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Review of the Effect of a Central-Force Problem on Planetary Motion

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Abstract

This paper discussed the questions that people usually asked about the night sky. Why doesn't the moon fall to the earth? Why do the planets move across the sky? Why doesn't the earth fly off into space rather than remaining in orbit around the sun? The study of the effects of central-force problem on planetary motion, led us to understand that a satellite or planet would tend to go off in a straight line if no central force were applied to it. A central force makes the satellite or planet deviate from a straight line and orbit earth or sun. This attractive force (Central-Force) is the gravitational force between earth and satellite, planet and sun. Indeed, central force is the most important force on the scale of planets, stars, and galaxies. It is responsible for holding our earth to together and for keeping the planets in orbit about the sun, since many naturally occurring forces are central. Examples include gravity and electromagnetism as described by Newton's Law of universal gravitation and coulomb's law respectively.

Keywords: Central-force, Gravitational force, Orbit, Planet

1.0 Introduction

In classical mechanics, the central-force motion problem is believed to determine the motion of a particle under the influence of a single central-force. The word "planet" comes from a Greek word meaning "wanderer" and indeed the planets continuously change their positions in the sky relative to the background of stars. One of the great intellectual accomplishments of the 16^{th} and 17^{th} centuries was the threefold realization that the earth is also a planet, that all planets orbit the sun, and that the apparent motions of the planets as seen from the earth can be used to precisely determine their orbits.
The first and second of these ideas were published by Nicolaus Copernicus in Poland in 1543. The nature of planetary orbits was deduced between 1601 and 1619 by the German astronomer and mathematician Johannes Kepler using a voluminous set of precise data on apparent planetary motions compiled by, his mentor, the Danish astronomer Tycho Brahe. By trial and error, Kepler discovered three empirical laws that accurately described the motions of the planets.
Kepler did not know why the planets moved in this way. Three generation later, when Newton turned his attention to the motion of the planets, he discovered that each of Kepler's laws can be derived and in fact, they are consequences of Newton's laws of motion and Newton's law of gravitation [1].

2.0 Equations of Motion of the Two-Body Problem

In the following discussion we shall assumed that as a planet or comet orbits the sun, the sun remains absolutely stationary. Of course, this can't be correct, because just as the sun of mass m_1 located at position vector $r_1$exerts a gravitational force on the planet, the planet of mass m_2 located at position vector $r_2$alsoexerts a gravitational force on the sun of the same magnitude but opposite direction, $f_{12} = -f_{21}$, that is Newton's third law. Suppose that there are no other forces in the problem. The equations of motion of two-body problem are thus [2]

$$m_1 \frac{d^2 r_1}{dt^2} = -f, \quad (1)$$

$$m_2 \frac{d^2 r_2}{dt^2} = f, \quad (2)$$

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Where $f = f_{21}$
Now, the center of mass of our system is located at

$$r_{cm} = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2} \qquad (3)$$

Hence, we can write

$$r_1 = r_{cm} - \frac{m_2}{m_1 + m_2} r, \qquad (4)$$

$$r_2 = r_{cm} + \frac{m_1}{m_1 + m_2} r, \qquad (5)$$

Where $r = r_2 - r_1$. Substituting the above two equations (4) and (5) into equation (1) and (2), and making the use of the fact that the center of mass of an isolated system does not accelerate, we find that both equations (1) and (2) yield

$$\mu \frac{d^2 r}{dt^2} = f, \qquad (6)$$

Where

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \qquad (7)$$

is called the reduced mass. Hence, we have effectively converted our original two-body problem into an equivalent one-body problem. In the equivalent problem, the force *f* is the same as that acting on both objects in the original problem (modulo a minus sign). However, the mass, μ, is different, and is less than either of m_1 or m_2 (which is why μ is called the "reduced" mass). We conclude that the dynamics of an isolated system consisting of two interacting point objects can always be reduced to that of an equivalent system consisting of a single point object moving in a fixed potential.

