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A Decomposition Algorithm for the Solution of Fractional Biological Population Model with Caputo Derivatives

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Abstract

In this paper, an Iterative Decomposition Method is applied to solve generalized Fractional population models, in which the derivatives are given in the sense of Caputo. The proposed method presents solutions as rapidly convergent infinite series of easily computable terms. Some examples are given to illustrate the efficiency and accuracy of the method. Solutions obtained compare favourably with known solutions.

Keywords: Iterative Decomposition Method, Biological Population Model, Fractional Derivatives, Caputo Derivative, Mittag-Leffler function.

1.0 Introduction

The use of Fractional differential equations in modeling of physical phenomena is gaining wide acceptance. This can be attributed to the established applications in diversed and varying areas of Science, Engineering and technology. Fractional derivatives and differential equations have been applied in such areas as anomalous transport in disordered systems, percolations in porous media, and diffusion of biological population [1]. Nonlinear oscillation of earthquakes [2, 3, 4] and chemical processes have been modeled using fractional differential equations. Most nonlinear differential equations cannot be solved analytically. Hence, getting effective approximate solution methods becomes very important. Certain known methods have been found to be efficient, only for relatively simple equations, while others do not work for arbitrary real orders, even as they work for rational orders.

Many analytic and numerical methods have been applied to solve linear and nonlinear fractional differential equation. The Adomian Decomposition method (ADM) [3], the Variational Iteration Method [5], the Homotopy Perturbation Method [6, 7], the Homotopy Analysis Method (HAM) [2, 6] are some of those methods which have successfully been applied. These methods have been applied to solve biological population models [2-5, 7-9]. However, each of these methods has a major drawback. For instance, the difficulty in computing the Adomian polynomial for nonlinear problem is an inherent drawback.

In this paper, we consider the consider the nonlinear fractional-order biological population model of the form

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{\partial^2 (u^2)}{\partial x^2} + \frac{\partial^2 (u^2)}{\partial y^2} + hu^a (1 - ru^b), \; t > 0, x, y \in \Re \tag{1}$$

With the given initial condition

$$u(x, y, 0) = f_0(x, y) \tag{2}$$

Where u denotes the population density, f represents the population supply due to births and deaths; h, a, b and r are real numbers. For some special values of a, b and r, (1) becomes the well-known Malthusian and Verhulst laws accordingly. For instance, for $a = 1$, $r = 0$, $h = c$, where c is a constant, (1) becomes the Malthusian model. Similarly, for a=1=b, (1) becomes the Verhulst model.

In this paper, we apply the Iterative Decomposition Method (IDM) to solve the various forms of (1) with the associated initial condition (2). The IDM has been successfully applied to solve many integer-order differential equations.

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For instance IDM has been applied to solve one-dimensional biharmonic equation [10], Delay differential equations [11] and Variationalproblems . In [11] the IDM was applied to multi-order fractional differential equations. The method does not require any form of linearization or discretization.

The rest of the paper is laid out as follows: In section 2, we provide some basic definitions and concepts. In section 3, the proposed method is explained, while in section 4, the method developed in section 3 is applied to solve some examples of fractional population model. In section 5, a conclusion is drawn.

2.0 Basic Definition

We now give some essential definitions which are helpful in the understanding of this paper.

Definition 1: A real function $f(t),\ t>0$, is said to be in the space $C_\mu,\ \mu\in\Re$, if there exists a real number $p>\mu$ such that $f(t)=t^p f_1(t)$, where $f_1(t)\in C(0,\infty)$, and it is said to be in the space $C^n{}_\mu$ if and only if $f^{(n)}\in C_\mu,\ n\in\mathrm{N}$.

Definition 2: The Riemann-Liouville fractional integral operator $J^\alpha\ (\alpha\ge 0)$ of a function $f\in C_\mu,\ \mu\ge -1$ is defined as

$$J^\alpha f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-x)^{\alpha-1} f(x)dx,\ \ (\alpha\ge 0) \tag{3}$$

Where $\Gamma(.)$ is the well known gamma function and

$$J^0 f(t)=f(t) \tag{4}$$

For the Riemann-Liouville fractional integral, we have

$$J^\alpha t^r=\frac{\Gamma(r+1)}{\Gamma(r+\alpha+1)}t^{\alpha+r} \tag{5}$$

Definition 3: The fractional derivative of $f(t)$ in the Caputo [1] sense is defined as

$$D^\alpha f(t)=J^{n-\alpha}D^n f(t)$$

$$=\frac{1}{\Gamma(n-1)}\int_0^t (t-x)^{n-\alpha-1} f^{(n)}(x)dx \tag{6}$$

For $n-1<\alpha\le n$, $n\in\mathrm{N}$, $t>0$

For the Caputo derivative, we have

$$D^\alpha t^r=\frac{\Gamma(r+1)}{\Gamma(r+1-\alpha)}t^{r-\alpha} \tag{7}$$

and $$\left(J^\alpha D^\alpha\right)f(t)=f(t)-\sum_{k=0}^{n-1}\frac{t^k}{k!}f^{(k)}(0) \tag{8}$$

