Journal of the Nigerian Association of Mathematical Physics
***Volume27*, (*July, 2014*), pp 85 – 100**

Flexural Motions under a Traversing Partially Distributed Load of a Uniform Rayleigh Beam with General Boundary Conditions

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Abstract

The dynamic analysis of a uniform Rayleigh beam resting on Winkler-type foundation and under uniform distributed moving masses is investigated in this paper. A procedure involving the generalized integral transformation with beam function as kernel, the use of properties of Heaviside function to express it in series form and a modification of the Struble's asymptotic technique was used to obtain an analytical solution valid for all variants of classical boundary conditions to the dynamical problem. The analytical solution and numerical analysis show that the critical speed for the moving distributed mass problem is reached earlier than that of the moving distributed force problem for both illustration examples considered. The results further show that an upward variations of rotatory inertia correction factor and foundation stiffness decrease the response amplitudes of the uniform Rayleigh beam whether the beam is traversed by moving distributed force or moving distributed mass.

1.0 Introduction

The flexural motions under moving masses of Beam-structures on elastic foundations have received great attention of researchers due to its wide applications in mechanical and civil Engineering over the years [1-5]. However, in most of the investigations, solution procedure fails to cover the entire range of practical problems likely to be encountered. In particular, solution techniques are not easily adjustable to the cases in which the end conditions are not simple ones. This shortcoming was first addressed by Sadiku and Leipholz [6]. The problem of elastic beam under the actions of moving concentrated masses was studied. A method capable of solving this problem for all classical boundary conditions was developed. Though, the theory developed in [6] is very versatile, its application is only limited to the case of beam executing flexural vibrations according to the simple Bernoulli-Euler theory of flexure. Nonetheless, it is known that during vibration, a typical element of a beam performs not only a translatory motion but also rotates [7]. Thus, there is a need to consider beams where motion is not governed by Bernoulli-Euler theory. To this end, Gbadeyan and Oni [8] developed a more robust technique which could be used to tackle the problems of Bernoulli-Euler beam under moving concentrated masses and also those of thick Rayleigh beams. Using this method and other approaches, the problem of beam structures under moving loads have been solved for various classical boundary conditions other than simple ones by several authors namely Oni and Omolofe [9], Oni and Awodola [10] , Oni and Ogunyebi [11].

It is remarked at this juncture, that in most of these investigations, the loads have been idealized as concentrated loads whereas, in practice, moving loads are actually distributed over a small segment or over the entire length of the structure. To this end, Esmailzadeh and Ghorashi [12] studied the moving-load-induced vibration problem using a moving uniform distributed mass model. They solved the problem by means of the conventional analytical approach, which is only suitable for the simple horizontal beam and will suffer much difficulty if the structures are complicated. Other recent works involving uniformly distributed moving mass model were carried out by Gbadeyan and Dada [13], Dada [14], Oni and Ogunyebi [11], Bogacz and Czyczula [15], Kargarnovin and Younesian [16], Sapountzakis and Tsiatus [17] and Jia-Jiang Wu [18]. These works, however, concentrated on numerical simulation or limited their consideration to structures having simple end supports. This paper, therefore, presents the dynamic analysis of the flexural motions under travelling distributed masses of uniform Rayleigh beams with general boundary conditions. The focus is on analytical procedures to generate closed form solutions to the dynamical problem. The moving distributed force problem is treated as special cases in the illustrative

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examples considered.This paper is sequel to an earlier one [19] which treated the problem of elastic uniform Rayleigh beam having simply supported boundary conditions.

2.0 Governing Equation

The motion of a finite uniform Rayleigh beam carrying a motion of distributed load of mass M travelling at constant speed c is considered. The position of the load along the beam is given by the single valued function of time x. The equation governing the transverse displacement of the Rayleigh beam V(x,t) neglecting damping and shear deformation effect is given by [1]

$$EI\frac{\partial^4 V(x,t)}{\partial x^4}+\mu\frac{\partial^2 V(x,t)}{\partial t^2}-\mu r^o\frac{\partial^4 V(x,t)}{\partial x^2\partial t^2}+KV(x,t)=MgH(x-ct)\left[1-\frac{1}{g}\frac{d^2V(x,t)}{dt^2}\right] \quad (2.1)$$

where x is the spatial coordinate, t is the time, $V(x,t)$ is the transversedisplacement, E is Young's modulus, I is the constant Moment of inertia of the beam, μ is the constant mass per unit length of the beam, $r^o$is the measure of rotatory inertia correction factor, K is the elastic foundation constant, a is the acceleration due to gravity. For this problem, the distributed load moving on the beam under consideration has mass commensurable with the mass of the beam. Consequently, the load inertia is not negligible but significantly affects the behaviour of the dynamical system. For our purpose we will take $x=ct$,where c the velocity of the distributed mass is, the time t is assumed to be limited to that interval of time within which the mass μ is on the beam, that is

$$0\le ct\le L \quad (2.4)$$

g is the acceleration due to gravity, $H(x-ct)$ is the Heaviside function defined as

$$H(x-ct)=\begin{cases}0, & x<ct\\ 1, & x>ct\end{cases} \quad (2.5)$$

and$\frac{d^2}{dt^2}$is the convective acceleration operator defined as

$$\frac{d^2}{dt^2}=\frac{\partial^2}{\partial t^2}+2c\frac{\partial^2}{\partial x\partial t}+c^2\frac{\partial^2}{\partial x^2} \quad (2.12)$$

because the mass M moves with constant velocity the boundary conditions of the above problem are assumed to be arbitrary and the initial conditions are given by

$$V(x,0)=V_t(x,0)=0 \quad (2.13)$$

3.0 Method of Solution

First,substituting (2.12) into (2.1), one obtains

$$EI\frac{\partial^4 V(x,t)}{\partial x^4}+\mu\frac{\partial^2 V(x,t)}{\partial t^2}-\mu r^0\frac{\partial^4 V(x,t)}{\partial x^2\partial t^2}+KV(x,t)+$$

$$MH(x-ct)\left(\frac{\partial^2 V(x,t)}{\partial t^2}+2c\frac{\partial^2 V(x,t)}{\partial x\partial t}+c^2\frac{\partial^2 V(x,t)}{\partial x^2}\right)=MgH(x-ct) \quad (3.1)$$

Equation (3.1) is a fourth order partial differential equation with constant coefficients. It is evident that exact closed form solution to this equation is impossible. To this end, an approximate analytical solution is desirable. Thus, a general approach is developed in order to solve the initial boundary value problem in equation (3.1). The important features of this technique are

i) It is applicable for all variants of classical boundary conditions often encountered in practice.

ii) It can also solve both thin and thick beam problems which earlier methods have been unable to tackle.

