Journal of the Nigerian Association of Mathematical Physics
Volume27, (July, 2014), pp 587 – 594
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On the Generalized Viscosity Solutions of Fully Nonlinear Parabolic Equations

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Abstract

Consider the fully nonlinear parabolic problem

$$u_t - f\left(D^2u, Du, u, x, t\right) = 0 \quad \in Q = \Omega x \quad]0, T[\quad T > 0,$$

where $u_t = D^2u = \left(\frac{\partial^2 u}{\partial x_i x_j}\right), \left(\frac{\partial u}{\partial x_i}\right)$. Ω ***is a bounded open set in*** R^n ***and n*** $\in Z_+$ ***is a positive integer. It is well known that the classical theory for viscosity solutions does not cover the case where f is discontinuous. This is because the straight forward method of comparing sub and super solutions does not work iff is discontinuous with respect to x and t. In order therefore, to obtain existence and uniqueness results for this class of problems, there is a need to introduce the concept of generalized viscosity solutions where the components of the equations are elements of the space of generalized functions. This is achieved using nonstandard methods involving classical estimates. No linearization off is assumed. We show that our solutions are consistent with the distributional solutions whenever they exist.***

1.0 Introduction

Crandall and Lions [1] developed a very successful method forproving the existence of solutions of nonlinear second order partialdifferential equations. Their method, which does not guaranteesmoothness of the solutions, applies to fully nonlinear equations (in whicheven the second order derivatives can enter in nonlinear fashion).Smoothness results were however obtained by Caffarelli [2] with anextension by Wang [3]. These methods produces, in particularregularity results for a broad range of nonlinear heat equations, includingthe Bellman equation.

$$u_t - \sup_{\alpha \in A}\left[\alpha_{ij}^{\alpha}(x, t)u_{ij} + b_i^{\alpha}(x, t)u_i + c^{\alpha}(x, t)u - g^{\alpha}(x, t)\right] = 0.$$

Thus, great successes have been achieved in the literature using bothclassical and non-classical methods in obtaining existence, uniquenessand regularity results for a large class of parabolic systems in the C^{∞} case. However, if the equations have singular elements, the classicaldistributional theory does not in general guarantee the existence ofsolutions of such problems. It therefore becomes necessary to considerinstead Colombeau algebras which provide a mathematical rigorousframework for simultaneously treating singular or distributional objects,non-linear operations and differentiation while at the same timedisplaying maximal consistency properties with respect to classicaloperations. In this paper, we introduce the concept of generalizedviscosity solutions defined on the space of generalized functionsintroduced in Colombeau [4] and Rosinger [5]. We obtain existence,uniqueness as well as consistence result for the fully nonlinear uniformlyparabolic equation

$$u_t - f\left(D^2u, Du, u, x, t\right) = 0 \quad \in Q = \Omega x \quad]0, T[\quad T > 0, \tag{1.1}$$

where $u_t = \frac{\partial u}{\partial t}, D^2u = \left(\frac{\partial^2 u}{\partial x_i x_j}\right), Du = \left(\frac{\partial u}{\partial x_i}\right)$. Ω is a bounded open set in R^n, n $\in Z_+$ is a positive integer. This equation arises often in chemical flow problems, gas dynamics and other physical processes and is known to have no weak

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solution in the classical distributional sense using the existing concept of Viscosity Solutions as given in Crandel and Lions [1]. Several methods exist in tackling the problem of existence of solutions for parabolic problems in the classical sense. One approach is to write the solutions down explicitly, another is to differentiate the equation to get equations for the derivatives as in Colombeau and Langlais [6]. However these methods do not apply to the general equation we are considering. The basic tools in our method are the methods of compactness and the use of classical estimates. In the sequel, we shall assume *f* to be uniformly elliptic, that is, (1.1) is uniformly parabolic. We briefly state the concept of generalized functions used.

