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Motion of Particles of Non-Zero Rest Masses Exterior To Homogeneous Spherical Mass Distributions

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Abstract

In this paper, we derive the equation of motion for test particles of non-zero rest masses based on the generalized exterior gravitational scalar potential. The time, radial, polar and azimuthal equations of motion for particles of non-zero rest masses moving in this gravitational field has been derived. The motion of the planets was confined to the equatorial plane of the Sun and the resulting radial equations of motion were solved to obtain a complete generalized radial equation of motion. The results is that the complete generalized radial equation of motion for particles of non - zero rest masses is augmented with correction terms of all order of c^{-2}which are for theoretical development and application.

Keywords: Generalized gravitational scalar potential, time, radial, polar and azimuthal equation of motion

1.0 Introduction

In 1915 Einstein published his dynamical geometrical theory of gravitation which is popularly known as General Relativity (GR). According to General Relativity theory, the observed gravitational attraction between masses results from their warping of space and time. This theory offered a resolution of the anomalous orbital precision of the orbit of the planet as well as the gravitational shift by the Sun [1,2]. The first exact solution to Einstein's geometrical gravitational theory of gravitation was carried out by Schwarzschild in 1916 based on Newton's dynamical gravitational scalar potential exterior to the body [1, 2] given by

$$f(r) = \frac{GM}{r}$$

where G is the universal gravitational constant and r is the mean distance from the Sun.
In this paper we derived the exact solution to Einstein's geometrical theory of gravitation whose tensor field varies with only the radial distance based on generalized dynamical gravitational scalar potential exterior to the body. Our analysis in this paper can be applied to static homogeneous spherical distribution of mass whose tensor field varies with only the radial distance. An example of such a distribution is a homogeneous distribution of mass within a spherical region which is rotating with uniform angular acceleration about a fixed diameter[3, 4].

2.0 Theoretical Analysis

Consider a spherical body of radius R and total rest mass M distributed uniformly with density ρ. The general field equations in the exterior region is given tensorially [5, 6] as

$$a_{\mu} = 0 \qquad (1)$$

where a_μ is Einstein's tensor. As usual, the Greek subscripts runs from 0 -3, with the 0th component representing the time coordinate, and the 1st, 2nd and 3rd components denote the space coordinate. In spherical coordinates$(0,1,2,3) \equiv (ct, r, \theta, \phi)$ Schwarzschild's metric is the solution of Einstein's gravitational field equations exterior to a static homogenous spherical body [7, 8, 9] given by

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$$g_{00} = 1 + \frac{2}{c^2} f(r) \quad (2)$$

$$g_{11} = -\left[1 + \frac{2}{c^2} f(r)\right]^{-1} \quad (3)$$

$$g_{22} = -r^2 \quad (4)$$

$$g_{33} = -r^2 \sin^2\theta \quad (5)$$

$$g_{\mu\nu} = 0 : otherwise \quad (6)$$

The corresponding contra-variant metric tensor for this gravitational field [5, 10] is given as

$$g^{00} = \left[1 + \frac{2}{c^2} f(r)\right]^{-1} \quad (7)$$

$$g^{11} = -\left[1 + 2f(r)\right] \quad (8)$$

$$g^{22} = r^2 \quad (9)$$

$$g^{33} = \left[-r^2 \sin^2\theta\right]^{-1} \quad (10)$$

$$g^{uv} = 0; otherwise \quad (11)$$

where $r > R$, the radius of the static spherical mass,
f(r)is pure Newtonian gravitational scalar potential in the space-time exterior to the static homogenous spherical distribution mass and is a function of the radial coordinate r only. The generalized dynamical gravitational scalar potential in the exterior region of the body defined in this field is given [11] as

$$f(r) = -\frac{GM}{r}\left[1 - \frac{GM}{c^2 R}\right] - \frac{G^2 M^2}{c^2 r^2} \quad (12)$$