In order to obtain a solution valid for all variants of classical boundary conditions, in the first instance, we adopt the method of generalized integral transform technique extensively described in [8].This integral transformation technique is given by

$$\overline{V}(m,t)=\int_0^L V(x,t)U_m(x)dx \quad (3.2)$$

with the inverse

$$V(x,t)=\sum_{m=1}^{\infty}\frac{\mu}{V_m}\overline{V}(m,t)U_m(x) \quad (3.3)$$

where

$$V_m = \int_0^L \mu U_m^2(x)dx \tag{3.4}$$

In equation (3.4), $U_m(x)$ is any function chosen such that the pertinent boundary conditions are satisfied. An appropriate selection of function for the beam problems are beam mode shapes. Thus, the mth normal mode of vibration of a uniform beam

$$U_m(x) = Sin\frac{\lambda_m x}{L} + A_m Cos\frac{\lambda_m x}{L} + B_m Sinh\frac{\lambda_m x}{L} + C_m Cosh\frac{\lambda_m x}{L} \tag{3.5}$$

is chosen as a suitable kernel of the integralin (3.2), where, $\lambda_m$is the mode frequency and $A_{m,}\ B_{m,}\ C_m$ are constants. The parameters $\lambda_m,\ A_m,\ B_m$ and C_m are obtained by substituting (3.5) into the appropriate boundary conditions.
Applying the generalized integral transforms (3.2), equation (3.1) can be written as

$$EIT_A(t) + EIG(0,L,t) + \mu\bar{V}_{tt}(m,t) - \mu r^0 T_B(t) + K\bar{V}(m,t) + T_C(t) +$$
$$2cT_D(t) + c^2T_E(t) = Mg\int_0^L H(x-ct)U_m dx \tag{3.6}$$

Where

$$G(0,L,t) = \left[\frac{\partial^3 V(x,t)}{\partial x^3}U_m(x) - \frac{\partial^2 V(x,t)}{\partial x^2}U_m'(x) + \frac{\partial V(x,t)}{\partial x}U_m''(x) - V(x,t)U_m'''(x)\right]_0^L \tag{3.7}$$

$$T_A(t) = \int_0^L V(x,t)\frac{\partial^4 U_m(x)}{\partial x^4}dx \tag{3.8}$$

$$T_B(t) = \int_0^L \frac{\partial^4 V(x,t)}{\partial x^2 \partial t^2}U_m(x)dx \tag{3.9}$$

$$T_C(t) = M\int_0^L H(x-ct)\frac{\partial^2 V(x,t)}{\partial t^2}U_m(x)dx \tag{3.10}$$

$$T_D(t) = M\int_0^L H(x-ct)\frac{\partial^2 V(x,t)}{\partial x \partial t}U_m(x)dx \tag{3.11}$$

$$T_E(t) = M\int_0^L H(x-ct)\frac{\partial^2 V(x,t)}{\partial x^2}U_m(x)dx \tag{3.12}$$

It is generally known that the natural modes in (3.5) satisfy the homogeneous differential equation

$$EI\frac{d^4 U_m(x)}{dx^4} - \mu\omega_m^2 U_m(x) = 0 \tag{3.13}$$

For the uniform beam, the parameter ω_m is the natural circular frequency defined by

$$\omega_m^2 = \frac{\lambda_m^4}{L^4}\frac{EI}{\mu} \tag{3.14}$$

From equation (3.13), it is straight forward to write

$$\int_0^L V(x,t)\frac{d^4 U_m(x)}{dx^4}dx = \frac{\mu}{EI}\omega_m^2\int_0^L V(x,t)U_m(x)dx \tag{3.15}$$

Thus by (3.2)

$$T_A(t) = \frac{\mu}{EI}\omega_m^2\bar{V}(m,t) \tag{3.16}$$

Noting that $\bar{V}(k,t)$ [14] is just the co-efficient of the generalized integral transforms, it is evident that

$$V(x,t)=\sum_{k=1}^{\infty}\frac{\mu}{V_k}\bar{V}(k,t)U_k(x) \tag{3.17}$$

Consequently,

$$\frac{\partial^2}{\partial x^2}V(x,t)=\sum_{k=1}^{\infty}\frac{\mu}{V_k}\bar{V}(k,t)\frac{d^2}{dx^2}U_k(x) \tag{3.18}$$

and

$$\frac{\partial^4}{\partial x^2\partial t^2}V(x,t)=\sum_{k=1}^{\infty}\frac{\mu}{V_k}\bar{V}_{tt}(k,t)\frac{d^2U_k(x)}{dx^2} \tag{3.19}$$

so that integral (3.9) becomes,

$$T_B(t)=\frac{\partial^2}{\partial t^2}\left\{\sum_{k=1}^{\infty}\frac{\mu}{V_k}\bar{V}(k,t)\int_0^L\frac{d^2U_k(x)}{dx^2}U_m(x)dx\right\} \tag{3.20}$$

In order to evaluate the integrals (3.10), (3.11) and (3.12), use is made of theFourier series representation ofthe Heaviside step function given by

$$H(x-ct)=\frac{1}{4}+\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{Sin\big((2n+1)\pi(x-ct)\big)}{2n+1},\quad 0<x<L \tag{3.21}$$

using equation (3.21), integral $T_C(t)$ becomes

$$T_C(t)=\frac{M\mu}{V_k}\sum_{k=1}^{\infty}\bar{V}_{tt}(k,t)\left[\frac{1}{4}\int_0^L U_k(x)U_m(x)dx+\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{Cos(2n+1)\pi ct}{2n+1}.\right.$$
$$\int_0^L Sin(2n+1)\pi x U_k(x)U_m(x)dx-\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{Sin(2n+1)\pi ct}{2n+1}.$$
$$\left.\int_0^L Cos(2n+1)\pi x U_k(x)U_m(x)dx\right] \tag{3.22}$$

In the same manner, it is straight forward to show that

$$T_D(t)=\frac{M\mu}{V_k}\sum_{k=1}^{\infty}\bar{V}_t(k,t)\left[\frac{1}{4}\int_0^L U_k'(x)U_m(x)dx+\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{Cos(2n+1)\pi ct}{2n+1}.\right.$$
$$\int_0^L Sin(2n+1)\pi x U_k'(x)U_m(x)dx-\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{Sin(2n+1)\pi ct}{2n+1}.$$
$$\left.\int_0^L Cos(2n+1)\pi x U_k'(x)U_m(x)dx\right] \tag{3.23}$$

and

$$T_E(t)=\frac{M\mu}{V_k}\sum_{k=1}^{\infty}\bar{V}(k,t)\left[\frac{1}{4}\int_0^L U_k''(x)U_m(x)dx+\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{Cos(2n+1)\pi ct}{2n+1}.\right.$$
$$\int_0^L Sin(2n+1)\pi x U_k''(x)U_m(x)dx-\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{Sin(2n+1)\pi ct}{2n+1}.$$
$$\left.\int_0^L Cos(2n+1)\pi x U_k''(x)U_m(x)dx\right] \tag{3.24}$$

Using the properties of the Heaviside function, the integral in the right hand side of equation (3.6) can be expressed as

$$T_F(t)=\frac{MgL}{\lambda_m}\left[-Cos\lambda_m+A_mSin\lambda_m+B_mCosh\lambda_m+C_mSinh\lambda_m+Cos\frac{\lambda_m ct}{L}\right.$$
$$\left.-A_mSin\frac{\lambda_m ct}{L}-B_mCosh\frac{\lambda_m ct}{L}-C_mSinh\frac{\lambda_m ct}{L}\right] \tag{3.25}$$