2.0 The Concept of Generalized Functions Used

The need of considering nonlinear PDE's, where either the respective coefficients or the data (initial or boundary conditions) are singular, necessitates the construction of an appropriate differential algebra, since classical linear distribution theory does not permit the treatment of such problems. J. F. Colombeau and others throughout the 1980's provided a rigorous mathematical setting which permits a wide range of nonlinear operations on distributional objects by construction of a differential algebra, the 'Colombeau algebra', which canonically contains the vector space of distributions as a subspace and the space of smooth functions as a faithful subalgebra. This results from the idea of constructing a space of functions for which derivatives as well as nonlinear operations are preserved, thus providing a means of generalization of the classical concept of a differentiable function. As an associative and a commutative algebra, it combines a maximum of favourable differential algebraic properties with a maximum of consistency properties with respect to classical operations. There is however not 'one' Colombeau algebra as there is 'one' space of tempered distributions, but the Colombeau algebra must often be adapted to the problem under consideration, using appropriate asymptotic conditions, which makes it possible to get existence, uniqueness as well as consistence results. However if these are too weak, one cannot draw any useful consequences from the object constructed. We give here a simplified construction of the algebra G(Ω) which permits restriction to subspaces and which depends on a givensystem of coordinates. The definitions given here can be found in Ifidon and Oghre [7]. Other more sophisticated construction can be found in oberguggenberger et al. [8] and diffeomorphism invariant construction has been reported more recently in Grosser et al. [9]

Definition 2.1. Let N_0 denotethe set of natural numbers including zero and D(F) the set of C^∞ functions defined on *R*, vanishing outside a variable compact subset of *R*.
Set

$$A_p(R) = \left\{\varphi \in D(R) \colon \int \varphi(x)\, dx = 1;\ \int x^k \varphi(x) dx = 0\right\}$$

whenever $1 \le k \le p;\ p \in N_0$

and

$$A_p(R^n) = \left\{\prod_{j=1}^{n} : \varphi(x_j);\ \varphi \in A_p(R)\right\}.$$

If $\varphi \in A_p(R^n)$ *and* $\varepsilon \in\ ]0,1[$, *we set* $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x\varepsilon^{-1})$. Denoteby Γ the set of all functions $\gamma : N_0 \to R^+$ such that $\gamma(p) \to +\infty$ whenever $p \to +\infty$.

We denote by E[Ω] the set of the functions of the form

$$u : A_0\ x\,\Omega \to R,\ (\varphi_\varepsilon, x) \to u_\varepsilon(x).$$

Given $u \in E[\Omega]$ and $\varphi \in A_p(R^n)$ we write $u(\varphi, x)$ for the value of $u(\varphi)$ at the point $x \in \Omega$. Observe that $E(\Omega)$ as defined above with point wise multiplication is an algebra. Also if *T*is a distribution then the convolution $(T^\bullet \varphi)$, whenever it is defined, is also a C^∞ function for $\varphi \in A_p(R^n)$ so that one has an imbedding

$$T \to [x \to [\varphi \to (T^\bullet \varphi)(x)]] \tag{2.1}$$

which makes $D'(\Omega)$ a subspace of $E[\Omega]$ which does not contain $C^\infty(\Omega)$ as a subalgebra, since the convolution product of u (u a continuous function on *R*) and $\varphi_3(x) = \frac{1}{\varepsilon^n}\varphi(x\varepsilon^{-1})$ namely $u^*\varphi_\varepsilon$ is not equal to *u* in general, no matter the smallness of ε. In other words, the inclusion of $C^\infty(\Omega)$ into $E[\Omega]$ and the inclusion as a subspace of *C* do not give the same result. In order to make the two inclusions above coherent, we define a subspace $E_M[\Omega]$of moderate elements of $E[\Omega]$as follows.

Definition2.2. A set $E_M[\Omega]$of $E[\Omega]$is called moderate if for every compact subset *K* of Ω and every differential operator $\partial^{\alpha} = \partial_{x_1}^{\alpha_n} \ldots \partial_{x_1}^{\alpha_n}(\alpha_i \in N_0), \alpha = \sum_{i=1}^{n}\alpha_i;$ there exists an element $q \in N_0$ such that for every $\varphi \in A_q(R^n)$there exist *c > 0* and η> 0 such that

$$\sup_{x\in K}\left|\partial^{\alpha}u(\varphi_{\varepsilon}, x)\right| \le c\varepsilon^{-q} \tag{2.2}$$

whenever 0 < ε <η

$E_M[\Omega]$thus defined is a differential algebra for componentwise operations.