$G$is the universal gravitational constant, M is the total mass of the distribution and c is the speed of light in vacuum. The coefficients of affine connection, defined by the metric tensor of space-time are found to be given in terms of the metric tensor [4] as

$$\Gamma^0_{10} = \Gamma^0_{01} = g^{00}.g_{00,1} \quad (13)$$

$$\Gamma^1_{00} = -\frac{1}{2} g^{11}.g_{00,1} \quad (14)$$

$$\Gamma^1_{11} = \frac{1}{2} g^{11}.g_{11,1} \quad (15)$$

$$\Gamma^1_{22} = -\frac{1}{2} g^{11}.g_{22,1} \quad (16)$$

$$\Gamma^1_{33} = -\frac{1}{2} g^{11}.g_{33,1} \quad (17)$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2} g^{22}.g_{22,1} \quad (18)$$

$$\Gamma^3_{31} = \Gamma^3_{13} = \frac{1}{2} g^{33}.g_{33,1} \quad (19)$$

$$\Gamma^2_{33} = -\frac{1}{2} g^{22}.g_{33,2} \quad (20)$$

Where, the comma denotes partial differentiation with respect to (0, 1, 2) = (ct,r, θ).
The coefficients of affine connection can now be written more explicitly in terms of (ct,r, θ, ϕ) as

$$\Gamma^0_{10} = \Gamma^0_{01} = \frac{1}{c^2}\left[1+\frac{2}{c^2}f(r)\right]^{-1}\frac{\partial F(r)}{\partial r} \qquad (21)$$

$$\Gamma^1_{00} = \frac{1}{c^2}\left[1+\frac{2}{c^2}f(r)\right]\frac{\partial F(r)}{\partial r} \qquad (22)$$

$$\Gamma^1_{11} = -\frac{1}{c^2}\left[1+\frac{2}{c^2}f(r)\right]\frac{\partial F(r)}{\partial r} \qquad (23)$$

$$\Gamma^1_{00} = \frac{1}{c^2}\left[1+\frac{2}{c^2}f(r)\right]\frac{\partial F(r)}{\partial r} \qquad (24)$$

$$\Gamma^1_{22} = r\left[1+\frac{2}{c^2}f(r)\right] \qquad (25)$$

$$\Gamma^1_{00} = -r\left[1+\frac{2}{c^2}f(r)\right]\sin^2\theta = \left[1+\frac{2}{c^2}f(r)\right] \qquad (26)$$

$$\Gamma^2_{33} = -sin\theta cos\theta \qquad (28)$$

$$\Gamma^3_{31} = \Gamma^3_{13} = r^{-1} \qquad (29)$$

$$\Gamma^\alpha_{\mu\nu} = 0; otherwise \qquad (30)$$

Thus, the gravitational field has eight non-zero coefficients of affine connections similar to Schwarzschild's solution to Einstein's field equation.

Einstein's equation of motion for particles of non-zero rest masses in gravitational fields are given [1, 7] by

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}\left(\frac{dx^\gamma}{d\tau^2}\right)\left(\frac{dx^\lambda}{d\tau}\right) = 0 \qquad (31)$$

where,

$\tau$is the proper time.

$x^\mu$are coefficient of space-time

$\Gamma^\mu$are the christoffel symbols or coefficients of affine connection.

Setting $\mu = 0,1,2,3$ into (31), it follows that Einstein's equation of motion for a planet, comet or asteroid (regarded as a particle) are given in polar coordinates as

$$\ddot{t} + \frac{2}{c^2}\left[1+\frac{2}{c^2}f(r)\right]\frac{\partial f}{\partial r}(r)\dot{t}\dot{r} = 0 \qquad (32)$$

$$\ddot{r} + \left[1+\frac{2}{c^2}f(r)\right]\frac{\partial f}{\partial r}(r)\dot{t}^2 - \frac{1}{c^2}\left[1+\frac{2}{c^2}f(r)\right]^{-1}\frac{\partial f}{\partial r}\dot{r}^2 -$$
$$r\left[1+\frac{2}{c^2}f(r)\right]\dot{\theta}^2 - r\sin^2\theta\left[1+\frac{2}{c^2}f(r)\right]\dot{\phi} \qquad (33)$$

$$\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - sin\theta cos\theta\dot{\phi}^2 = 0 \qquad (34)$$

$$\ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} = 0 \qquad (35)$$

where a dot denotes one time differentiation.

These are the generalized radial equation of motion for particles of non – zero masses according to Einstein's geometrical theory of gravitation.