Substituting (3.16), (3.20), (3.22), (3.23), (3.24), and (3.25) into (3.6) and simplifying yield

$$\bar{V}_{tt}(m,t)+\left(\omega_m^2+\frac{K}{\mu}\right)\bar{V}(m,t)-r^0\sum_{k=1}^{\infty}\bar{V}_{tt}(k,t)G_B(k,m)+\Gamma_0 L\left\{\sum_{k=1}^{\infty}\left[\frac{1}{4}\bar{V}_{tt}(k,t).\right.\right.$$

$$G_D(k,m)+\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{Cos(2n+1)\pi ct}{2n+1}\bar{V}_{tt}(k,t)G_E(n,k,m)-\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{Sin(2n+1)\pi ct}{2n+1}\bar{V}_{tt}(k,t).$$

$$G_F(n,k,m)+\frac{c}{2}\bar{V}_t(k,t)G_G(k,m)+\frac{2c}{\pi}\sum_{n=0}^{\infty}\frac{Cos(2n+1)\pi ct}{2n+1}\bar{V}_t(k,t)G_H(n,k,m)-$$

$$\frac{2c}{\pi}\sum_{n=0}^{\infty}\frac{Sin(2n+1)\pi ct}{2n+1}\bar{V}_t(k,t)G_I(n,k,m)+\frac{c^2}{4}\bar{V}(k,t)G_B(k,m)+\frac{c^2}{\pi}\sum_{n=0}^{\infty}\frac{Cos(2n+1)\pi ct}{2n+1}.$$

$$\left.\left.\bar{V}(k,t)G_J(n,k,m)-\frac{c^2}{\pi}\sum_{n=0}^{\infty}\frac{Sin(2n+1)\pi ct}{2n+1}\bar{V}(k,t)G_L(n,k,m)\right]\right\}=\frac{PL}{\mu\lambda_m}[-Cos\lambda_m$$

$$+A_mSin\lambda_m+B_mCosh\lambda_m+C_mSinh\lambda_m+Cos\frac{\lambda_m ct}{L}+A_mSin\frac{\lambda_m ct}{L}-B_mCosh\frac{\lambda_m ct}{L}$$

$$\left.-C_mSinh\frac{\lambda_m ct}{L}\right] \tag{3.26}$$

where

$$\Gamma_0=\frac{M}{\mu L} \tag{3.27}$$

$$P=Mg \tag{3.28}$$

$$G_B(k,m)=\frac{1}{V_k}\int_0^L U_k''(x)U_m(x)dx \tag{3.29}$$

$$G_D(k,m)=\frac{1}{V_k}\int_0^L U_k(x)U_m(x)dx \tag{3.30}$$

$$G_E(n,k,m)=\frac{1}{V_k}\int_0^L Sin(2n+1)\pi x U_k(x)U_m(x)dx \tag{3.31}$$

$$G_F(n,k,m)=\frac{1}{V_k}\int_0^L Cos(2n+1)\pi x U_k(x)U_m(x)dx \tag{3.32}$$

$$G_G(k,m)=\frac{1}{V_k}\int_0^L U_k'(x)U_m(x)dx \tag{3.33}$$

$$G_H(n,k,m)=\frac{1}{V_k}\int_0^L Sin(2n+1)\pi x U_k'(x)U_m(x)dx \tag{3.34}$$

$$G_I(n,k,m) = \frac{1}{V_k}\int_0^L Cos(2n+1)\pi x U_k'(x)U_m(x)dx \qquad (3.35)$$

$$G_J(n,k,m) = \frac{1}{V_k}\int_0^L Sin(2n+1)\pi x U_k''(x)U_m(x)dx \qquad (3.36)$$

and

$$G_L(n,k,m) = \frac{1}{V_k}\int_0^L Cos(2n+1)\pi x U_k''(x)U_m(x)dx \qquad (3.37)$$

Equation (3.26) is the transformed equation governing the problem of a uniform Rayleigh beam on a constant elastic foundation when under the action of travelling distributed load. This coupled non-homogeneous second order differential equation holds for all general boundary conditions. In what follows, two special cases of equation (3.26) are discussed.

4.0 Solution of The Transformed Equation

(i) Case I

If we neglect the inertia term, we have the classical case of a moving force problem. Under this assumption, $\Gamma_0 = 0$ and equation (3.26) after some simplifications and rearrangements becomes

$$\bar{V}_{tt}(m,t) + \left(\omega_m^2 + \frac{K}{\mu}\right)\bar{V}(m,t) - r^0\sum_{k=1}^{\infty}\bar{V}_{tt}(k,t)G_B(k,m) = \frac{PL}{\mu\lambda_m}[-Cos\lambda_m + A_m Sin\lambda_m +$$

$$B_m Cosh\lambda_m + C_m Sinh\lambda_m \quad + Cos\frac{\lambda_m ct}{L} - A_m Sin\frac{\lambda_m ct}{L} - B_m Cosh\frac{\lambda_m ct}{L} - C_m Sinh\frac{\lambda_m ct}{L}] \qquad (4.1)$$

Evidently, an exact analytical solution to this equation is not possible. To this end, a modification of the asymptotic method due to Struble already alluded to shall be used to tackle this problem. Consequently, equation (4.1) is rearranged to take the form

$$\bar{V}_{tt}(m,t) + \frac{\omega_{mf}^2}{[1-\varepsilon_0 LG_B(m,m)]}\bar{V}(m,t) - \frac{\varepsilon_0}{[1-\varepsilon_0 LG_B(m,m)]}\left[\sum_{\substack{k=1\\k\neq m}}^{\infty} L\,\bar{V}_{tt}(k,t)G_B(k,m)\right]$$

$$= \frac{PL}{\mu\lambda_m[1-\varepsilon_0 LG_B(m,m)]}[-Cos\lambda_m + A_m Sin\lambda_m + B_m Cosh\lambda_m + C_m Sinh\lambda_m$$

$$+Cos\frac{\lambda_m ct}{L} - A_m Sin\frac{\lambda_m ct}{L} - B_m Cosh\frac{\lambda_m ct}{L} - C_m Sinh\frac{\lambda_m ct}{L}] \qquad (4.2)$$

where,

$$\omega_{mf}^2 = \omega_m^2 + \frac{K}{\mu}, \qquad \varepsilon_0 = \frac{r^0}{L} \qquad (4.3)$$

By this technique, one seeks the modified frequency corresponding to the frequency of the free system due to the presence of the effect of the rotatory inertia correction factor R_0. An equivalent free system operator defined by the modified frequency then replaces equation (4.2). Thus, the right hand side of equation (4.2) is set to zero, we then consider a parameter $\varepsilon_1 < 1$ for any arbitrary ratio ε_0, defined as

$$\varepsilon_1 = \frac{\varepsilon_0}{\varepsilon_0 + 1} \qquad (4.4)$$

so that

$$\varepsilon_0 = \varepsilon_1 + o(\varepsilon_1^2) \qquad (4.5)$$

But,

$$\frac{1}{1-\varepsilon_0 LG_B(m,m)} = 1 + \varepsilon_1 LG_B(m,m) \qquad (4.6)$$

where

$$|\varepsilon_1 L G_B(m,m)| < 1 \qquad (4.7)$$

Setting $\varepsilon_1 = 0$, a situation corresponding to the case in which the rotatory inertia correction factor is regarded as negligible is obtained, then the solution of (4.2) can be written as

$$\bar{V}(m,t) = A_0 Cos\left[\omega_{mf} t - B_0\right] \qquad (4.8)$$

where $A_0,\ \omega_{mf}\ and\ B_0$ are constants.
Furthermore as $\varepsilon_1 < 1$, Struble's technique requires that the asymptotic solution of the homogeneous part of equation (4.2) be of the form [16]

$$\bar{V}(m,t) = \eta(m,t) Cos\left[\omega_{mf} t - \alpha(m,t)\right] + \varepsilon_1 \bar{V}(k,t) + o(\varepsilon_1^2) \qquad (4.9)$$