Definition 2.3. An ideal $\tilde{N}[\Omega]$of $E_M[\Omega]$is called Null if for every differential operator ∂^{α} , there exist $N \in N_0$ and $\gamma \in \Gamma$such that for all $q \ge N$ and each $\phi \in A_q(R^n)$there exist *c > 0*and η> 0 such that

$$\sup_{x\in K}\left|\partial^{\alpha}u(\phi_{\varepsilon}, x)\right| \le c\varepsilon^{-N} \text{ whenever } 0 < \varepsilon < \eta. \tag{2.3}$$

Observe that the elements in $\tilde{N}[\Omega]$have afaster decay than any power of ε as ε→ 0. For simplicity we shall drop γ in our definition of $\tilde{N}$.

Definition 2.4. Set

$$G(\Omega) = \frac{E_M[\Omega]}{\tilde{N}[\Omega]}.$$

The Colombeau algebra $G(\Omega)$ is thus defined as the quotient of the subspaces of the moderate elements of $E[\Omega]$ with respect to the negligible elements of $E[\Omega]$. Whether $u(\phi_{\varepsilon}, x) \in E[\Omega]$ ismoderate or not is determined by testing the asymptotic behaviour of *u* on scaled test functions $\phi_{\varepsilon}(x) = \frac{1}{\varepsilon^n}\phi(x\varepsilon^{-1})$ as the scaling parameter ε tends to zero; for moderateness we require slow growth $O(\varepsilon^{-N})$ for some fixed *N* for negligibility we required fast decay $O(\varepsilon^{-N})$ for all n in all derivatives on compact sets. A diffeomorphism invariance theory of the algebra is achieved via diffeomorphism invariance of the testing process itself. The algebra $G(\Omega)$ has the following properties: the space of distributions, $D'(\Omega)$ is contained in $G(\Omega)$ through the formula (2.1). furthermore, if $T \in C^{\infty}(\Omega)$ then (2.1) defines the same element in $G(\Omega)$ as the constant embedding $T \to \left[x \to \left[\phi \to (T^*\phi)(x)\right]\right]$. This males $C^{\infty}(\Omega)$ a subalgebra of $G(\Omega)$. The derivation of $G(\Omega)$, defined on representatives by $(\partial x_i\, u)(\phi, x) = \partial x_i\left(u(\phi, x)\right)$extends differentiation on $D'(\Omega)$ by $G(\Omega)$ is finer than the usual topology of $D'(\Omega)$. As a consequence, if $T_n \to T$ in $D'(\Omega)$ for the topology induced by $G(\Omega)$, then $T_n \to T$ or the usual topology of $D'(\Omega)$(Biagioni and Colombeau [10]. Finally, the restriction at the time $t = t_0$ is defined as the class of the map $(\phi^{\otimes n-1}, x) \to (\phi^{\otimes n}, x, t_0)$ if $\phi \in A_q(R^n)$ and if $u(\phi^{\otimes n}, x, t_0)$ is a representative of *u*.

3.0 General Existence and Uniqueness Result

The following definitions are useful.

Definition 3.1. We say a generalized function on Q is a real valued (positive function) if it has a real valued (positive representative.

Definition 3.2. A generalized function $u \in G(Q)$ is called a generalized constant if it has a representative which is constant for each $\varepsilon > 0$.

Definition 3.3. An element $H \in G(\Omega)$ is said to be non-negative if it has a nonnegative representative.

Definition 3.4. An element $H \in G(\Omega)$ with representative $h(\phi_\varepsilon, x, t)$ is said to be of bounded type, if there exists $c > 0, \eta > 0$ such that $\sup_{(x,t)\in K} |h(\varphi_\varepsilon, x, t)| \le c, \varepsilon < \eta$.