3.0 The Decomposition Algorithm for Fractional Biological Population Model

For the fractional biological population model (1), with the associated initial condition (2), the fractional derivative is invertible. Applying the inverse fractional operator (J^α) to both sides of (1) yields

$$u(x,y,t)=f_0(x,y)+J^\alpha\left\{\frac{\partial^2(u^2)}{\partial x^2}+\frac{\partial^2(u^2)}{\partial y^2}+hu^a\left(1-ru^b\right)\right\} \tag{9}$$

The Iterative decomposition [] suggests that the solution could be decomposed into the infinite series of convergent terms.

$$u(x,y,t)=\sum_{n=0}^{\infty}u_n(x,y,t) \tag{10}$$

The second term in the RHS of (9) can be decomposed as

$$\mathrm{u}(x,y,t)= f_0(x,y)+ J^{\alpha}\sum\left[\sum_{n=0}^{\infty}\left\{\frac{\partial^2}{\partial x^2}\left(\sum_{j=0}^{n}u_j\right)^2+\frac{\partial^2}{\partial y^2}\left(\sum_{j=0}^{n}u_j\right)^2+h\left(\sum u_j{}^{\alpha}\right)\left(1-r\left(\sum u_j\right)^b\right)\right\}\right]$$

$$-\left\{\frac{\partial^2}{\partial x^2}\left(\sum_{j=0}^{n-1}u_j\right)^2+\frac{\partial^2}{\partial y^2}\left(\sum_{j=0}^{n-1}u_j\right)^2+h\left(\sum_{j=0}^{n-1}u_j\right)^{\alpha}\left(1-r\left(\sum_{j=0}^{n-1}u_j\right)^b\right)\right\} \tag{11}$$

From (10) and (11), we then have

$$\sum_{n=\infty}u_n(x,y,t)= f_0(x,y)+$$

$$J^{\alpha}\left[\sum_{n=0}^{\infty}\left\{\frac{\partial^2}{\partial x^2}\left(\sum_{j=0}^{n}u_j\right)^2+\frac{\partial^2}{\partial y^2}\left(\sum_{j=0}^{n}u_j\right)^2+h\left(\sum u_j{}^{\alpha}\right)\left(1-r\left(\sum_{j=0}^{n}u_j\right)^b\right)\right\}\right.$$

$$-\left\{\frac{\partial^2}{\partial x^2}\left(\sum_{j=0}^{n-1}u_j\right)^2+\frac{\partial^2}{\partial y^2}\left(\sum_{j=0}^{n-1}u_j\right)^2+h\left(\sum_{j=0}^{n-1}u_j\right)^{\alpha}\left(1-r\left(\sum_{j=0}^{n-1}u_j\right)^b\right)\right\} \tag{12}$$

Taking

$u_0 = f_0(x,y)$ and

u_{n+1} = Second term of (12) (13)

We may then approximate the solution by the truncated series

$$\Phi_N = \sum_{n=0}^{N-1}y_n \tag{14}$$

And $\lim_{N\to\infty}\Phi_N(x,y,t)=u(x,y,t)$ (15)

4.0 Numerical Examples

To illustrate the efficiency and accuracy of the Iterative Decomposition Method, for nonlinear biological population model, we consider the following examples.

Example 4.1

Consider the nonlinear fractional biological population model [2, 9]

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}}=\frac{\partial^2(u^2)}{\partial x^2}+\frac{\partial^2(u^2)}{\partial y^2}+hu \tag{16}$$

Subject to the initial condition

$$u(x,y,0)=\sqrt{xy} \tag{17}$$

Problem (16)-(17) is readily obtained from (1) in the case a=1, r=0 which corresponds to the Malthusian model.

