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Solution of Cooper Pair Problem Using Perturbation Theory

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Abstract

A Cooper pair problem was solved using perturbation theory. The energy of a two-electron bound state (cooper pair) in a superconductor is one of the main results of the theory of superconductivity. The analysis of this work shows that the energy of cooper pair depend on the potential v_o that appear in the exponential term at the Fermi surface.

Keywords: Fermi surface, cooper pair, phonon, Crystal lattice

1.0 Introduction

In this paper we are going to solve the cooper problem, which is the problem of weak attractive electrons in the Fermi surface, that form a bond state of two electrons.

The cooper pair is composed of two electrons with opposite spin and momentum [1], such that the spin wave function should be antisymmetric [2].At low temperature they condense and their Bose-Einstein condensate is a super fluid and this is the amazing state that we know as superconductor [3].

The phenomenon of superconductivity depends on the transition temperature and isotope effect of material [4]. This phenomenon dramatically illustrates the differences between systems of quantum mechanical particles that obey Bose-Einstein statistics instead of Fermi-Dirac (FD) statistics. At room temperature, electrons, which have spin $1/2$, are distributed among their possible energy states according to FD statistics. At very low temperatures, because of their low kinetic energy the electrons pair up to form spin-0 Cooper electron pairs, named after the American physicist Leon Cooper. Since these electron pairs have zero spin, they behave as bosons, and promptly condense into the same ground state. A large energy gap between this ground state and the first excited state ensures that any current is "frozen in." This causes the current to flow through without resistance, which is one of the defining properties of superconducting materials [5].

The cooper pair's problem has not been attacked using perturbation theory before [6].The solution of cooper problem will involve the same method used in single particle quantum mechanics and it is very simple technically[7].

We must content our self here with a sketch outline derivation of the fundamental principle of the whole theory-the demonstration first given by Bardeen, Cooper and Schrieffer (BCS), that the ground state of an assembly of mutual attracting fermions is separated by an energy gap from lowest excited levels of the energy spectrum.

The superconducting state of the BCS theory is a radically new type of quantum state, which requires special treatment.

The starting point for the full discussion of superconductivity would be the identification of mechanism for an attractive force between the electrons. However efficiently it may be screened, the direct coulomb interaction between like charges is always repulsive. But the positively charge ions provide sources of attraction for the electrons, and may be used as intermediaries: the exchange of phonons between electrons can give rise to an effective electron–electron interaction which is complicated in form but which is negative –attractive-when the energy difference of the two electron states is small[8].

The origin of electron attraction in natural superconductors is subtle actually. The physical picture of what happens is far from simple and seems to elude or involve a common quarsi particle description [9 – 11].

2.0 Theoretical Analysis

In order to understand how an attraction between two electrons can occur, we must consider the interaction of an electron with a lattice as shown in Figure 1.

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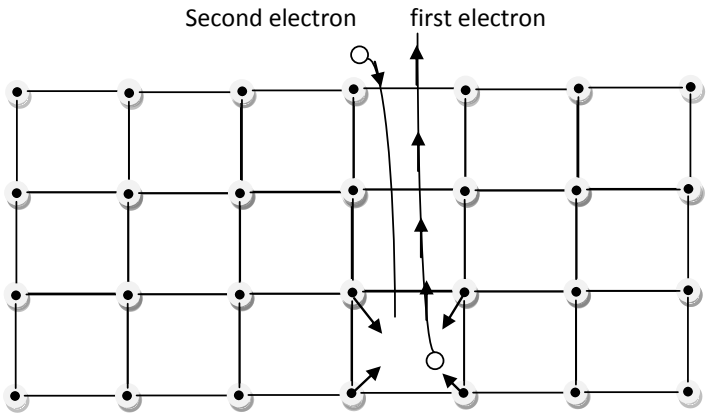

Figure1: Simple crystal lattice pattern with ion in each corner or at each lattice point and electrons are moving through.
Whenever an electron passes the region in a lattice where we have positive ions charge, the electron polarizes the lattice, so it will take the lattice time to relax. When another electron passes the same region the positive ions will attract again and this results to the weak effective phonon- mediated attraction between the electrons, so in some ways they exchanges elastic waves which are called phonons.
The cooper pair formation can be summarized in five steps as follows:

1. An electron moves through the positively charged ion cores of the lattice and gets attracted causing distortion.
2. The distortion of the lattice causes the local area to gain a small positive charge.
3. The area of positive charge can attract an electron towards the first electron.
4. This attraction can overcome the repulsive coulumbic forces and create a binding between the two electrons.
5. The pair of electrons can then travel through the lattice as a single entity known as cooper pair.