where $\eta(m,t)$ and $\alpha(m,t)$ are slowly varying functions of time
To obtain the modified frequency, equation (4.9) and its derivatives are substituted into the homogeneous part of equation (4.2) and taking into account (4.5) one obtains,

$$-2\dot{\eta}(m,t)\omega_{mf} Sin\left[\omega_{mf} t - \alpha(m,t)\right] + 2\eta(m,t)\dot{\alpha}(m,t)\omega_{mf} Cos\left[\omega_{mf} t - \alpha(m,t)\right]$$

$$+\omega_{mf}^2 \varepsilon_1 L G_B(m,m)\eta(m,t) Cos\left[\omega_{mf} t - \alpha(m,t)\right] = 0 \qquad (4.10)$$

retaining terms to $o(\varepsilon_1)$ only.
The variational equations are obtained by equating the coefficients of $Sin\left[\omega_{mf} t - \alpha(m,t)\right]$ and $Cos\left[\omega_{mf} t - \alpha(m,t)\right]$terms on both sides of equation (4.10) to zero. Thus we have

$$-2\dot{\eta}(m,t)\omega_{mf} = 0 \qquad (4.11)$$

$$\eta(m,t)\omega_{mf}\left[2\dot{\alpha}(m,t) + \omega_{mf}\varepsilon_1 L G_B(m,m)\right] = 0 \qquad (4.12)$$

Solving equations (4.11) and (4.12) respectively, one obtains

$$\eta(m,t) = A_m \qquad (4.13)$$

$$\alpha(m,t) = -\frac{\omega_{mf}\varepsilon_1 L G_B(m,m)}{2} t + B_m \qquad (4.14)$$

where A_m and B_m are constants.
Therefore when the effect of the rotatory inertia correction factor is considered, the first approximation to the homogeneous system is

$$\bar{V}(m,t) = A_m Cos\left[\gamma_{mf} t - B_m\right] \qquad (4.15)$$

where

$$\gamma_{mf} = \omega_{mf}\left(1 + \frac{\varepsilon_1 G_B(m,m)}{2}\right) \qquad (4.16)$$

represents the modified natural frequency due to the effect of rotatory inertia correction factor. Thus to solve the non-homogeneous equation (4.2), the differential operator which acts on $\bar{V}(m,t)$ and $\bar{V}(k,t)$ is replaced by the equivalent free system operator defined by the modified frequency γ_{mf}, i.e

$$\bar{V}_{tt}(m,t) + \gamma_{mf}^2 \bar{V}(m,t) = P_0[-Cos\lambda_m + A_m Sin\lambda_m + B_m Cosh\lambda_m + C_m Sinh\lambda_m$$

$$+Cos\frac{\lambda_m ct}{L} - A_m Sin\frac{\lambda_m ct}{L} - B_m Cosh\frac{\lambda_m ct}{L} - C_m Sinh\frac{\lambda_m ct}{L}] \quad (4.17)$$

where

$$P_0 = \frac{PL}{\mu\lambda_m(1-\varepsilon_1 LG_B(m,m))} \quad (4.18)$$

Using the method of Laplace transformationin conjunction with the initial condition, it is not difficult to show that the solution to equation (4.17) is given by

$$\bar{V}(m,t) = P_0\left[\frac{Q(m,c)(1-Cos\gamma_{mf}t)}{\gamma_{mf}} + \frac{Cos\alpha_c t - Cos\gamma_{mf}t}{\gamma_{mf}^2 - \alpha_c^2}\right.$$

$$-\frac{A_m(\gamma_{mf}Sin\alpha_c t - \alpha_c Sin\gamma_{mf}t)}{\gamma_{mf}(\gamma_{mf}^2-\alpha_c^2)} - \frac{B_m(Cosh\alpha_c t - Cos\gamma_{mf}t)}{\alpha_c^2+\gamma_{mf}^2}$$

$$\left.-\frac{C_m(\gamma_{mf}Sinh\alpha_c t - \alpha_c Sin\gamma_{mf}t)}{\gamma_{mf}(\alpha_c^2+\gamma_{mf}^2)}\right] \quad (4.19)$$

where

$$\alpha_c = \frac{\lambda_m c}{L} \quad (4.20)$$

$$Q(m,c) = [-Cos\lambda_m + A_m Sin\lambda_m + B_m Cosh\lambda_m + C_m Sinh\lambda_m] \quad (4.21)$$

which on inversion gives

$$V(x,t) = \sum_{m=1}^{\infty}\frac{P_0}{V_m}\left[\frac{Q(m,c)(1-Cos\gamma_{mf}t)}{\gamma_{mf}} + \frac{Cos\alpha_c t - Cos\gamma_{mf}t}{\gamma_{mf}^2 - \alpha_c^2}\right.$$

$$-\frac{A_m(\gamma_{mf}Sin\alpha_c t - \alpha_c Sin\gamma_{mf}t)}{\gamma_{mf}(\gamma_{mf}^2-\alpha_c^2)} - \frac{B_m(Cosh\alpha_c t - Cos\gamma_{mf}t)}{\alpha_c^2+\gamma_{mf}^2}$$

$$\left.-\frac{C_m(\gamma_{mf}Sinh\alpha_c t - \alpha_c Sin\gamma_{mf}t)}{\gamma_{mf}(\alpha_c^2+\gamma_{mf}^2)}\right].\left(Sin\frac{\lambda_m x}{L} + A_m Cos\frac{\lambda_m x}{L} + B_m Sinh\frac{\lambda_m x}{L} + \right.$$

$$\left.C_m Cosh\frac{\lambda_m x}{L}\right) \quad (4.22)$$

Equation (4.22) represents the transverse displacement response to a distributed force moving at constant velocity of a uniform Rayleigh beam resting on elastic foundation and having arbitrary support end conditions.

(ii) Case II

If the inertia effect of the moving mass is not negligible, then$\Gamma_0 \neq 0$ and the solution to the entire equation (3.26) is sought. This is termed the moving mass problem. An exact analytical solution to equation (3.26) is not possible. Thus, as in the previous section, the modified Struble's asymptotic method is employed to get an approximate analytical solution. To this end, equation (3.26) is simplified and rearranged to take the form

$$\bar{V}_{tt}(m,t) + \frac{c\Gamma_0 LR_2(m,m,t)}{1+\Gamma_0 LR_1(m,m,t)}\bar{V}_t(m,t) + \frac{c^2\Gamma_0 LR_3(m,m,t)+\gamma_{mf}^2}{1+\Gamma_0 LR_1(m,m,t)}\bar{V}(m,t)$$

$$+\Gamma_0 L\sum_{\substack{k=1\\k\neq m}}^{\infty}\frac{[R_1(k,m,t)\bar{V}_{tt}(k,t) + cR_2(k,m,t)\bar{V}_t(k,t) + c^2R_3(k,m,t)\bar{V}(k,t)]}{1+\Gamma_0 LR_1(m,m,t)}$$

$$= \frac{\frac{\Gamma_0 g L^2}{\lambda_m}\left[Q(m,c) + Cos\frac{\lambda_m ct}{L} - A_m Sin\frac{\lambda_m ct}{L} - B_m Cosh\frac{\lambda_m ct}{L} - C_m Sinh\frac{\lambda_m ct}{L}\right]}{1 + \Gamma_0 L R_1(m,m,t)} \tag{4.23}$$