3.0 Conservation Theorems-First Integral of the Motion

The system which we wish to discuss may be considered to consist of a particle of mass μ which moves in a central-force field described by the potential function *U(r).* Since the potential energy does not depend on the orientation, the system possesses spherical symmetry.
Thus Kepler's second law-that sector velocity is constant-means that angular momentum is constant. It is easy to see why the angular momentum of the planet must be constant. According to $\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = \vec{\tau}$, the rate of change of $\vec{L}$ equals the torque of the gravitational force $\vec{F}$ acting on the planet:
In our situation, $\vec{r}$ is the vector from the sun to the planet, and the force $\vec{F}$ is directed from the planet to the sun. So this vector always lie along the same line, and its vector product $\vec{r} \times \vec{F}$ is zero.
Hence, $d\vec{L}/dt = 0$, a conclusion that does not depend on the $1/_{r^2}$ behavior of the force; angular momentum is conserved for any force that acts always along the line joining the particles to a fixed point. Such a force is called a central force.
The Lagrangian for such a system may be written as

$$L = \frac{1}{2} m_1 |\dot{r}_1|^2 + \frac{1}{2} m_2 |\dot{r}_2|^2 - U(r) \qquad (8)$$

Where m_1 *and* m_2 are the masses of two objects while $\dot{r}_1$ and $\dot{r}_2$ are the derivatives of position vectors with respect to time.

If we Substitute equations (4) and (5) into the equation (8) we get,

$$L = \frac{1}{2} \mu |\dot{r}|^2 - U(r) \qquad (9)$$

The Lagrangian, equation (9), and the total Energy E, may be expressed in plane polar coordinates as,

$$L = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - U(r)$$
$$E = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + U(r) \qquad (10)$$

We may substitute $\dot{\theta} = \frac{l}{\mu r^2}$ into equation (10) and obtain the expression for $\dot{r}$ as,

$$\dot{r} = \pm\sqrt{\frac{2}{\mu}(E - U(r)) - \frac{l^2}{\mu^2 r^2}} \qquad (11)$$

Integrating equation (11) we have,

$$\theta(r) = \int \frac{\left(\frac{l}{r^2}\right) dr}{\sqrt{2\mu\left(E - U(r) - \frac{l^2}{2\mu r^2}\right)}} \qquad (12)$$

where we have used the fact that $\dot{\theta} = \frac{d\theta}{dr}\cdot\frac{dr}{dt} \equiv \frac{l}{\mu r^2}$ *and hence* $d\theta = \left(\frac{l}{\mu r^2}\right)\frac{dr}{\dot{r}}$

4.0 Orbits in a Central Field

In physics, an orbit is the gravitationally curved path of one object around a point or another body, for example, the gravitational orbit of a planet around a star. The radial velocity of a particle moving in a central field is given by equation (11). This equation indicates that $\dot{r}$ will vanish at the roots of the radical, i.e. at points for which equation (11) becomes,

$$E - U(r) - \frac{l^2}{2\mu r^2} = 0 \qquad (13)$$

We May compute from equation (12) the change in the angle θ which results from one complete transit of r from r_{min} to r_{max} and back to r_{min}. Since the motion is symmetrical in time, this angular change is twice that which would result from the passage from r_{min} to r_{max}, thus [3]

$$\Delta\theta = 2\int_{r_{min}}^{r_{max}} \frac{l dr/r^2}{\sqrt{2\mu\left(E - U(r) - \frac{l^2}{2\mu r^2}\right)}} \qquad (14)$$

5.0 Centrifugal Energy and the Effective Potential

In many situations relativistic effects can neglected, and Newton's law give a highly accurate description of the motion. Then the gravitational force between each pair of bodies is proportional to the product of their masses and decreases inversely with the square of the distance between them. To this Newtonian approximation, for a system of two point masses or spherical bodies, only influenced by their mutual gravitation (the two-body problem), the orbit can be exactly calculated. In the expression above for $\dot{r}, \Delta\theta$, etc, a common term is the radical [3],

$$\sqrt{E - U(r) - \frac{l^2}{2\mu r^2}} = radical \qquad (15)$$

The last term in the radical has the dimensions of energy, and since $\dot{\theta} = \frac{l}{\mu r^2}$, can also be written as

$$\frac{l^2}{2\mu r^2} = \frac{1}{2}\mu r^2\dot{\theta}^2 \qquad (15a)$$

If we interpret this quantity as a "potential energy" and denote it by U_c, then it is given by,

$$U_c \equiv \frac{l^2}{2\mu r^2} \qquad (16)$$

Then the "force" that must be associated with U_c is

$$f_c = -\frac{\partial U_c}{\partial r} = \frac{l^2}{\mu r^3} = \mu r\dot{\theta}^2$$

Thus, we see that the term $\frac{l^2}{2\mu r^2}$of equation (13) can be interpreted as the centrifugal potential energy of the particle, and as such, can be included with $U(r)$ in an effective potential defined by

$$V(r) \equiv U(r) + \frac{l^2}{2\mu r^2}$$

Therefore, the effective potential function for gravitational attraction is

$$V(r) = -\frac{k}{r} + \frac{l^2}{2\mu r^2} \qquad (17)$$

6.0 Planetary Motion and Kepler's Problem

In astronomy, Kepler's three laws of planetary motion are:

i. "The orbit of every planet is an ellipse with the sun at a focus".

ii. "A line joining a planet and the sun sweeps out equal areas during equal intervals of time".

iii. "The square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit".