Phonon- Mediated Attraction
When electrons interact by exchanging phonons, the energy and momentum must be conserved in that process.

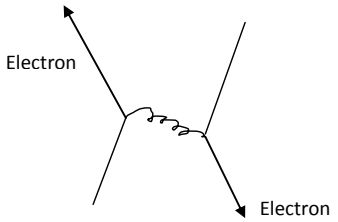

Figure 2: Feynman diagram shows that as electrons move around the crystal lattice they exchange phonons.

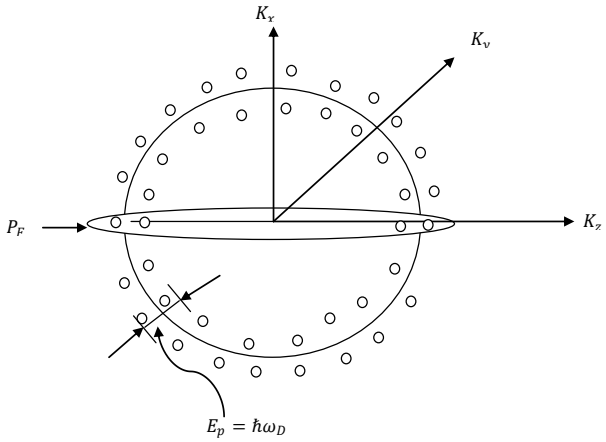

Figure 3: Spherical Fermi surface

In order to understand this interaction and the typical energy involve in such process we need to consider the spherical Fermi surface in Figure 3.

What we know is the typical energy of the Fermi electron E_F and $E_F = K_B T_F$. When converted to temperature we get

$$T_F = \frac{E_F}{K_B} \approx 10{,}000K \quad (1)$$

$$Where: E_F = fermi\ energy, T_F = fermi\ temperature, K_B = Bolzman\ constant, and$$

$$T_p = \frac{E_p}{K_B} = \frac{\hbar\omega_D}{K_B} \approx 400\ k \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$Where\ T_p = phonon\ temperature, E_p = Phonon\ energy$$

Equations (1) and (2) show that the phonon energy available is pretty low compared to the Fermi-energy of electrons.

3.0 Analysis of the Interaction

The exact interaction is complicated so we will use a simple model. Consider the given potential

$$V(p) = \begin{cases} -V_o & if\ \frac{p^2}{2m} - E_F < \hbar\omega_D \\ 0, & otherwise \end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

This corresponds to local attraction in free space; it is momentum dependent interaction, *V(P)*, between the two electrons attracted to each other and located at a narrow shell.

This two electrons exchange phonon energy $\hbar\omega_D$. If these two electrons are located beyond this narrow region the attraction is zero. The problem of one particle in a quantum well has been solved [4]. Now two particles attracted to each other by the potential in momentum space will now be was considered.

The two-particle Schrödinger equation is given by

$$\left[\frac{\hbar^2\nabla^2}{2\mu} - V_o\delta(r)\right]\psi(r) = E\psi(r) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

$$\left[\frac{\hbar^2\nabla^2}{m} - V_o\delta(r)\right]\psi(r) = E\psi(r) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Where μ is reduce mass of the two electrons which is defined by

$$\mu = \frac{m_1 m_2}{m_{1+m_2}} = \frac{m}{2}\ \ if\ m_1 = m_2 = m$$

$Now\ V(r) = -V_o\delta(r)$ is called delta potential energy function, which has bound state and negative energy.

The only way that we can solve this problem is by changing the position space of equations (4) and (5) into momentum space version. So we introduce Fourier transform and specifically for Dirac delta function we used

$$F[\delta(r)] = \widetilde{\delta(r)} = \int_{-\infty}^{\infty} \delta(r)e^{ipr}dr = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

$$\delta(r) = \int_{-\infty}^{\infty} e^{-ipr}\frac{d^3p}{(2\pi)^3} = \int_p\ e^{-ipr} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

The representation of the wave function $\psi(r)$in a form of Fourier transform in momentum-space is given by

$$\psi(r) = \int_{-\infty}^{\infty} \psi(p)e^{-ipr}\frac{d^3p}{(2\pi)^3} = \int_p\ \psi(p)e^{-ipr} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

$$\psi''(r) = \int_p (-ip)^2\psi(p)e^{-ipr} = (-ip)^2\int_p \psi(p)e^{-ipr} \ldots\ldots\ldots\ldots\ldots\ldots (9)$$

$(-ip)^2\psi(p)$ is called Fourier transform of $\psi''(r)$.Since $\delta(r)$ is sharply peaked at r = 0, therefore

$$\psi(0) = \int_{-\infty}^{\infty} \psi(p)\frac{d^3p}{(2\pi)^3} = \int_p \psi(p) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

$$\delta(r)\psi(r) = \delta(r)\psi(0) = \psi(0)\int_p\ e^{-ipr} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