Definition 3.5. An element $H \in G(\Omega)$ is said to be of L^P-type, $1 \le p < +\infty$, with respect to the variable (x, t), if it has a representative $h(\varphi_\varepsilon, x, t)$ with the property that $\forall K \subset Q$ compact, $\exists M > 0$ and $N \in Z_+$ such that $\forall \varphi \in A_N(Q), \exists \gamma > 0$ with

$$\sup_{(x,t)\in K} \left\{ \int_Q |h(\varphi_\varepsilon, x, t)|^p dQ \right\}^{\frac{1}{p}} \le M \;\; 0 < \varepsilon < \gamma$$

Definition 3.6. A smooth real valued function $f(M, P, u, x, t)$: $Q \; x \; R^n \; x \; [0, T] \to R$ is said to be uniformly elliptic if there exist a representative $F(M, P, u, x, t)$ of *f* and a positive real number A such that

$$|F(M, P, u, x, t) - F(N, P, u, x, t)| \le A \|M - N\|. \tag{3.1}$$

Definition 3.7. Fix *T > 0* and let $u(x, t): Q_1 = R^n \; x \; [0, T] \to R$ with representative $u(\phi_\varepsilon, x, t) \in G(Q_1)$ be a generalized function. We call *u* a generalized viscosity solution to the system (1.1) if $u(x, .) = u_0$ *and* $\exists \varphi \in D(Q_1)$ with representative $\varphi(\phi_\varepsilon, x, t)$ such that the conditions (3.2), (3.2), (3.4), (3.5) hold.

$$\phi_t(\varphi_\varepsilon, x, t) - \min \left\{ \begin{array}{l} f(D^2\phi(\varphi_\varepsilon, x^+, t^+), D\phi(\varphi_\varepsilon, x^+, t^+), \phi(\varphi_\varepsilon, x^+, t^+), x^+, t^+) \\ f(D^2\phi(\varphi_\varepsilon, x^+, t^+), D\phi(\varphi_\varepsilon, x^+, t^+), \phi(\varphi_\varepsilon, x^+, t^+), x^+, t^+) \end{array} \right\} \le 0 \tag{3.2}$$

whenever

$$\sup(u(\varphi_\varepsilon, x, t) - \phi(\varphi_\varepsilon, x, t)) \in \tilde{N}[Q_1] \tag{3.3}$$

at a point $(x_0, \; t_0) \in Q_1$.

$$\phi_t(\varphi_\varepsilon, x, t) - \max \left\{ \begin{array}{l} f(D^2\phi(\varphi_\varepsilon, x^+, t^+), D\phi(\varphi_\varepsilon, x^+, t^+), \phi(\varphi_\varepsilon, x^+, t^+), x^+, t^+) \\ f(D^2\phi(\varphi_\varepsilon, x^+, t^+), D\phi(\varphi_\varepsilon, x^+, t^+), \phi(\varphi_\varepsilon, x^+, t^+), x^+, t^+) \end{array} \right\} \ge 0 \tag{3.4}$$

whenever

$$\inf(u(\varphi_\varepsilon, x, t) - \phi(\varphi_\varepsilon, x, t)) \in \tilde{N}[Q_1] \tag{3.5}$$

at a point $(x_0, \; t_0) \in Q_1$.

We say $u(x, t) \in G(Q_1)$ is a generalized sub viscosity solution of (1.1) if (3.2) and (3.3) hold and a generalized super viscosity solution of (1.1) if (3.4) and (3.5) hold.