where

$$R_1(m,m,t) = \frac{1}{4}G_D(m,m) + \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{Cos(2n+1)\pi ct}{2n+1}G_E(n,m,m)$$
$$-\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{Sin(2n+1)\pi ct}{2n+1}G_F(n,m,m) \tag{4.24}$$

$$R_2(m,m,t) = \frac{1}{2}G_G(m,m) + \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{Cos(2n+1)\pi ct}{2n+1}G_H(n,m,m)$$
$$-\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{Sin(2n+1)\pi ct}{2n+1}G_I(n,m,m) \tag{4.25}$$

$$R_3(m,m,t) = \frac{1}{4}G_B(m,m) + \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{Cos(2n+1)\pi ct}{2n+1}G_J(n,m,m)$$
$$-\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{Sin(2n+1)\pi ct}{2n+1}G_L(n,m,m) \tag{4.26}$$

and

$$G_D(m,m) = G_D(k,m)|_{k=m} \qquad G_E(n,m,m) = G_E(n,k,m)|_{k=m}$$
$$G_F(n,m,m) = G_F(n,k,m)|_{k=m} G_G(m,m) = G_G(k,m)|_{k=m}$$
$$G_H(n,m,m) = G_H(n,k,m)|_{k=m} G_I(n,m,m) = G_I(n,k,m)|_{k=m}$$
$$G_B(m,m) = G_B(k,m)|_{k=m} G_J(n,m,m) = G_J(n,k,m)|_{k=m}$$
$$G_L(n,m,m) = G_L(n,k,m)|_{k=m} \tag{4.27}$$

Like in the previous case, the homogeneous part of equation (4.23) is considered and a modified frequency corresponding to the frequency of the freesystem due to the presence of the moving mass M is sought. An equivalent free system operator defined by the modified frequency then replaces equation (4.23). Thus, a parameter $\Gamma_1 < 1$is considered for any arbitrary mass ratio $\Gamma_0$defined as

$$\Gamma_1 = \frac{\Gamma_0}{1 + \Gamma_0} \tag{4.28}$$

It can be shown that

$$\Gamma_0 = \Gamma_1 + o(\Gamma_1^2) \tag{4.29}$$

and

$$\frac{1}{1 + \Gamma_1 L R_1(m,m,t)} = 1 - \Gamma_1 L R_1(m,m,t) + o(\Gamma_1^2) \tag{4.30}$$

where

$$|\Gamma_1 L R_1(m,m,t) + o(\Gamma_1^2)| < 1 \tag{4.31}$$

Following the same arguments with those in the previous section, Struble's technique is used to obtain

$$\Omega_m = \frac{8\gamma_{mf}^2 - \Gamma_1 L\{\gamma_{mf}^2 G_D(m,m) - c^2 G_B(m,m)\}}{8\gamma_{mf}} \tag{4.32}$$

as the modified frequency corresponding to the frequency of the free system due to the presence of the moving mass.

Thus, to solve the non-homogeneous equation (4.23), the differential operator which acts on $\bar{V}(m,t)$ and $\bar{V}(k,t)$ is replaced by the equivalent free system operator defined by the modified frequency Ω_m, i.e

$$\bar{V}_{tt}(m,t) + \Omega_m^2 \bar{V}(m,t) = \frac{\Gamma_1 L^2 g}{\lambda_m}\Big[Q(m,c) + Cos\frac{\lambda_m ct}{L} - A_m Sin\frac{\lambda_m ct}{L} - B_m Cosh\frac{\lambda_m ct}{L} - C_m Sinh\frac{\lambda_m ct}{L}\Big] \tag{4.33}$$

This is analogous to equation (4.17). Thus, using similar argument as in the previous section, the solution to equation (4.33) can be obtained as

$$V(x,t) = \sum_{m=1}^{\infty}\frac{\Gamma_1 L^2 g}{\lambda_m V_m}\Big[Q(m,c)\left(\frac{1-Cos\Omega_m t}{\Omega_m}\right) + \frac{Cos\alpha_c t - Cos\Omega_m t}{\Omega_m^2 - \alpha_c^2} - A_m\left(\frac{\Omega_m Sin\alpha_c t - \alpha_c Sin\Omega_m t}{\Omega_m(\Omega_m^2 - \alpha_c^2)}\right) - B_m\left(\frac{Cosh\alpha_c t - Cos\Omega_m t}{\alpha_c^2 + \Omega_m^2}\right) - C_m\left(\frac{\Omega_m Sinh\alpha_c t - \alpha_c Sin\Omega_m t}{\Omega_m(\alpha_c^2 + \Omega_m^2)}\right)\Big].\left(Sin\frac{\lambda_m x}{L} + A_m Cos\frac{\lambda_m x}{L} + B_m Sinh\frac{\lambda_m x}{L} + C_m Cosh\frac{\lambda_m x}{L}\right) \tag{4.34}$$

Equation (4.34) now represents the transverse displacement response to a distributed mass moving at constant velocity of a uniform Rayleigh beam resting on an elastic foundation which is valid for all variants of classical boundary conditions. For a particular boundary condition, it is only necessary to compute the constants A_m, B_m, C_m and λ_m from the beam functions using the boundary conditions and then substitute back into equation (4.34)

5. 0 Illustrative Examples

In this section, practical examples of classical boundary conditions are selected to illustrate the analyses presented in this paper.

5.1 Clamped-Clamped Uniform Rayleigh Beam

In this case, both ends of the uniform Rayleigh beam are clamped. Thus, both deflection and slope vanish and we have the boundary conditions given by

$$V(0,t) = 0 = V(L,t) \quad and \quad \frac{\partial V(0,t)}{\partial x} = 0 = \frac{\partial V(L,t)}{\partial x} \tag{5.1}$$

and for normal modes

$$U_m(0) = 0 = U_m(L) \quad and \quad \frac{\partial U_m(0)}{\partial x} = 0 = \frac{\partial U_m(L)}{\partial x} \tag{5.2}$$

which implies

$$U_k(0) = 0 = U_k(L) \quad and \quad \frac{\partial U_k(0)}{\partial x} = 0 = \frac{\partial U_k(L)}{\partial x} \tag{5.3}$$

Applications of (5.2)to (3.5) yields

$$A_m = \frac{Sinh\lambda_m - Sin\lambda_m}{Cos\lambda_m - Cosh\lambda_m} = \frac{Cos\lambda_m - Cosh\lambda_m}{Sin\lambda_m + Sinh\lambda_m} = -C_m \quad and \quad B_m = -1 \tag{5.4}$$

The frequency equation becomes

$$Cos\lambda_m Cosh\lambda_m = 1 \quad (5.5)$$

It follows from equation (5.5) that

$$\lambda_1 = 4.73004, \quad \lambda_2 = 7.85320, \quad \lambda_3 = 10.99561 \quad (5.6)$$

Expression for $A_k,\ B_k,\ C_k$ and the corresponding frequency equation are obtained by a simple interchange of m with$k$ in (5.4) and (5.5). Substituting (5.4) and (5.5) into equations (4.22) and (4.34) one obtains the displacement response respectively to a distributed moving force and a distributed moving mass of a Clamped-Clamped uniform Rayleigh beam on elastic foundation.