Substituting equations (7), (8), (9) and (11) into equation (5), gives
the momentum-space version of equation (5) which is

$$\frac{p^2\psi(p)}{m} - v_o \int_{p-p_F<\hbar K_D} \frac{d^3p}{(2\pi\hbar)^3}\psi(p) = E\psi(p) \ldots\ldots\ldots\ldots\ldots\ldots (12)$$

Where $\psi(p)$Is the two-particle wave function in the momentum space,
$v_o$is the effective interaction constant (attraction), and

$$\int_{p-p_F<\hbar K_D} \frac{d^3p}{(2\pi\hbar)^3}\psi(p) = \psi(o) \qquad (13)$$

Equation (13) shows that the constraints in the integral correspond to electrons interacting in a narrow shell near the Fermi surface. This means that we limit the integration over momentum in the vicinity of the Fermi surface, such that there is possibility of the phonon exchange.

Equations (4) and (12) are basically the cooper pairing problem. The result of this calculation is going to be the pairing of two electrons into a bound state.

However the interesting thing that happens is that because electrons are playing roll in this pairing or interactions with phonon, they exist in the vicinity of Fermi-surface and this surface is two-dimensional. This gives rise to the sense of reducing dimensionality. We see that in real space we started with 3-dimentional system, so effectively what we are dealing here is reduced dimension and this give rise to appearance of bound state in a much unexpected way.

It is clear that the phonon's momentum is much smaller than that of electrons

$\hbar k_D \gg p_F$.

Where k_D is the typical phonon's wave vector, called Debye wave vector.

The energy of the two particles is

$$E = 2E_F + \Delta \qquad (14)$$

Where $E_F = \frac{P_F^2}{2m}$, isthe Fermi energy, and

Δ, is the main quantity of interest, the pairing energy of two electrons.

This pairing will be possible if and only if a solution to equation (12) exist with $\Delta < 0$

(That is, we can rewrite it as $-\Delta$, with $\Delta > 0$). Hence,

$$E = 2E_F - \Delta \qquad (15)$$

So we can have a bound state by doing so the energy. Which physically implies that electrons can find a way to lower their energy via pairing and becoming bosons, the cooper pairs: the latter eventually condenses and their Bose-Eistein condensate is a superconductor.

Using the shorthand notation for the integral over P in equation (13) as $\int_p$,we can re-write the latter as follows

$$\left(\frac{p^2}{m} - E\right)\psi(p) = v_o \int_p \psi(p) \dots\dots\dots\dots\dots\dots\dots\dots\dots (16)$$

$$\left[\frac{p^2}{m} - 2E_F - \Delta\right]\psi(p) = v_o \int_p \psi(p) \dots\dots\dots\dots\dots (17)$$

By dividing both sides of (11) by the expression in the square brackets in (17) and then integrating over the momentum, and then it becomes:

$$\psi(p) = v_o \int_p \frac{\psi(p)}{\left[\frac{p^2}{m} - 2E_F - \Delta\right]} \dots\dots\dots\dots\dots (18)$$

$$\int_p \psi(p) = v_o \int_p \psi(p) \int_p \frac{1}{\left[\frac{p^2}{m} - 2E_F - \Delta\right]} \dots\dots\dots\dots\dots (19)$$

The term in the left-hand side (an unknown constant) cancels out the corresponding integral in the right hand side, leading to the following self-consistency equation. This is one of the key equations of the Bardeen–Cooper-Schrieffer (BCS) theory of the super conductivity often called the BCS self-consistency equation.

$$1 = \int_p \frac{v_o}{\left[\frac{p^2}{m} - 2E_F - \Delta\right]} \dots\dots\dots\dots\dots (20)$$

Since all the action happens near the Fermi surface, i.e. $p \sim p_F$ (p_F Fermi-momentum) as follows from the constraints in equation (20) we can simplify the integral as follows:

$$\int_{p-p_F<\hbar\omega_D} \frac{d^3p}{(2\pi\hbar)^3} = \frac{1}{(2\pi\hbar)^3} \int_{p-p_F<\hbar\omega_D} d\Omega_p p^2 dp \dots\dots\dots\dots (21)$$

Since nothing depends on the angle, we can integrate over $d\Omega_p$ leading to a factor of 4π.