Applying the inverse operator of the fractional differential operator to both sides of (16) and using the initial condition we have

$$u(x,y,t) = \sqrt{xy} + J^{\alpha}\left\{u^2{}_{xx} + u^2{}_{yy} + hu\right\} \tag{18}$$

Taking $u_0 = \sqrt{xy}$ (19)

Then, $u_1 = J^{\alpha}\left\{(u_0{}^2)_{xx} + (u_0{}^2)_{yy} + hu_0\right\}$

$$= h\,\frac{\sqrt{xy}\,t^{\alpha}}{\Gamma(\alpha+1)}$$

$$u_2 = J^{\alpha}\left\{h\left(\sqrt{xy} + h\sqrt{xy}\,\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right)\right\} - J^{\alpha}\left\{hu_0\right\} \tag{20}$$

$$= \frac{h^2\sqrt{xy}}{\Gamma(2\alpha+1)}t^{2\alpha}$$

$$u_3 = \frac{h^3\sqrt{xy}}{\Gamma(3\alpha+1)}t^{3\alpha} \tag{21}$$

$$u_4 = \frac{h^4\sqrt{xy}}{\Gamma(4\alpha+1)}t^{4\alpha} \tag{22}$$

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$$u_n = \frac{h^n\sqrt{xy}}{\Gamma(n\alpha+1)}t^{n\alpha} \tag{23}$$

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Then, $u(x,y,t)$ can be approximated as

$$u(x,y,t) = \sum_{n=0}^{\infty} u_n$$

$$= \sqrt{xy}\left\{1 + \frac{ht^{\alpha}}{\Gamma(\alpha+1)} + \frac{h^2t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{h^3t^{3\alpha}}{\Gamma(3\alpha+1)} + \ldots\right\}$$

$$= \sqrt{xy}\left\{\sum_{n=0}^{\infty}\frac{(kt^{\alpha})^n}{\Gamma(n\alpha+1)}\right\} \tag{24}$$

From the definition of Mittag-Leffler function,

$$u(x,y,t) = \sqrt{xy}\;E_{\alpha}\left(ht^{\alpha}\right) \tag{25}$$

Which is the exact solution [2 - 4].

Example 2

Consider the generalized biological population model for a=1, r=0, h=1 in (1) [3, 4]

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}} = \frac{\partial^2(u^2)}{\partial x^2} + \frac{\partial^2(u^2)}{\partial y^2} + u \tag{26}$$

Subject to the initial condition

$$u(x,y,0) = \sqrt{\sin x \sinh y} \tag{27}$$

Applying the inverse operator to both sides of (26),

$$u(x,y,t)=u(x,y,0)+J^{\alpha}\left\{\frac{\partial^2(u^2)}{\partial x^2}+\frac{\partial^2(u^2)}{\partial y^2}+u\right\} \qquad (28)$$

$$=\sqrt{\sin x \sinh y}+J^{\alpha}\left\{(u^2)_{xx}+(u^2)_{yy}+u\right\} \qquad (29)$$

Taking $u_0=\sqrt{\sin x \sinh y}$, we have

$$u_1=J^{\alpha}\left\{(u_0{}^2)_{xx}+(u_0{}^2)_{yy}+u_0\right\}$$

$$=\sqrt{\sin x \sinh y}\ \frac{t^{\alpha}}{\Gamma(\alpha+1)} \qquad (30)$$

$$u_2=J^{\alpha}\left\{\sqrt{\sin x \sinh y}+\sqrt{\mathrm{sinxsinhy}}\ \frac{t^{\alpha}}{\Gamma(\alpha+1)}\right\}-J^{\alpha}\{u_0\} \qquad (31)$$

$$=\sqrt{\sin x \sinh y}\ \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$u_3=\sqrt{\sin x \sinh y}\ \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \qquad (32)$$

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$$u_n=\sqrt{\sin x \sinh y}\ \frac{(t^{\alpha})^n}{\Gamma(n\alpha+1)} \qquad (33)$$

Then, $u(x,y,t)=\sqrt{\sin x \sinh y}\left\{1+\frac{\mathrm{t}^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\ldots+\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}\right\}$ (34)

$$=\sqrt{\sin x \sinh y}\ E_{\alpha}(t^{\alpha}) \qquad (35)$$

which agrees with the solutions in [2 – 5, 9].

Or $u(x,y,t)=\sqrt{\sin x \sinh y}\ e^{t}$ as $\alpha\to 1$ [4, 9]

Example 3

Consider the fractional biological population model [5]

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}}=\frac{\partial^2(u^2)}{\partial x^2}+\frac{\partial^2(u^2)}{\partial y^2}+hu^{-1}-hr \qquad (36)$$

Subject to the initial condition

$$u(x,y,0)=\sqrt{\left(\frac{hr}{4}x^2+\frac{hr}{4}y^2+y+5\right)} \qquad (37)$$

The exact solution for the special case $\alpha=1$ is given in [5]

$$u(x,y,t)=\sqrt{\left(\frac{hr}{4}x^2+\frac{hr}{4}y^2+y+2ht+5\right)} \tag{38}$$