These three mathematical laws were discovered by German Mathematician and astronomer Johannes Kepler (1517-1630), and used by him to describe the motion of planets in the solar system. They describe the motion of any two bodies orbiting each other. Planetary system; planets, dwarf planets, asteroids (minor planets), comets, and space debris orbit the central star in elliptical orbits.

A comet in a parabolic or hyperbolic orbit about a central star is not gravitationally bound to the star and therefore is not considered part of the star's planetary system. To date, no comet has been observed in our solar system with a distinctly hyperbolic orbit. Bodies which are gravitationally bound to one of the planets in a planetary system, either natural or artificial satellites, follow orbits about that planet.

The equation for the path of a particle moving under the influence of a central force whose magnitude is inversely proportional to the square of the distance between the particle and the force can be obtained from equation (12),

$$\theta(r) = \int \frac{\left(\frac{l}{r^2}\right) dr}{\sqrt{2\mu\left(E - \frac{k}{r} - \frac{l^2}{2\mu r^2}\right)}} \qquad (18)$$

The integral can be easily evaluated if the variable is changed to $\mu \equiv \frac{1}{r}$. If the origin of θ is defined so that integration constant is zero, we find

$$\cos\theta = \frac{\frac{l^2}{\mu k}\frac{1}{r} - 1}{\sqrt{1 + \frac{2El^2}{\mu k^2}}} \qquad (19)$$

Let us now define the following constants:

$$\propto \equiv \frac{l^2}{\mu k}$$

$$\varepsilon \equiv \sqrt{1 + \frac{2El^2}{\mu k^2}}$$

This can be written as

$$\frac{\propto}{r} = 1 + \varepsilon\cos\theta \qquad (20)$$

This is the equation for a conic section with one focus at the origin, the quantity ε is called eccentricity and 2α is termed the latus rectum of the orbit.

The minimum value for r occurs when cosθ is a maximum, i.e., for θ = 0. Thus, the choice of zero for the constant in equation (20) corresponds to measuring θ fromr_{min}, which position is called the peri-center; r_{max} corresponds to the apo-center. The general term for turning points is apsides.

For the case of planetary motion, the orbits are ellipses with major and minor axes (a and b; respectively) given by

$$a = \frac{\propto}{1-\varepsilon^2} = \frac{k}{2|E|} \qquad (21)$$

$$b = \frac{\propto}{\sqrt{1-\varepsilon^2}} = \frac{l}{\sqrt{2\mu|E|}} \qquad (22)$$

The major axis depends only on the energy of the particle, whereas the minor axis is a function of both first integral of the motion, $E$and l. The geometry of elliptic orbits in terms of the parameters ∝, ε, a, b is shown in Figure where $P\ and\ P'$ are the foci. From this diagram, we see that the apsidal distances (r_{min} and r_{max} as measured from foci to the orbit) are given by:

$$r_{min} = a(1-\varepsilon) = \frac{\propto}{1+E} \qquad (23)$$

$$r_{max} = a(1+\varepsilon) = \frac{\propto}{1-E} \qquad (24)$$

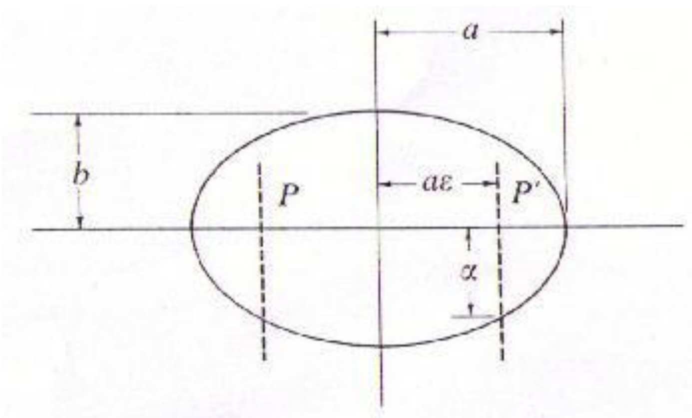

Fig. 1: Geometry of ellipse

In order to find the period for elliptic motion, rewrite equation (15a) for the areal velocity as

$$dt = \frac{2\mu}{l}dA$$

Since the entire area A of the ellipse is swept out in one complete period τ

$$\int_0^\tau dt = \frac{2\mu}{l}\int_0^A dA$$

$$\tau = \frac{2\mu}{l}A \qquad (25)$$

Now the area of an ellipse is given by A = πab, and using a and b from equations (21) and (22), we find