We can also introduce a new variable which has the physical meaning of the electron energy relative to the threshold-Fermi-energy.

$$\xi = \frac{p^2}{2m} - E_F \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (22)$$

$$2\xi = \frac{p^2}{m} - 2E_F \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (23)$$

And approximating the remaining momentum terms in the numerator as $p \sim p_F$ which is

$$p^2 dp = mpd\left(\frac{p^2}{2m}\right) \approx mp_F d\xi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (24)$$

Equation (20) becomes

$$1 = \frac{1}{(2\pi\hbar)^3}(4\pi)\int_0^{\hbar\omega_D} \frac{v_o mp_F d\xi}{\left[\frac{p^2}{m} - 2E_F - \Delta\right]} \ldots\ldots\ldots\ldots\ldots (25)$$

Substituting (23) in to (25) gives

$$1 = \frac{v_o mp_F}{(2\pi\hbar)^3}(4\pi)\int_0^{\hbar\omega_D} \frac{d\xi}{[2\xi - \Delta]} \ldots\ldots\ldots\ldots\ldots (26)$$

Let

$$\alpha = 2\xi - \Delta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (27)$$

$$\frac{d\alpha}{d\xi} = 2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (28)$$

$$\frac{d\alpha}{2} = d\xi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (29)$$

$$Then\ (26) becomes \quad \rightarrow \quad 1 = \frac{v_o mp_F}{2(2\pi\hbar)^3}(4\pi)\int_0^{\hbar\omega_D} \frac{d\alpha}{[\alpha]} \ldots\ldots\ldots\ldots\ldots (30)$$

$$1 = \frac{v_o mp_F}{2(2\pi\hbar)^3}(4\pi)\ln\alpha \mathrm{I}_0^{\hbar\omega_D} \ldots\ldots\ldots\ldots\ldots (31)$$

$$1 = \frac{v_o mp_F}{2(2\pi\hbar)^3}(4\pi)\ln(2\xi - \Delta)\mathrm{I}_0^{\hbar\omega_D} \ldots\ldots\ldots\ldots\ldots (32)$$

$$1 = \frac{v_o mp_F}{2(2\pi\hbar)^3}(4\pi)[\ln(2\hbar\omega_D - \Delta) - \ln(2(0) - \Delta)] \ldots\ldots\ldots\ldots\ldots (33)$$

Let

$$N_o = \frac{mp_F}{2(2\pi\hbar)^3}(4\pi) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (34)$$

$$Then\ (33) becomes\ 1 = N_o v_o[\ln(2\hbar\omega_D - \Delta) - \ln(2(0) - \Delta)] \qquad (35)$$

$$1 = N_o v_o \ln\left[\frac{2\hbar\omega_D - \Delta}{-\Delta}\right] \ldots\ldots\ldots\ldots\ldots (36)$$

Assuming that $2\hbar\omega_D \gg \Delta, we\ get$

$$1 \approx N_o v_o \ln\left[\frac{2\hbar\omega_D}{-\Delta}\right] \ldots\ldots\ldots\ldots\ldots (37)$$

$$\frac{1}{N_o v_o} \approx \ln\left[\frac{2\hbar\omega_D}{-\Delta}\right] \ldots\ldots\ldots\ldots\ldots (38)$$

Taking exponential of both sides we get

$$e^{\left(\frac{1}{N_o v_o}\right)} \approx \frac{2\hbar\omega_D}{-\Delta} \ldots\ldots\ldots\ldots\ldots (39)$$

$$-\Delta \approx \frac{2\hbar\omega_D}{e^{\left(\frac{1}{N_o v_o}\right)}} \ldots\ldots\ldots\ldots\ldots (40)$$

$$\Delta \approx -2\hbar\omega_D e^{\left(\frac{-1}{N_o v_o}\right)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (41)$$

4.0 Result and Discussion

Equation (41) is obtained as one of the main results of the theory of superconductivity, the energy of two-electrons bound into a state (cooper pair) in a superconductor.

From the Equation (41) we see that the electrons are finding ways to lower their energy by forming cooper pair. They basically form the superconductor.

This small interaction potential energy v_o that appear in the exponential term has great effect. For instance,

i. If it is small then the energy of the bound state (called superconducting gap) is also small.
ii. If we did not have the exponential term, we remain with phonon energy. this implies that we can have superconductivity at very high temperature(room temperature)
iii. The exact calculation as proposed by Yuri[2] has an extra factor due to the fact that all the matrix elements are calculated in the renormalized state.
iv. Equation (41) shows that$\Delta \neq 0$ at any interaction strength. Because it only happens in the vicinity of the Fermi surface, this also shows that the Fermi surface is very important in this study.

If potential v_o is small, Δ also is small. Equation (41) as a function of v_o is a very unusual function. We have already seen that this function is special because it does not have Taylor expansion so there is no way one can approach this result by doing so but through "perturbation theory" This may be one of the reasons why it took so long for people to figure out the key to superconductivity.

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