Lemma 1. Assume (3.2), (3.3), (3.4) and (3.5) hold. Let *u* be a viscosity solution of (1.1) taking initial data $u_0 \in D(\Omega)$. Let ϕ be a continuously differentiable test function with representative $\varphi(\phi_\varepsilon, x, t)$. if $\sup(u(\phi_\varepsilon, x, t) - \varphi(\phi_\varepsilon, x, t)) \in \tilde{N}[Q_1]$ at a point (x_0, T) where $x_0 \notin \{x_1, x_2, \ldots x_M\}$, then we may write

$$\phi_t(\varphi_\varepsilon, x_0, T) - F\left(D^2\phi, D\phi, \phi, x_0 \; T^-\right) < 0$$

Proof. Assume that $\sup(u(\varphi_\varepsilon, x, t) - \phi(\varphi_\varepsilon, x, t)) \in \tilde{N}(Q_1)$ at a point (x_0, T), without loss of generality it is plausible to assume that the maximum is strict. Let us define, for α> 0

$$\phi_\alpha\ (\varepsilon, x, t) = \phi(\varphi_\varepsilon, x, t) + \frac{\alpha}{(T-t)}$$

for each α, $\sup(u(\varphi_\varepsilon, x, t) - \phi_\alpha(\varepsilon, x, t)) \in \tilde{N}(Q_1)$ at a point $(x_\alpha\ , t_\alpha)$ such that

$$x_\alpha \to x_0 \ and \ t_\alpha \to T \ as \ \varepsilon \to 0$$

Since *u* is a viscosity solution.

$$\phi_{\alpha,t}(\varepsilon, x_\alpha, t_\alpha) - F(D^2\phi_\alpha, D\phi_\alpha, \phi_\alpha, x_\alpha, t_\alpha) < 0$$

we also have that

$$\phi_a, x(\varepsilon, x, t) = \phi_x(\varphi_\varepsilon, x, t)$$

and

$$\phi_a, x(\varepsilon, x, t) = \phi_x(\varphi_\varepsilon, x, t) + \frac{\alpha}{(T-t)^2}$$

thus

$$\phi_t(\varphi_\varepsilon, x_\alpha, t_\alpha) - F(D^2\phi_\alpha, D\phi_\alpha, \phi_\alpha, x_\alpha, t_\alpha) < 0$$

For *r> 0* define sets in $R^n \ x \ R$ as follows

$$Q_r = \{|x| < r\}\ x\ (-r^2, 0] = B_r\ (-r^2, 0]$$

and

$$\partial_p Q_r = \partial B_r\ x\ (-r^2, 0] \cup B_r\ (-r^2, 0].$$

Theorem 1. There exists a generalized viscosity solution to (1.1) if $u_0 \in G(\Omega)$ with compact support.

Proof. Let $u(x,t) \in C(Q)$ with representative $u(\phi_\varepsilon, x, t)$ to be classical viscosity solution to (1.1) with initial data $u_0(x)$. Let $u_0(\phi_{\varepsilon,}, x, 0):\ ]0,1[\ x\ \Omega \to R$ be a representative of $u_0(x)$, replacing $u_0(\phi_\varepsilon, x, 0)\ by\ \gamma u_0(\phi_\varepsilon, x, 0)$ withγ a C^∞ function on Ω, identical to 1 on a neighborhood of supp $u_0, \gamma \in D(\Omega)$, then $\gamma u_0(\phi_\varepsilon, x, 0)$ is also a representative of u_0. Therefore we may assume $u_0(\phi_\varepsilon, x, 0) = 0$ if x is outside some neighborhood of supp u_0. Now since $u(\varphi_\varepsilon, x, t) \in C(Q)$ is the viscosity solution to (1.1) with initial data $u(\varphi_\varepsilon, x, t)\ at\ t = 0$ we conclude that there is an element $u(\varphi_\varepsilon, x, t) \in E(Q)$ satisfying (1.1) thus from Lemma 1, there exist $\Phi(\phi_\varepsilon, x, t) \in C^2(Q)$ satisfying

$$\Phi_t(\varphi_\varepsilon, x, t) - F(D^2\Phi, D\Phi, \Phi, x, t) < 0, \tag{3.8}$$

$$\Phi(\varphi_\varepsilon, x, 0) = u_0 \tag{3.9}$$

at a point $(x_0, t) \in Q_1$. To show that $u(\phi_\varepsilon, x, t)$ so defined is a generalized viscosity solution to (1.1) we need to show that $\sup(u(\phi_\varepsilon, x, t) - \varphi(\phi_\varepsilon, x, t)) \in \tilde{N}(Q_1)$. Consider

$$w(\varphi_\varepsilon, x, t) = u(\varphi_\varepsilon, x, t) - \Phi(\varphi_\varepsilon, x, t) - \alpha t \tag{3.10}$$

where $\alpha > 0$ is a constant. Thus

$$w_t(\varphi_\varepsilon, x, t) = u_t(\varphi_\varepsilon, x, t) - \Phi_t(\varphi_\varepsilon, x, t) - \alpha. \tag{3.11}$$