$$\tau = \frac{2\mu}{l}\pi ab = \frac{2\mu}{l}\pi\frac{k}{2|E|}\frac{l}{\sqrt{2\mu|E|}}$$

$$= \pi k\sqrt{\frac{\mu}{2}} \cdot |E|^{-3/2}$$

We also note that the minor axis may be written as

$b = \sqrt{\alpha a}$

Therefore, since $\alpha = \frac{l^2}{\mu k}$, the period τ may also be expressed as

$$\tau^2 = \frac{4\pi^2\mu}{k}a^3 \qquad (26)$$

This result, that the square of the period is proportional to the cube of the major axis of the elliptic orbit, is known as Kepler's third law [4].

7.0 Kepler's Equation

In classical mechanics, Kepler's problem is a special case of the two-body problem, in which the two bodies interact by a central force F that varies in strength as the inverse square of the distance r between them. The force may be either attractive or repulsive. The "problem" to be solved is to find the position or speed of the two bodies over time given their masses and initial positions and velocities. Using classical mechanics, the solution can be expressed as a Kepler orbit using six orbital elements. Figure 4 shows Kepler's construction. The motion takes place in the elliptical orbit with the force centre located at the focus 0 which is also the origin for a rectangular coordinate system. In this system the equation of the orbit is

$$\frac{(x + a\varepsilon)^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (27)$$

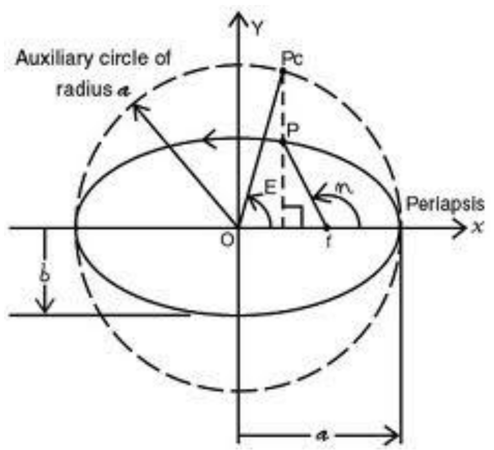

Fig. 2: Kepler's construction

Next, we circumscribe the ellipse with a circle of radius, a, and project the point, p, onto the circle at point, pc. The angle between the x-axis and line connecting the centre of the circle with the point, pc, is called the eccentric anomaly, E, and is defined by

$$\cos E = \frac{x + a\varepsilon}{a} \qquad (28a)$$

$$\sin E = \frac{y}{b} \qquad (28b)$$

From these relationsof equations (28a) and (28b) we may write

$$x = a(\cos E - \varepsilon) \qquad (28c)$$

$$y = b \sin E \qquad (28d)$$

The 'eccentric anomaly' E is useful to compute the position of a point moving in a keplerian orbit. As for instance, if the body passes the periaston is at coordinates $x = a(\cos E - \varepsilon)$, $y = 0$ at time$t = 0$.

Adding equations (28c) and (28d), we find

$$x = a(\cos E - \varepsilon) \qquad (28e)$$

Squaring equation (28e), we find

$$r^2 = a^2(1 - \varepsilon \cos E)^2 \qquad (28f)$$

So that in terms of the eccentric anomaly, the radius is

$$r = a(1 - \varepsilon \cos E) \qquad (29)$$

To obtain an explicit relationship between $E \; and \; m(\equiv \theta)$. first, we rewrite equation (20), with the help of (21), as

$\varepsilon r \cos m = a(1-\varepsilon^2) - r$ (29a)

If we add εr to both sides of equation (29a), we have

$\varepsilon r(1+\cos m) = (1-\varepsilon)\{a(1+\varepsilon) - r\}$ (29b)

Substituting for r from equation (29) in the right-hand side of this expression, we find

$\varepsilon r(1+\cos m) = (1-\varepsilon)\{a(1+\varepsilon) - a(1-\varepsilon\cos E)\}$ (29c)

Or,

$r(1+\cos m) = a(1-\varepsilon)(1+\cos E)$ (29d)

If we subtract εr from both sides of equation (29a) and substitute for r from (29) and then simplify, we obtain

$r(1-\cos m) = a(1+\varepsilon)(1-\cos E)$ (29e)

Upon dividing equation (29e) by Eqn. (29d) we get;

$$\frac{1-\cos m}{1+\cos m} = \frac{1+\varepsilon}{1-\varepsilon}\cdot\frac{1-\cos E}{1+\cos E} \qquad (29f)$$

We may use the half – angle formula for the tangent to write equation (29f) as

$$\tan\frac{m}{2} = \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\cdot\tan\frac{E}{2} \qquad (29g)$$

This gives E uniquely in terms of m. Therefore $m(t)$ may be easily obtained once $E(t)$ is found. In order to calculate $E(t)$ we may transform equation (29f) into an equation (29g) for E by computing the integrand in terms of E.