5.2 One End Clamped - One End Free Uniform Rayleigh Beam

Next at $x = 0$, the beam is taken to be clamped and at the end $x = L$, the beam is free. Thus, the boundary conditions of the uniform Rayleigh beam can be written as,

$$V(0,t) = 0 = \frac{\partial V(0,t)}{\partial x} \quad \text{and} \quad \frac{\partial^2 V(L,t)}{\partial x^2} = 0 = \frac{\partial^3 V(L,t)}{\partial x^3} \quad (5.7)$$

thus for normal modes

$$U_m(0) = 0 = \frac{dU_m(0)}{dx} \quad \text{and} \quad \frac{\mathrm{d}^2 \mathrm{U_m(L)}}{dx^2} = 0 = \frac{\mathrm{d}^3 \mathrm{U_m(L)}}{dx^3} \quad (5.8)$$

which implies

$$U_k(0) = 0 = \frac{dU_k(0)}{dx} \quad \text{and} \quad \frac{\mathrm{d}^2 \mathrm{U_k(L)}}{dx^2} = 0 = \frac{\mathrm{d}^3 \mathrm{U_k(L)}}{dx^3} \quad (5.9)$$

Applications of (5.8) and (5.9) to (3.5) yields

$$A_m = -\frac{Sin\lambda_m + Sinh\lambda_m}{Cos\lambda_m + Cosh\lambda_m} = \frac{Cos\lambda_m + Cosh\lambda_m}{Sin\lambda_m + Sinh\lambda_m} = -C_m \quad and \quad B_m = -1 \quad (5.10)$$

and the frequency equation for both end conditions is

$$Cos\lambda_m Cosh\lambda_m = -1 \quad (5.11)$$

such that

$$\lambda_1 = 1.875, \quad \lambda_2 = 4.694, \quad \lambda_3 = 7.855 \quad (5.12)$$

Using (5.10) and (5.11) in equations (4.22) and (4.34), one obtains the displacement response respectively to a distributed moving force and distributed moving mass of a clamped-free uniform Rayleigh beam resting on elastic foundation.

6.0 Discussion of the Analytical Solutions

At this point, it is important to establish conditions under which resonance occurs for an undamped system such as this. Resonance takes place when the motion of the vibrating structure becomes unbounded.

Equation (4.34) clearly show that, the uniform Rayleigh beam traversed by a distributed force moving with a constant velocity will attain resonance at

$$\gamma_{mf} = \alpha_c \quad (6.1)$$

while the same beam under the action of a moving distributed mass experiences resonance effect whenever

$$\Omega_m = \alpha_c \quad (6.2)$$

but

$$\Omega_m = \frac{8\gamma_{mf}^2 - \Gamma_1 L\{\gamma_{mf}^2 G_D(m,m) - c^2 G_B(m,m)\}}{8\gamma_{mf}} \quad (6.3)$$

which implies that

$$\Omega_m = \gamma_{mf}\left[1 - \frac{\Gamma_1 L}{8\gamma_{mf}}\left\{G_D(m,m) - \frac{c^2 G_B(m,m)}{\gamma_{mf}^2}\right\}\right] = \alpha_c \qquad (6.4)$$

Equations (6.1) and (6.4) show that for the same natural frequency, the critical speed for the same system of a uniform Rayleigh beam resting on a constant foundation and traversed by a moving distributed force is greater than that traversed by a moving distributed mass. Thus resonance is reached earlier in the moving distributed mass system than in the moving distributed force system.

7.0 Numerical Calculations and Discussions

In order to present the calculations of practical interests in dynamic of structures and Engineering design for illustrative examples considered, the uniform Rayleigh beam is taken to be of length L=12.192m. The load velocity, c=8.128ms^{-1} and E=2109$\times 10^9$kg/m. The values of the rotatory inertia correction factor r^0 are varied between 0.005 and 9.5, while the values of the foundation moduli constant K are varied between 0 and $4000000Nm^2$.The flexural vibrations of the uniform Rayleigh beam are calculated and graphs are plotted for beam response against time for various values of rotatory inertia correction factor r^0 and foundation moduli K.

In Figure 7.1, the transverse displacement response of the uniform clamped-clamped Rayleigh beam to distributed forces for various values of Foundation moduli K and fixed value of rotatory inertia correction factor $r^0 = 5$ are displayed. It is seen from this figure that as the values of the foundation moduli increase, the response amplitude of the clamped-clamped beam decrease. The same result is obtained when the clamped-clamped Rayleigh beam is traversed by moving distributed masses as shown in Figure 7.3. The response of the clamped-clamped uniform Rayleigh beam to distributed forces for various values of rotatory inertia correction factor r^0 and fixed value of foundation modulus$K = 40000$are shown in Figure 7.2. It is seen that the deflection of the beam decreases with increase in the rotatory inertia correction factor. The same result and analysis are obtained when the clamped-clamped beam is acted upon by moving distributed masses as shown in figure 7.4. Figure 7.5 depicts the comparison of the transverse displacement response for moving distributed force and moving distributed mass cases of the uniform clamped-clamped Rayleigh beam for fixed values of foundation modulus$K = 400000$ and rotatory inertia correction factor $r^0 = 5$. Clearly, the response amplitude of the moving distributed mass is greater than that of the moving distributed force problem.In Figure 7.6, the transverse displacement response of the uniform clamped-free Rayleigh beam to moving distributed forces for various values of Foundation moduli K and fixed value of rotatory inertia correction factor $r^0 = 5$ are displayed. It is seen that as the values of the foundation moduli increases, the response amplitude of the clamped-free beam under the action of distributedforces decreases. The same behaviour characterizes the deflection profile of the clamped-free Rayleigh beam under the action of moving distributed masses for various foundation moduli as is depicted in Figure 7.8. Furthermore, the deflection profile of the clamped-free uniform Rayleigh beam under moving distributed forces for various values of rotatory inertia correction factor r^0 and fixed value of foundation modulus$K = 40000$ is shown in Figures 7.7. It is seen that the deflection of the beam decreases with increase in the rotatory inertia correction factor. The same result and analysis are obtained when the cantilever beam is traversed by moving distributed masses as shown in figure 7.9. Finally, Figure 7.10 depicts the comparison of the transverse displacement response for moving distributed force and moving distributed mass cases of the uniform clamped-free Rayleigh beam for fixed values of foundation modulus $K = 400000$ and rotatory inertia correction factor $r^0 = 5$. It is observed that the response amplitude of the moving distributed force is greater than that of the moving distributed mass problem.

8.0 Conclusion

A closed form solution valid for all variants of classical boundary conditions of the dynamical system is presented for the displacement response to a travelling distributed mass of a finite uniform Rayleigh beam resting on an elastic foundation. The solution technique is based on the generalized integral transformation, the representation of the Heaviside function in series form and a modification of Struble's asymptotic method often used in treating homogeneous and non-homogeneous nonlinear oscillatory systems. Numerical calculation and representation in plotted curves for both illustrative examples considered depict the following interesting results:

(i) the critical speed for the same system consisting a finite uniform Rayleigh beam resting on an elastic foundation and traversed by a moving distributed force is greater than that traversed by a moving distributed mass. Hence, resonance is reached earlier in the moving distributed mass system than in the moving distributed force system.

(ii) as the rotatory inertia correction factor r^0 increases, the transverse displacement response of the beam model decreases.

(iii) for both moving distributed force and moving distributed mass problems, the response amplitudes of the Rayleigh beam decrease as the foundation modulus K increases and finally,

(iv) for fixed values of foundation modulus K and rotatory inertia correction factor r^0, the response amplitude of the moving distributed mass problem is greater than that of the moving distributed force problem. This confirms the result already reported in literature for cases when the travelling load is modelled as concentrated loads.