For pure radial motion $\dot{\theta} = \dot{\phi} = 0$

The radial equation of motion (33) becomes

$$\ddot{r}+\left[1+\frac{2}{c^2}f(r)\right]\frac{\partial f}{\partial r}(r)\dot{t}^2-\frac{1}{c^2}\left[1+\frac{2}{c^2}f(r)\right]^{-1}\frac{\partial f}{\partial r}(r)\dot{r}^2=0 \qquad (36)$$

Substituting

$$\left[1+\frac{2}{c^2}f(r)\right]\frac{\partial f}{\partial r}=\left[\frac{k}{r^2}-\frac{k^2}{c^2Rr^2}+0\left(c^{-4}\right)\right] \qquad (37)$$

and

$$\frac{1}{c^2}\left[1+\frac{2}{c^2}f\right]^{-1}\frac{\partial f}{\partial r}=\left[-\frac{k^2}{c^2r^2}+0(c^{-4})\right] \qquad (38)$$

into the pure radial equation of motion (36) we obtain

$$\ddot{r}+\left[\frac{k}{r^2}-\frac{k^2}{c^2Rr^2}+0\left(c^{-4}\right)\right]\dot{t}-\left[\frac{k^2}{c^2r^2}+0\left(c^{-4}\right)\right]\dot{r}=0 \qquad (39)$$

This is the generalized pure radial equation of motion for particles of non – zero rest masses according to generalized Einstein's geometrical gravitational theory of gravitation.

Consider the motion of a planet whose motion is confined to the equatorial plane of the sun. $\theta=\frac{\pi}{2}$

The Schwarzschild's equations of motion reduces to

$$\ddot{t}+\frac{2}{c^2}\left[1+\frac{2}{c^2}f(r)\right]\frac{\partial f}{\partial r}\dot{t}\dot{r}=0 \qquad (40)$$

$$\ddot{r}+\left[1+\frac{2}{c^2}f(r)\right]\frac{\partial f}{\partial r}\dot{t}^2-\frac{1}{c^2}\left[1+\frac{2}{c^2}f(r)\right]^{-1}\frac{\partial f}{\partial r}\dot{r}^2-r\dot{\phi}^2\left[1+\frac{2}{c^2}f(r)\right]=0 \qquad (41)$$

$$\ddot{\phi}+\frac{2}{r}\dot{r}\dot{\phi}=0 \qquad (42)$$

Under the experimentally established condition that

$$t\to\tau; r\to\infty \qquad (43)$$

The exact solution of (40) is given by

$$\dot{t}^2=A\left[1+\frac{2}{c^2}f(r)\right]^{-1} \qquad (44)$$

where, A is the constant of motion.

The exact solution of (42) is given by

$$\dot{\phi}=\frac{l}{r^2} \qquad (45)$$

$$\dot{r}^2=\frac{2k}{r}+\frac{l^2}{r^2} \qquad (46)$$

where, l is the constant of motion corresponding to the angular momentum per unit mass of the planet.

The radial equation of motion in equatorial plane of the sun (41) can be written explicitly as

$$\ddot{r}=-\frac{k}{r^2}+\frac{k^2}{c^2Rr^2}+\frac{2k^2}{c^2r^3}-\frac{3kl^2}{c^2r^4}+\frac{l^2}{r^3}=0 \qquad (47)$$

Equation (47) is the complete generalized radial equation of motion in the equatorial plane of the sun. This equation reduces to the limit c^0, to the corresponding pure Newtonian dynamical equation of motion and satisfies the well known Equivalence Principles in Physics. In general it contains Post – Newton and Post – Einstein corrections of all orders of c^{-2} which are for theoretical development and experimental investigations and applications.

3.0 Remarks and Conclusion

The time, radial, polar and azimuthal equations of motion for particles of non- zero rest masses exterior to astrophysically spherical distribution of mass whose tensor field varies with only the radial distance based on generalized dynamical gravitational scalar potential were found to be equations (32), (33), (34) and (35) respectively. The generalized pure radial equation of motion and the complete generalized radial equation of motion in the equatorial plane of the sun are found to be equations (39) and (47)
The immediate consequences of the results obtained in this paper are;

(i) The complete generalized radial equation of motion in the equatorial plane of the Sun (47) can be transformed to obtain a generalized radial speed and radial acceleration interms of radial coordinate.

(ii) The generalized radial acceleration can be transformed to obtain a generalized Einstein's dynamical planetary equation of motion and hence the planetary parameters such as orbital angular frequency, angular momentum per unit mass, eccentricity, amplitude, time period and aphelion and perihelion distance.

(iii) The coefficient of affine connection obtained can be used to construct the Riemmann-Christoffel, Ricci and Einstein's tensor for this Field and the Einstein's Field equations for this gravitational Field can be obtained.

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