) directly we have : $2w=d+s+3$. Hence $w=1/2(d+s+3)$, as required

3.0 Conclusion

This paper achieved the minimum nilpotency class and the corresponding irreducible representation and maximum nilpotency class and its corresponding irreducible representations for any non abelian group by using the centre. With $w<3$, the number of irreducible representations is less than the nilpotency class. The order of the group is then at most p^3 for a prime *p*. This situation is reversed if $w \geq 4$.For this to occur the order of the group is at least p^4. These results agree with those obtained by Jelten and Momoh [12], where $3\leq w\leq7$ and $4\leq w\leq6$ and $p= 2$ and$p \geq 3$.

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