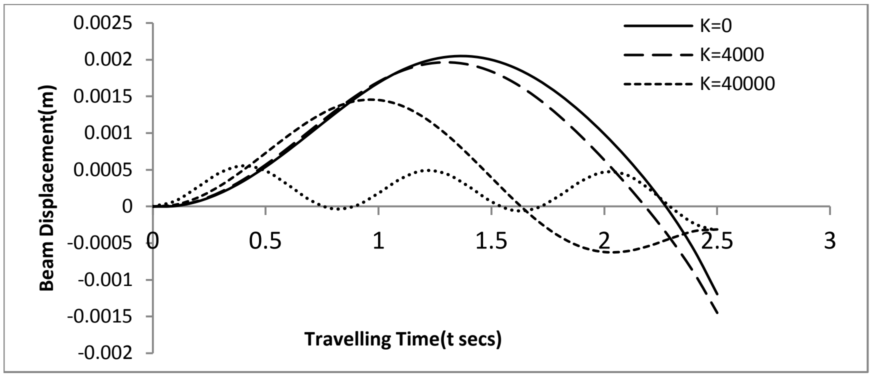

Fig 7.1: Displacement response todistributed forces of uniform clamped-clamped Rayleighbeam for various values of foundation moduli K.

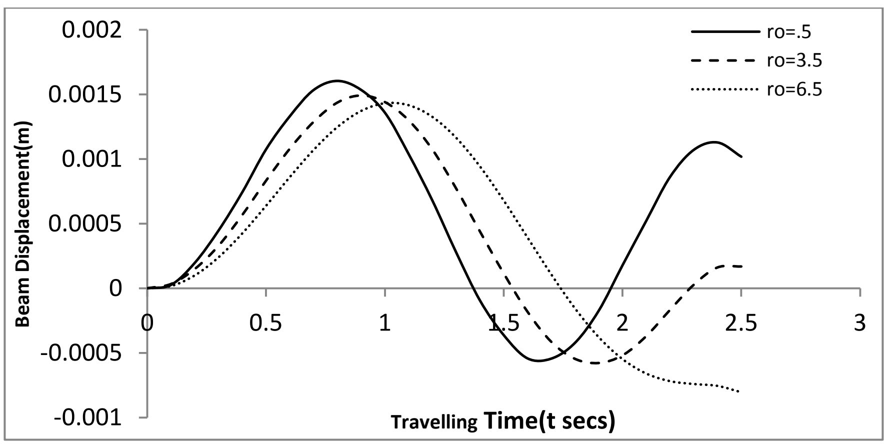

Fig 7.2: Displacement response to distributed forces of uniform clamped-clamped Rayleigh beam for various values of rotatory inertia correction factor r_0.

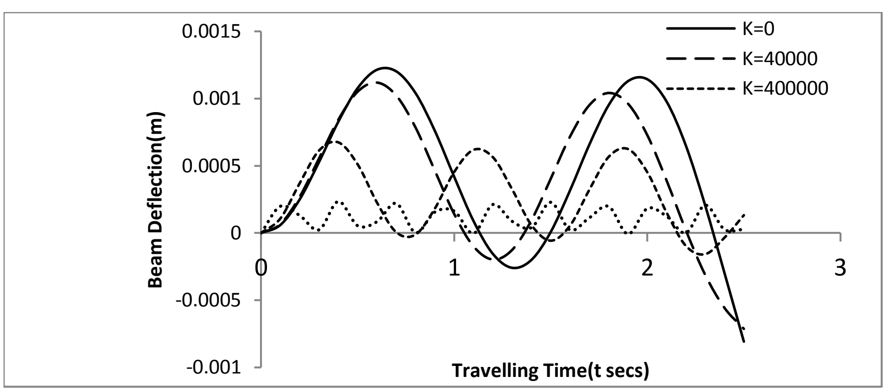

Fig 7.3: Deflection profile of uniform clamped-clamped Rayleigh beam under action of action of distributed masses for various values offoundationmoduli K.

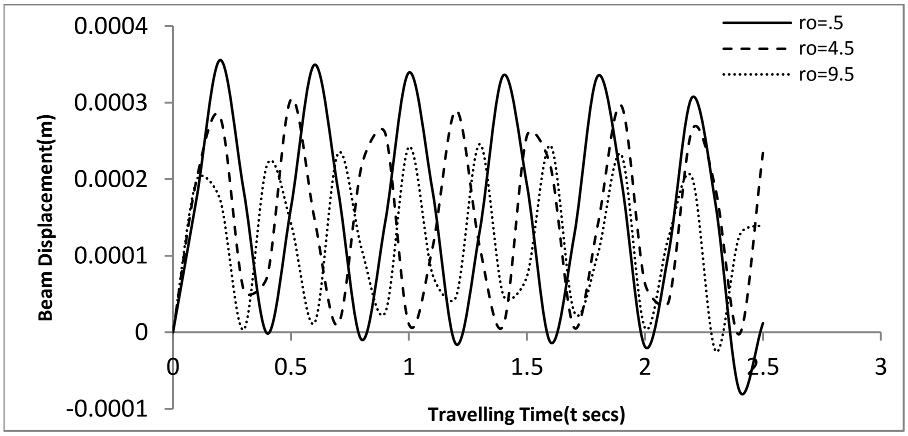

Fig 7.4: Deflection profile of uniform clamped-clamped Rayleigh beam under action of distributed massesfor various values of Rotatory inertia correction factor Ro.

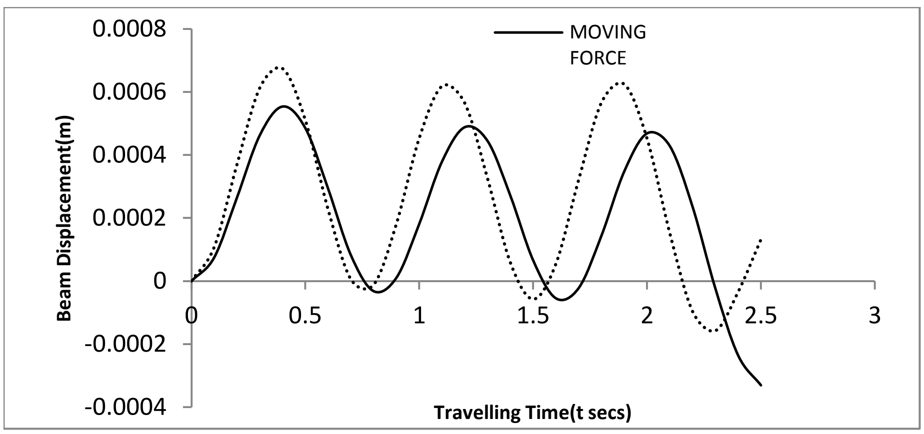

Fig 7.5: Comparison of displacementresponse to distributed force and distributed mass cases of uniformclamped-clamped Rayleigh beam for fixed values of K=400000 and r^0=5.