Therefore $w(\phi_\varepsilon, x, t)$ satisfies an equation of the form

$$w_t(\varphi_\varepsilon, x, t) \le F(D^2\Phi + D^2 w\ D\Phi, Dw, \Phi + w, x, t) - F(D^2\Phi + D\Phi, \Phi + x, t) + \alpha$$

since (1.1) is uniformly parabolic we have

$$\left|w_t(\varphi_\varepsilon, x, t)\right| \le A\left\|D^2 w\right\| + \left|F(M, P+q, N, x, t) - F(M, P\ N, x, t)\right| + \left|F(M, P\ N, x, t) - F(M, P+q, N, x, t)\right| + |\alpha|$$

taking limits as q → 0 and expanding the second term on the right hand side in a Taylor series about q, we have

$$\left|w_t(\varphi_\varepsilon, x, t)\right| \le A\left\|D^2 w\right\| + (|M|+1)\, S_h(x,t) + |\alpha| + \frac{|q|^2}{2} \sup_{|p-\zeta|>|q|} \left\|\frac{\partial^2}{\partial p^2} F(M, \zeta, M, x, t)\right\|.$$

where

$$S_h(x, t) = \sup_{|q|\le h} \frac{\left|F(M, P+q, N, x, t) - F(M, P\ N, x, t)\right|}{|M|+1}$$

Therefore

$$\left|w_t(\varphi_\varepsilon, x, t)\right| \le A\left\|D^2 w\right\| + (|M|+1) S_h(x,t) + |\alpha| + B|q|^2,$$

where B does not depend on $\phi \in A_p$ provided *p* is large enough (and $\varepsilon > 0$ is small enough depending on ϕ) . Following arguments in Colombeau and Langlais [6] we have that for $u \in C^{2,\alpha}(Q_r)$ and any integer *k* there exists a polynomial ρ_k such that

$$\left\|w_t(\varphi_\varepsilon, x, t)\right\|_c 2k, \alpha_{(Q_r)} \le \rho_k(|M|+1)\left\|S_h(x,t)\right\|_c 2k+1_{(Q_r)} + B|q|^2.$$

If we chooseα appropriately, then following Wang [3] and the explicit bounds defining $S_h(x,t)$ we can write

$$\left\|w_t(\varphi_\varepsilon, x, t)\right\|_c 2k, \alpha_{(Q_r)} \le \rho_k\left(C(|M|+1)r^\alpha + B|q|^2\right)$$

where α is a constant depending on *A*, *B*. Since *u* is a viscosity solution to (1.1) we have that $q \in \tilde{N}(Q_r) \Rightarrow B|q|^2 \in \tilde{N}(Q_r)$.Also $\exists \eta > 0$ such that

$$C(|M|+1)r^\alpha \in \tilde{N}(Q_r)\ for < r < \varepsilon < \eta.$$

Similarly if $\Phi(\phi_\varepsilon, x, t) \in C^2(Q)$ satisfying

$$\Phi_t(\varphi_\varepsilon, x, t) - F(D^2\Phi, D\Phi, \Phi, x, t) = 0$$

$$\Phi(\varphi_\varepsilon, x, 0) = u_0$$

we can show that $\left(u(\phi_\varepsilon, x, t) - \varphi(\phi_\varepsilon, x, t)\right) \in \tilde{N}(Q_1)$.

Theorem 2. The solution of (1.1) is unique.