Differentiating equation (29g) yields

$$dm = \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\cdot\frac{cos^2\left(\frac{m}{2}\right)}{cos^2\left(\frac{E}{2}\right)}\cdot dE \qquad (29h)$$

We may write

$$r = a(1-\varepsilon)\cdot\frac{1+\cos E}{1+\cos m} \qquad (29i)$$

$$Or \quad r = a(1-\varepsilon)\cdot\frac{cos^2\left(\frac{E}{2}\right)}{cos^2\left(\frac{m}{2}\right)} \qquad (29j)$$

Where we have used the half-angle formula for the cosine functions. In order to express r^2dm in terms of$E$, we take one factor of r from equation (29e), to get

$$r^2dm = \{a(1-\varepsilon\cos E)\}\cdot\left\{a(1-\varepsilon)\cdot\frac{cos^2\left(\frac{E}{2}\right)}{cos^2\left(\frac{m}{2}\right)}\right\}\cdot\left\{\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\cdot\frac{cos^2\frac{m}{2}}{cos^2\frac{E}{2}}\cdot dE\right\} \qquad (30)$$

$$= a^2\cdot\sqrt{1-\varepsilon^2}\cdot(1-\varepsilon\cos E)dE \qquad (30a)$$

This result of equation (30a) may be written as

$$\frac{\pi ab}{\tau}\cdot t = \frac{a^2\sqrt{1-\varepsilon^2}}{2}\cdot\int_m^E(1-\varepsilon cosE)dE \qquad (30b)$$

Integrating equation (30b), and again using $b = a\sqrt{1-\varepsilon^2}$, we have the result as

$$\frac{2\pi t}{\tau} = E - \varepsilon\sin E \qquad (30c)$$

The quantity $\frac{2\pi t}{\tau}$ is called the mean anomaly since it measures the angular deviation of a body moving in a circular orbit with periodτ. Following astronomical practice, we denote the mean anomaly by$M$. Thus,

$M = E - \varepsilon\sin E$ (31)

This is Kepler's equation.

Kepler's equation may be used to obtain a simple expression for the velocity of a body in its orbit in terms of the magnitude of the radius vector. We may write

$V^2 = \dot{x}^2 + \dot{y}^2$ (32)

For x and y, the square of the velocity becomes

$V^2 = a^2E^2sin^2 + a^2(1-\varepsilon E^2cos^2E)$ (33)

$= a^2E^2(1-\varepsilon^2cos^2E)$ (34)

If we differentiate Kepler's equation with respect to the time, we have

$$\frac{2\pi}{\tau} = E(1-\varepsilon\cos E) \qquad (35)$$

Solving this equation (35) for φ and substituting, we obtain

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$$V^2 = \left(\frac{2\pi}{\tau}\right)^2 a^2 \frac{1-\varepsilon^2 cos^2 E}{(1-\varepsilon\cos E)^2} \qquad (36)$$

$$= \left(\frac{2\pi}{\tau}\right)^2 a^2 \frac{1+\varepsilon\cos E}{1-\varepsilon\cos E} \qquad (36a)$$

$$= \left(\frac{2\pi}{\tau}\right)^2 a^2 \frac{2-(1-\varepsilon cos E)}{1-\varepsilon\cos E} \qquad (36b)$$

Substituting $\frac{r}{a} = 1-\varepsilon\cos E$, into equation (36b) we obtain the result as

$$V^2 = \left(\frac{2\pi}{\tau}\right) a^3 \left(\frac{2}{r}-\frac{1}{a}\right) \qquad (37)$$

Finally, Kepler's third law may be used to reduce this expression to [5]

$$V^2 = \frac{k}{\mu}\left(\frac{2}{r}-\frac{1}{a}\right) \qquad (38)$$

8.0 Conclusion

In classical mechanics, central force problem is a special case of the two-body problem, in which the two bodies interact by a central force F that varies in strength as the inverse square of the distance r between them. The force may be either attractive or repulsive. An understanding of central force motion is necessary for the design of satellites and space vehiclesand the study of the effects of central-force on planetary motion, made us to understand that satellite or planet would tend to go off in a straight line if no central force were applied to it.Therefore, it is responsible for holding our earth to together and for keeping the planets in orbit about the sun.

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