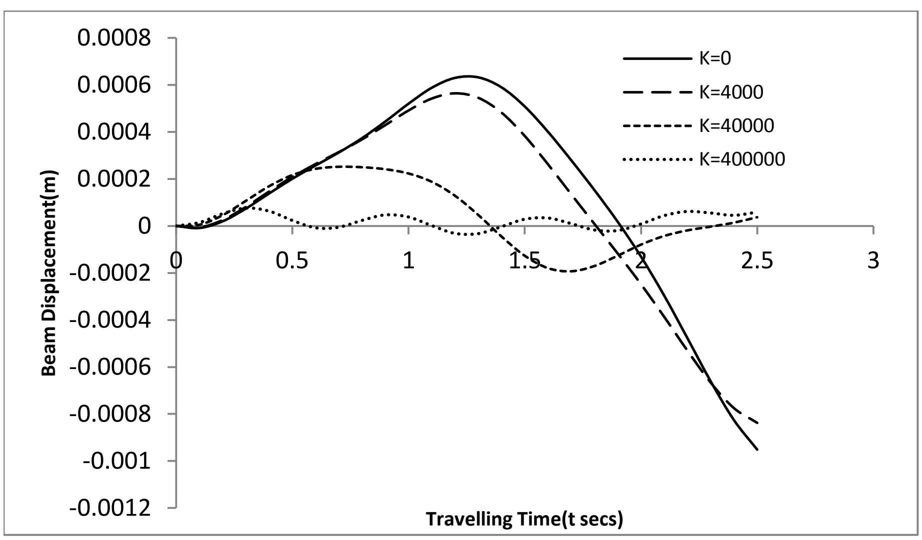

Fig 7.6: Displacement response of uniform clamped-free Rayleigh beam under action of distributed forcesfor various values of foundation moduli K.

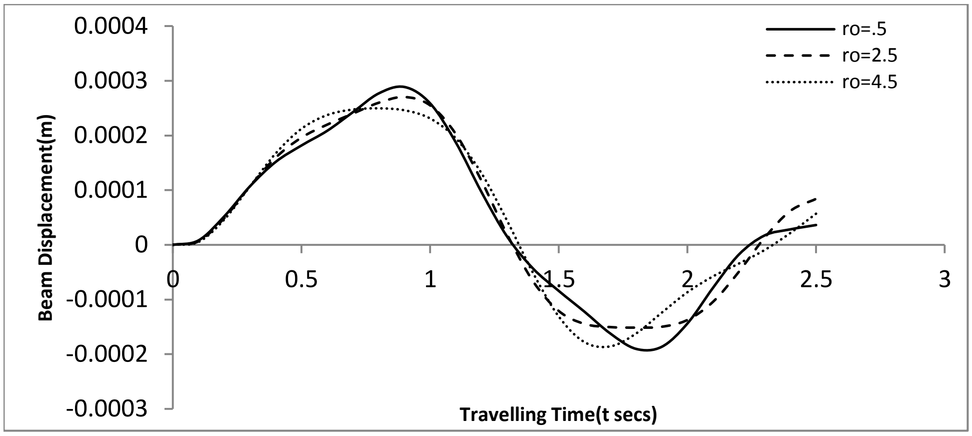

Fig 7.7: Displacement response of uniform clamped-free Rayleigh beam under action of distributed forces for various values of rotatory inertia correction factor r_0.

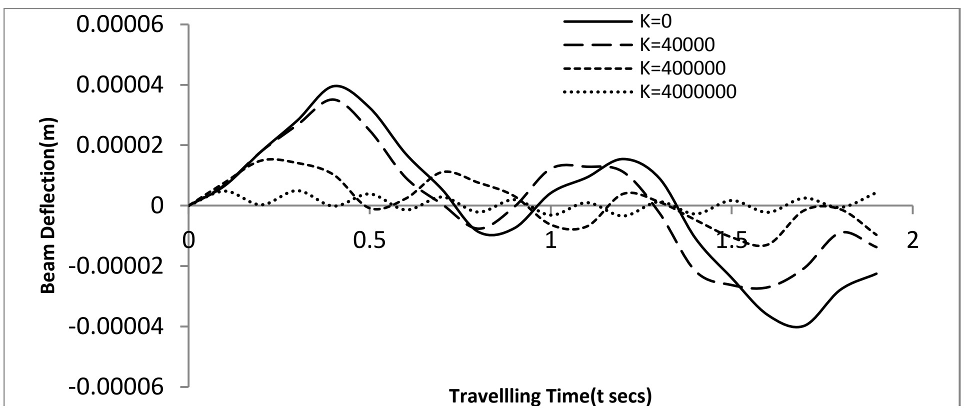

Fig 7.8: Deflection profile of uniformclamped-free Rayleigh beam under action of distributedmasses for various values of foundation moduli K.

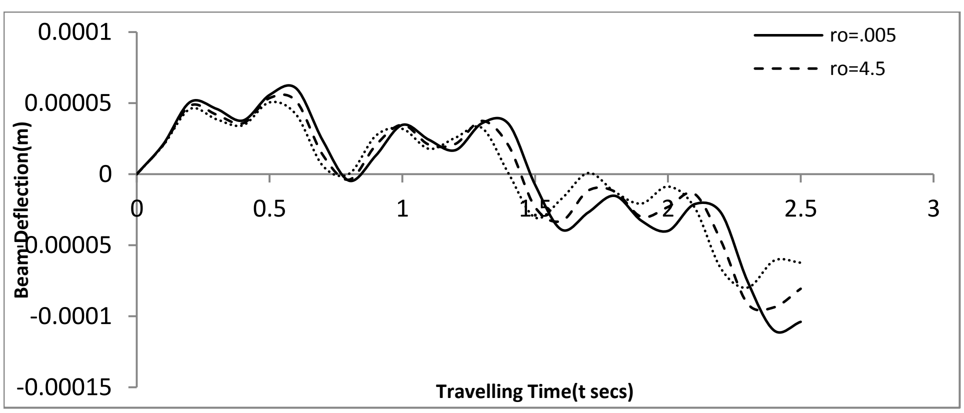

Fig 7.9: Deflection profile of uniform clamped -free Rayleigh beam under action of distributed masses for various values of rotatory inertia correction factor r^0

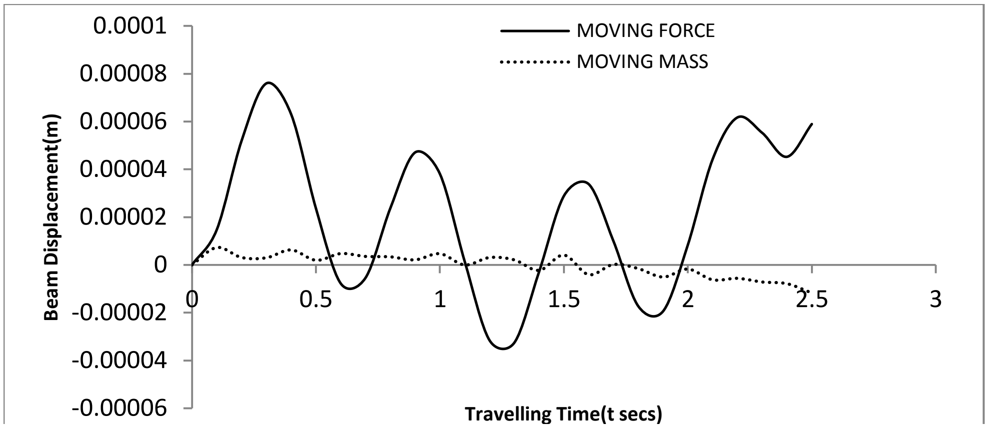

Fig 7.10: Comparison of displacement response to distributed force anddistributed mass cases of uniformclamped-free Rayleigh beam forfixed values of K=400000 and r^0=5

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