Proof. Let $u, v \in G(Q)$ with representatives $u(\phi_\varepsilon, x, t)$ and $u(\phi_\varepsilon, x, t)$ respectively, be two generalized viscosity solutions to (1.1) satisfying the same initial conditions, then their difference satisfies

$$\left(u(\varphi_\varepsilon, x, t) - v(\varphi_\varepsilon, x, t)\right)_t - \left(F(D^2u, Du, u, x, t) - D(D^2v, Dv, v, x, t)\right) = g(\varphi_\varepsilon, x, t)$$

where $g \in \tilde{N}(Q)$.

We have the estimate

$$\left|D^2u(\varphi_\varepsilon,\ x, t)\right|^p + \left|D^2v(\varphi_\varepsilon,\ x, t)\right|^p \le c\left(\int_Q \left|g(\varphi_\varepsilon,\ x, t)\right|^p dxdt + 1\right)$$

$$+ \int_0^t \left|F(D^2u, Du, u, x, t)\right|^p dt + \int_0^t \left|F(D^2v, Dv, v, x, t)\right|^p dt$$

where c is a constant depending on g. *p > n + 1*

Now, $\Phi(\varphi_\varepsilon, x, t)$ satisfies

$$u_t(\varphi_\varepsilon, x, t) - F(D^2u, Du, u, x, t) \quad = \quad h(\varphi_\varepsilon, x, t) \tag{3.19}$$

and $v(\phi_\varepsilon, x, t)$ satisfies

and $\upsilon(\varphi_\varepsilon, x, t) - F(D^2v, Dv, v, x, t) = s(\varphi_\varepsilon, x, t)$, (3.20)

where $h \in \tilde{N}[Q]$ *and* $\delta \in \tilde{N}[Q]$.

Integrating (3.21)

$$|u(\varphi_\varepsilon, x, t)|^p \le |u_0(\varphi_\varepsilon, x, t)|^p + \int_0^t |F(D^2u, Du, u, x, t)|^p dt + \int_0^t |h(\varphi_\varepsilon, x, t)|^p dt. \tag{3.21}$$

$$|v(\varphi_\varepsilon, x, t)|^p \le |v_0(\varphi_\varepsilon, x)|^p + \int_0^t |F(D^2v, Dv, v, x, t)|^p dt + \int_0^t |\delta(\varphi_\varepsilon, x, t)|^p dt \tag{3.22}$$

(3.21) + (3.22) yields

$$|u(\varphi_\varepsilon, x, t)|^p + |v(\varphi_\varepsilon, x, t)|^p \le |u_0(\varphi_\varepsilon, x)|^p + |v_0(\varphi_\varepsilon, x)|^p$$
$$+ \int_0^t |F(D^2u, Du, u, x, t)|^p dt$$
$$+ \int_0^t |F(D^2v, Dv, v, x, t)|^p dt + \int_0^t |h(\varphi_\varepsilon, x, t)|^p dt + \int_0^t |s(\varphi_\varepsilon, x, t)|^p dt.$$

Therefore

$$|u(\varphi_\varepsilon, x, t) - v(\varphi_\varepsilon, x, t)|^p \le |u_0(\varphi_\varepsilon, x) - v_0(\varphi_\varepsilon, x)|^p + \int_0^t |F(D^2u, Du, u, x, t)|^p dt$$
$$+ \int_0^t |F(D^2v, Dv, v, x, t)|^p dt + \int_0^t |h(\varphi_\varepsilon, x, t)|^p dt + \int_0^t |s(\varphi_\varepsilon, x, t)|^p dt$$

Using this in (3.18) gives

$$|D^2u(\varphi_\varepsilon, x, t)|^p + |D^2v(\varphi_\varepsilon, x, t)|^p \le |u_0(\varphi_\varepsilon, x) - v_0(\varphi_\varepsilon, x)|^p$$

$$|u(\varphi_\varepsilon, x, t) - v(\varphi_\varepsilon, x, t)|^p$$
$$+ c\left(\int_0^t |g(\varphi_\varepsilon, x, t)|^p dxdt + 1\right) + \int_0^t |h(\varphi_\varepsilon, x, t)|^p dt + \int_0^t |s(\varphi_\varepsilon, x, t)|^p dt.$$