Proceeding as in earlier examples, we have

$$u_0=\sqrt{\left(\frac{hr}{4}x^2+\frac{hr}{4}y^2+y+5\right)}$$

$$u_1=\frac{h}{\sqrt{\left(\frac{\text{hr}}{4}x^2+\frac{\text{hr}}{4}y^2+y+5\right)}}\cdot\frac{t^{\alpha}}{\Gamma(\alpha+1)} \tag{39}$$

$$u_2=\frac{-2h^2}{\left\{\sqrt{\left(\frac{\text{hr}}{4}x^2+\frac{\text{hr}}{4}y^2+y+5\right)}\right\}^3}\cdot\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \tag{40}$$

$$u_3=\frac{3h^3}{\left\{\sqrt{\left(\frac{\text{hr}}{4}x^2+\frac{\text{hr}}{4}y^2+y+5\right)}\right\}^5}\cdot\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \tag{41}$$

$$u_4=\frac{-4h^4}{\left\{\sqrt{\left(\frac{\text{hr}}{4}x^2+\frac{\text{hr}}{4}y^2+y+5\right)}\right\}^7}\cdot\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} \tag{42}$$

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Then we may approximate $u(x,y,t)$ as

$$u(x,y,t)=\sqrt{\left(\frac{hr}{4}x^2+\frac{hr}{4}y^2+y+5\right)}+\frac{ht^{\alpha}}{\sqrt{\left(\frac{\text{hr}}{4}x^2+\frac{\text{hr}}{4}y^2+y+5\right)}}\cdot\frac{1}{\Gamma(\alpha+1)}$$

$$\frac{-2h^2}{\left\{\sqrt{\left(\frac{\text{hr}}{4}x^2+\frac{\text{hr}}{4}y^2+y+5\right)}\right\}^3}\cdot\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\dots \tag{43}$$

$$u(x,y,t)=u_0+\frac{ht^{\alpha}}{u_0}\sum_{n=0}^{\infty}\frac{n+1}{\Gamma((n+1)\alpha+1)}\left(\frac{-ht^{\alpha}}{u_0{}^2}\right)^n \tag{44}$$

As $\alpha\to 1,$ we have

$$u(x,y,t) = u_0 + \frac{ht}{u_0}\exp\left(\frac{-ht}{u_0^{\ 2}}\right) \tag{45}$$

5.0 Conclusion

In this work, we have applied the Iterative Decomposition method to solve fractional biological population models, subject to some given initial conditions. The method gives a significant improvement over some existing method, in terms of ease of computation of the solution terms. The solutions obtained compare favourably with known/ existing solutions.

References

[1] Podlubny, I, Fractional Differential Equations, Academic Press, New York, 1999.

[2] Arafa, a.A.M., Rida, S.Z; Mohamed, H., Homotopy Analysis Method for solving Biological Population Model, Commun. Theor. Phys. 56:797-800, 2011.

[3] El-Sayed, A.M.A., Rida, S.; Arafa, A.A.M; Exact solutions of Fractional –Order Biological Population Model, Commun. Theor. Phys. 25:992-996, 2009.

[4] He, J.H; Some Applications of nonlinear Fractional Differential Equations and their Applications, Bull. Sci. Technol., 15(2) : 86-90,1999.

[5] Hilfer, R; Applications of Fractional Calculus in Physics World. Scientific, Singapore, 2000.

[6] Kumar,D; Singh, Jagdev, Sushila, Application of Homotopy Analysis Transform Method to Fractional Biological Population Model, Romanian Reports in Physics, 65(1): 63-75, 2013.

[7] Liu, Li,Z, Yueyin, Z; Homotopy Perturbation Method to Fractional Biological Population Equation, Fractional Diff. Equations, 1(1): 117-124, 2011.

[8] Momani, S; and Odibat, Z; Homotopy Perturbation Method for nonlinear Partial Differential Equations of Fractional Order, Phys. Lett. A. 365 : 345-350, 2007.

[9] Prajapati, R.N; Mohan, R; AcouplingTecnique for Analytical Solution of Time Fractional Biological Population Model, J. Engr. Comp. Applied Sci. , 2(2) : 30-38, 2013.

[10] Taiwo, O.A; Odetunde, O.S and Adekunle, Y.A; Numerical Approximation of One Dimensional Biarmonic Equations by an Iterative Decomposition Method, Int. J. Math. Sci; 20(1) : 37-44, 2009.

[11] Taiwo, O.A; and Odetunde, O.S; Approximation of Multi-Order Fractional Differential Equations by an Iterative Decomposiion Method, Amer. J. Engr. Sci. Tech. Res. 1(2) : 1-9, 2013