Since u is a generalized viscosity solution of (1.1) $\exists \Phi \in D(Q)$ such that

$$\sup(u(\varphi_\varepsilon, x, t) - \Phi(\varphi_\varepsilon, x, t)) \in \tilde{N}(Q)$$

Therefore

$$\|u(\varphi_\varepsilon, x, t) - v(\varphi_\varepsilon, x, t)\|_{W^{2,p}} \le |u_0(\varphi_\varepsilon, x) - v_0(\varphi_\varepsilon, x)|_{L^p(\Omega)}$$
$$+ \|u(\varphi_\varepsilon, x, t) - \Phi(\varphi_\varepsilon, x, t)\|_{L^p(\Omega)}$$
$$+ \|v(\varphi_\varepsilon, x, t) - \Phi(\varphi_\varepsilon, x, t)\|_{L^p(\Omega)} + c\left(\int_Q |g(\varphi_\varepsilon, x, t)|^p dxdt + 1\right)$$
$$+ \delta(\varepsilon, p, A, B)$$

therefore

$$\left\| u(\varphi_\varepsilon, x, t) - v(\varphi_\varepsilon, x, t) \right\|_{W^{2,p}} \to 0 \; as \; \varepsilon \to 0$$

Since

$$\left(u_0(\varphi_\varepsilon, x) - v_0(\varphi_\varepsilon, x)\right) \in \tilde{N}[Q] \; and \; \delta(\varepsilon, p, A, B) \in \tilde{N}[Q].$$

4.0 Coherence of the Solution

Uniqueness as defined in *G(Q)* does not imply uniqueness in the distributional sense since the equality in *G(Q)* is too strong a way for defining equality in *D'(Q)*. In order to make the concept of equality in *G(Q)* coherent with the equality in *D'(Q*. Thus two elements $u, v \in G(Q)$ with representatives $u(\varphi_\varepsilon, x, t) - v(\varphi_\varepsilon, x, t)$ are said to be associated, if for every test function $\Psi \in D(G)$

$\int_Q \left(u(\varphi_\varepsilon, x, t) - \upsilon(\varphi_\varepsilon, x, t)\right) \Psi(x, t)\, dxdt \to 0 \; as \; \varepsilon \to 0$. We may also have association of elements in *G(Q)*with their corresponding elements in *D'(Q)*. See Oberguggenberge [14].

Coherence with the distribution solution can be achieved through Theorem 2 and Definition (3.3) and (3.5). This ensures that the corresponding solutions to (1.1) are indeed associated in *G(Q)*.

Theorem 3. Let $u_0(\phi_\varepsilon, x)$ be an elements in $G(\Omega)$ with compact support, which admits an element $w_0(x) \in D'(\Omega)$ with representative $w_0^*(\phi^{\otimes n-1}) \to w_0(\phi_\varepsilon, x)$ as an associated distribution, then the corresponding solutions of (1.1) are associated with each other in *G(Q)*.

Proof. Choose $\upsilon \in C^2(Q)$ to be the classical solution to the parabolic problem

$$\upsilon_t - f\left(D^2\upsilon, 0, 0, 0, 0\right) = 0, \quad \upsilon\big|_{Q_{\frac{1}{2}}} = u,$$

Where *u* is the classical $W^{2,p}$ viscosity solution of (1.1), then following Wang [16] $u \to \upsilon$ in *C*. By Theorem 1, *u* is moderate and has representative $u(\varphi_\varepsilon, x, t)$ in *G(Q)*. By continuous dependence we have

$$u(\varphi_\varepsilon, x, t) \to \upsilon \quad \text{in } C.$$

On the other hand $w^*\left(\varphi^{\otimes n}, x, t\right) \to \upsilon$ in *C*.

Therefore, by Definitions (3.3) and (3.6) we have

$$\left(u(\varphi_\varepsilon, x, t) \to w^*\left(\varphi^{\otimes n}, x, t\right)\right) \in \tilde{N}[Q].$$

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